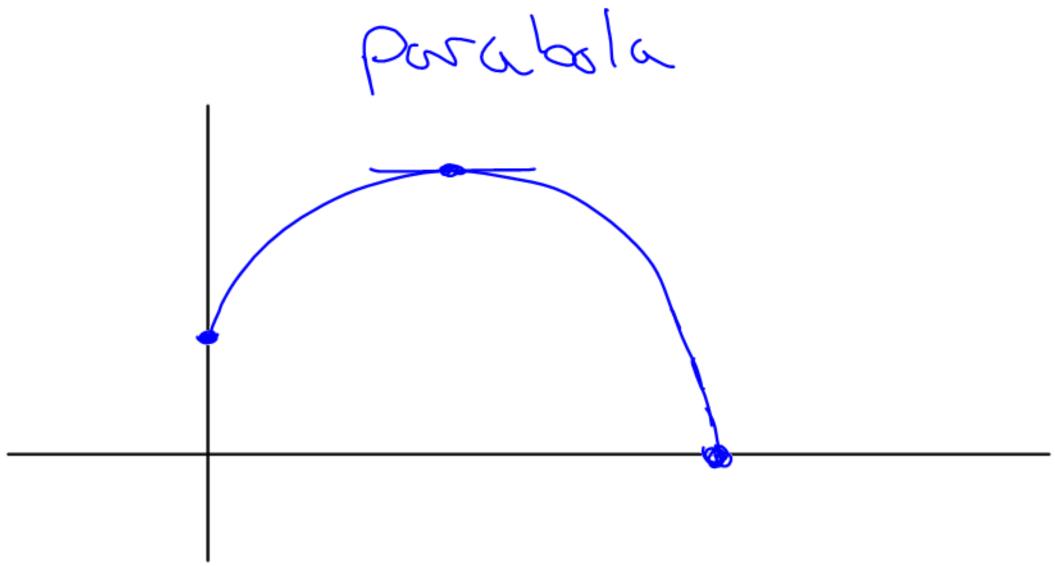
Week 4: Inverse Trig, Tangent Lines and Linearization

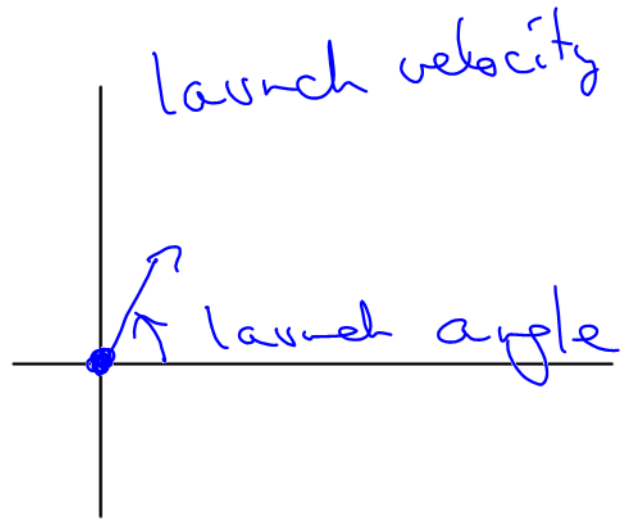
Goals:

- Review inverse trigonometric functions and their derivatives.
- Create and use linearization/tangent line formulas.

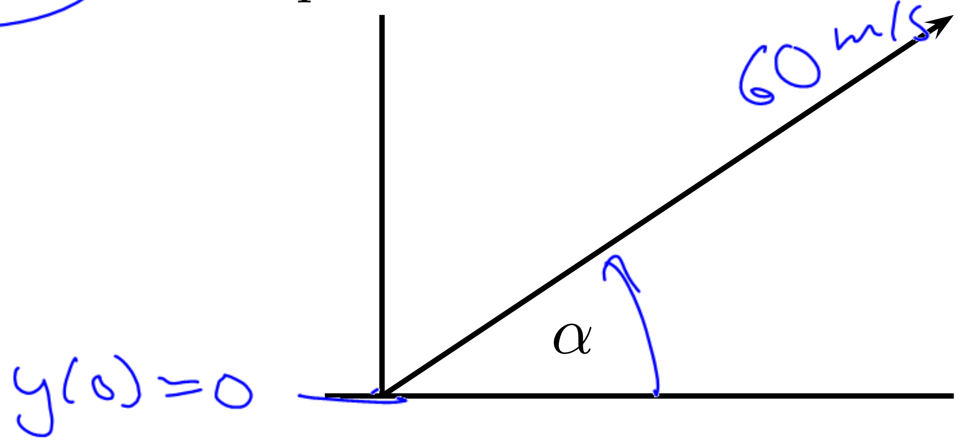
Projectile Motion Using Vectors and Inverse Trig

To introduce our study our inverse trigonometric functions, we turn to a problem that examines the path of an object moving under the influence of the force of gravity. It may be a problem that you have seen in some version already. Here we will emphasize **vector notation** in the solution.



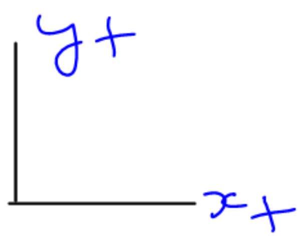


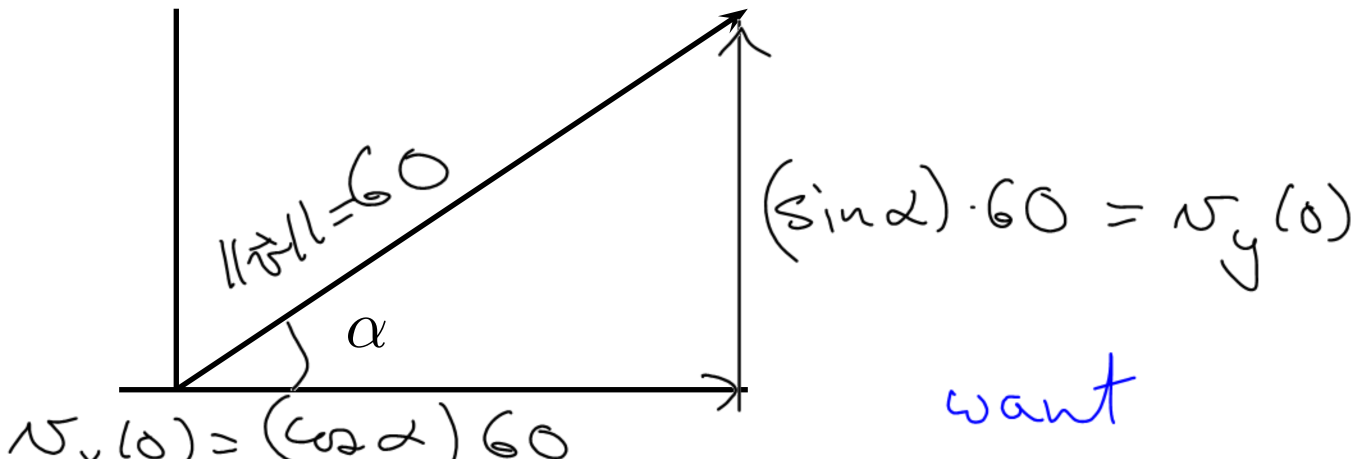
A projectile is fired from the origin, with angle of elevation α (radians) and speed 60 meters per second.



Assume that air resistance is negligible and that the only force on the projectile is gravity (producing a downward acceleration of $g = 9.8$ meters per second per second).

Problem. Draw a free-body diagram for the projectile.





want

Problem. Find a (vector-valued) formula for the position of the projectile at time t.

$$\vec{a}(t) = \langle 0, -9.8 \rangle \quad m/s^2$$

$\frac{d}{dt}$?

$$\vec{v}(t) = \langle a, -9.8t + b \rangle$$

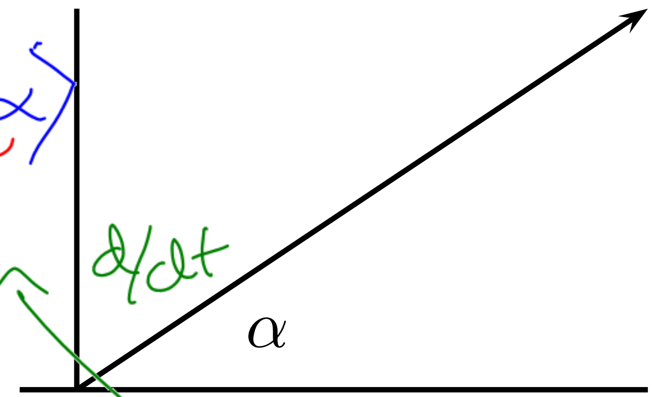
a const
b const

$a, b = ?$ Use initial velocity to find a, b

At time $= 0$, speed $= 60$ m/s, angle α

x of $\vec{v}(0)$: $\boxed{a = 60\cos\alpha}$ | y of $\vec{v}(0) = \boxed{b = 60\sin\alpha}$

$$\vec{v}(t) = \langle \underbrace{60\cos\alpha}_{a,\ \text{const}},\ -9.8t + \underbrace{60\sin\alpha}_{b} \rangle$$



d/dt ??   d/dt   d/dt ✓   d/dt

$$\vec{r}(t) = \langle [60\cos\alpha]t + c,\ -\frac{9.8t^2}{2} + [60\sin\alpha]\cdot t + d \rangle$$

Use $\vec{r}(0) = \langle 0, 0 \rangle$ Start trajectory at $(0,0)$

x at $t=0$: $c = 0$ | y at $t=0$: $d = 0$

so

$$\vec{r}(t) = \langle 60\cos\alpha \cdot t,\ -\frac{9.8t^2}{2} + 60\sin\alpha \cdot t \rangle$$

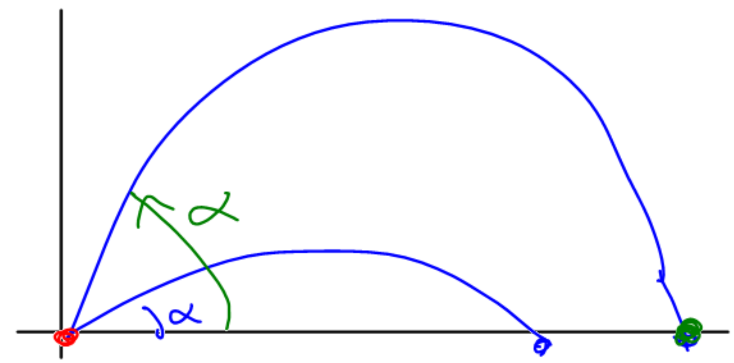
vector-valued function for the trajectory.

Projectile Motion Using Vectors - Application

fixed speed at $t=0$ of 60 m/s

$$\mathbf{r}(t) = \left\langle \underbrace{60\cos(\alpha)t}_{x(t)},\ \underbrace{\frac{-9.8}{2}t^2 + 60\sin(\alpha)t}\right\rangle$$

Problem. If launched at angle α, how far down range will the projectile land?



Solve

for 0 $y = \frac{-9.8t^2}{2} + 60\sin\alpha \cdot t$

$0 = t\left[-4.9t + 60\sin(\alpha)\right]$

$\langle x, 0 \rangle$

y

$t=0$, and $-4.9t + 60\sin\alpha = 0$

launch

$60\sin\alpha = 4.9t$

$\boxed{t = \frac{60\sin\alpha}{4.9}}$ (need x, rather than t)

(Range continued) $$\mathbf{r}(t) = \left\langle 60\cos(\alpha)t,\ \frac{-9.8}{2}t^2 + 60\sin(\alpha)t \right\rangle$$

at impact

$$x = 60\cos(\alpha)\left[\frac{60\sin\alpha}{4.9}\right]$$

$$\boxed{x = \frac{3600}{4.9}\cos(\alpha)\sin(\alpha)}$$

For any launch angle, we can find/compute the impact location.!

find ?

Problem. At what angle α should the projectile be launched so that it will land 300 meters down range?

want

$$\overset{300}{\cancel{x_{land}}} = \frac{3600}{4.9}\underbrace{\cos(\alpha)\sin(\alpha)}, \text{ solve for } \alpha$$

$$300 = \frac{3600}{4.9}\left(\frac{\sin(2\alpha)}{2}\right)$$

land x

$$\frac{300 \cdot 9.8}{3600} = \sin(2\alpha) \rightarrow 0.81\dot{6} = \sin(2\alpha)$$

g

$(\text{speed} @ t=0)^2$

$$\frac{0.956}{2} = \frac{2\alpha}{2} \quad (\text{rad})$$

Mem:

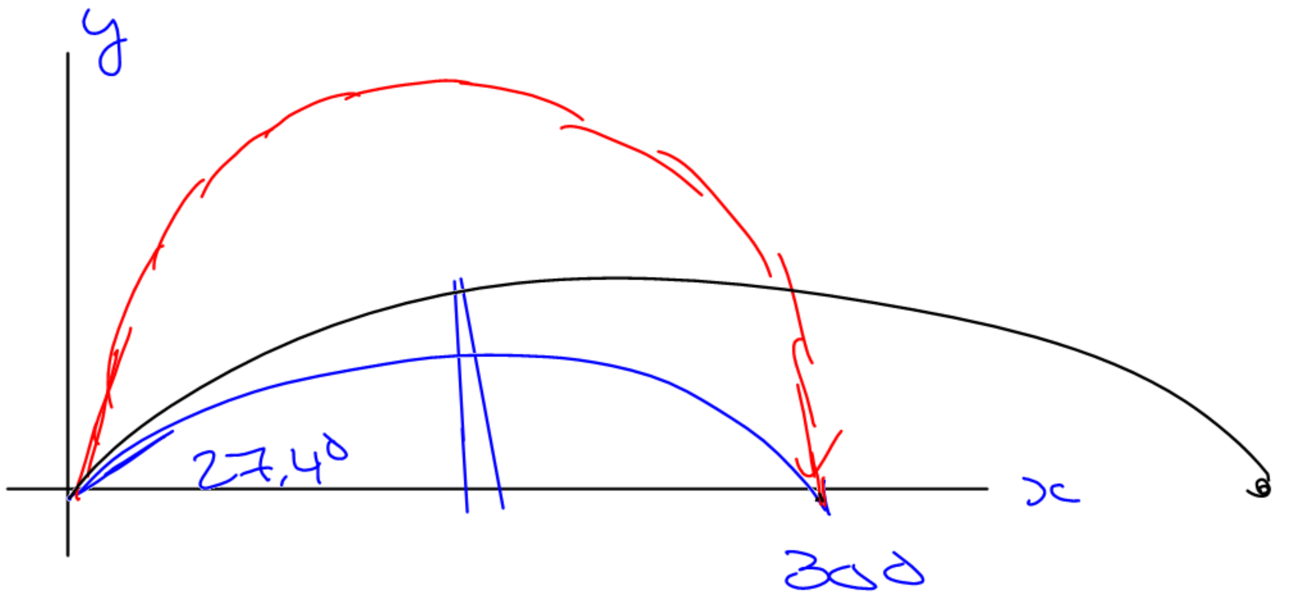
$$\sin^2 t + \cos^2 t = 1$$

$$\alpha = 0.478 \text{ rad} \times \frac{180}{\pi}$$

or $\boxed{\alpha = 27.4°.}$

~~Useful trig identity~~: $\underbrace{\sin(2\alpha)}_{2} = \cancel{2}\underbrace{\cos(\alpha)\sin(\alpha)}$

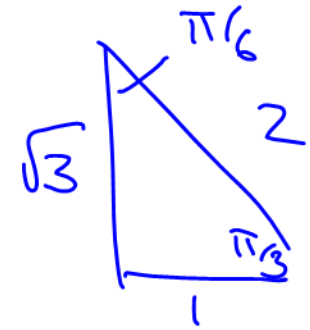
Problem. Consider your answer to the previous launch angle question: is there something suspicious or incomplete about your answer?



There is a second angle we didn't find!

Inverse Trigonometric Functions

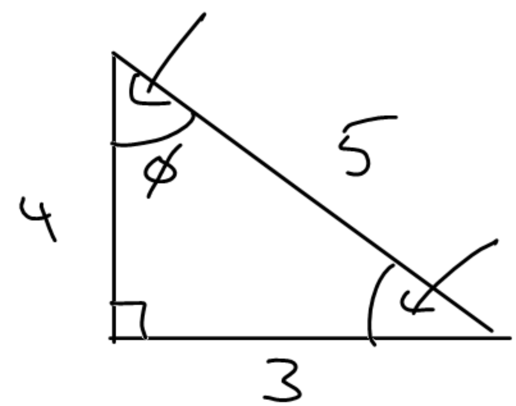
Example: *Evaluate* $\sin\left(\frac{\pi}{3}\right)$.



$$\sin(\pi/3) \cong 0.866\ldots$$
$$= \frac{\sqrt{3}}{2}$$

Example: *Draw a right-angle triangle with a hypotenuse of length 5, and other side lengths of 3 and 4.*

$$\cos(\phi) = \frac{4}{5} \rightarrow \phi = 0.635 \text{ rad}$$



$$\sin(\theta) = \frac{4}{5}$$

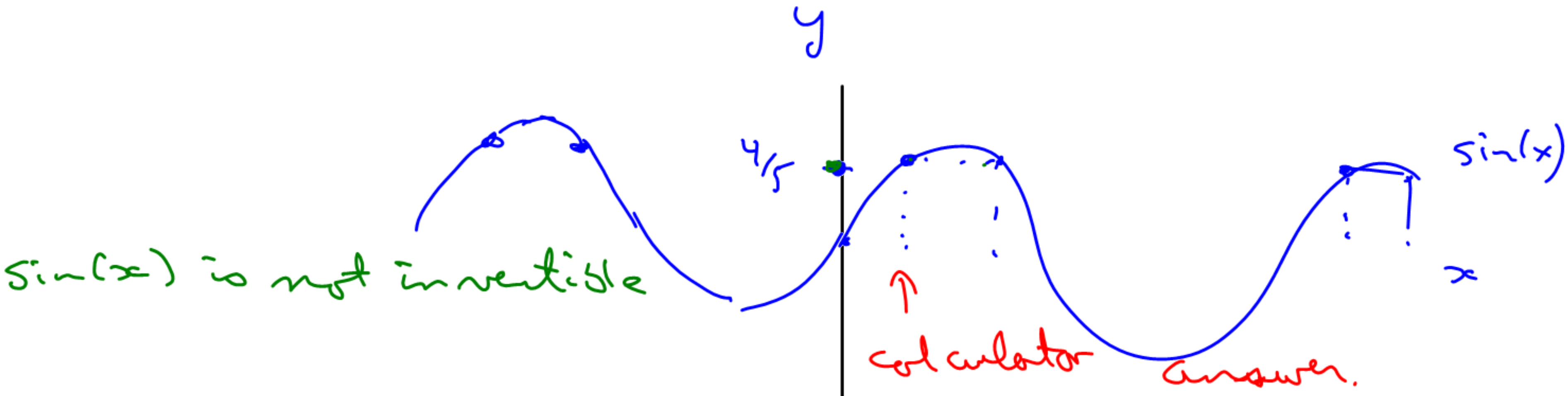
$$\theta = 0.927 \text{ rad}$$

Determine the missing angles in the triangle.

"inverse" not reciprocal, $\frac{1}{-}$)

Most students would use the "SHIFT + sin" or "$\sin^{-1}$" button combination on a calculator to find the missing angles in the previous question.

Example: *Why should you (as a mathematician) be suspicious of such an easy implementation of the inverse of the sine function?*



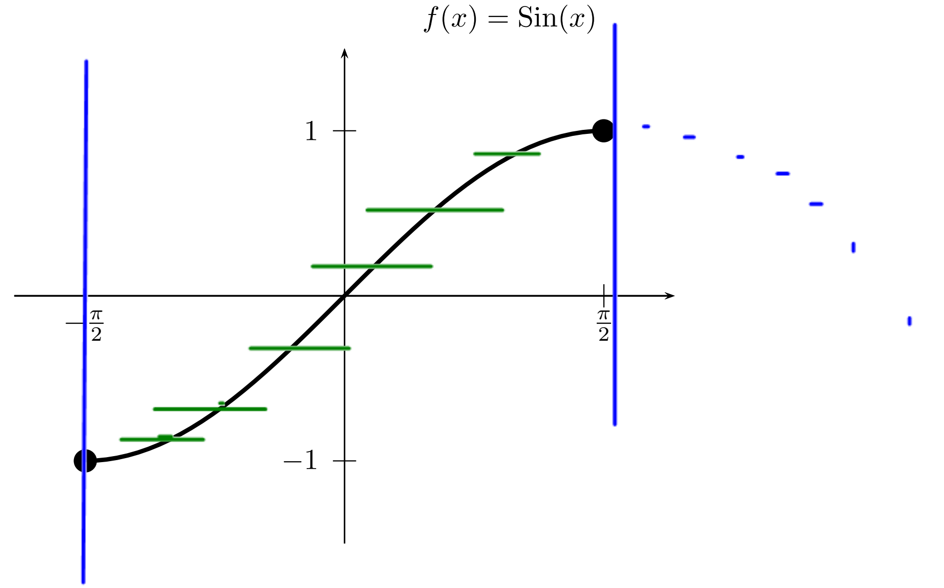
How can we remove the obstacle to an inverse of sine? (Clearly, there must be a way since the calculator is doing **something**!)

Sine and arcsine

For convenience we call this new function $\mathrm{Sin}\,(x)$, where

$$\mathrm{Sin}\,(x) = \sin(x)$$

provided $\underbrace{-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}}$.



passes the horizontal line test.

Example: Sin(x) *has an inverse: what are two notations for this inverse function?*

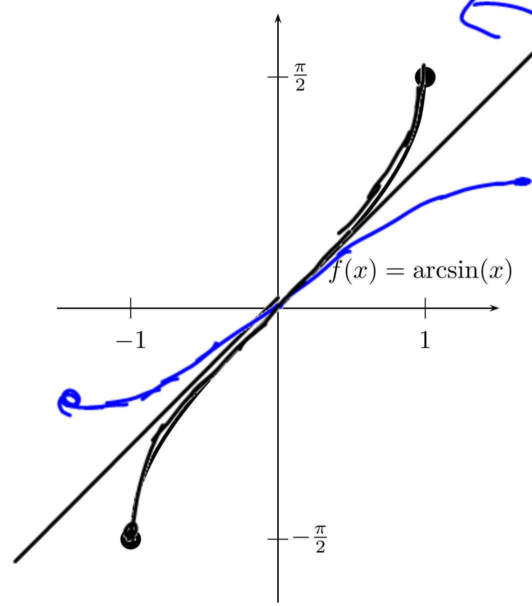
$\sin^{-1}(x)$

or

$\boxed{\text{arcsin}}(x)$

indivisible

inverse



reflection.

The domain of arcsin is:

acceptable inputs to arcsin (ratios → angles)

↳ domain $[-1, 1]$

The range of arcsin is:

(output)

$[-\pi/2, \pi/2]$

Sine and Arcsine as Inverses

like $e^{\ln(x)} = x$

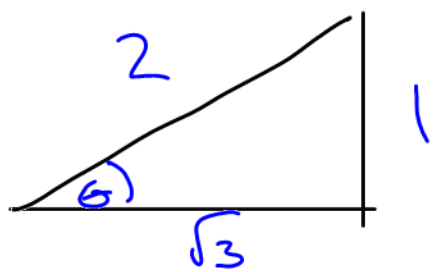
Since arcsin undoes what sin does, and vice-versa, the following equations are true, but only for the specified values of x:

$$\arcsin(\sin x) = x, \quad \text{for} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin(\arcsin x) = x, \quad \text{for} \quad -1 \leq x \leq 1.$$

Example: *What is the value of* arcsin(0.5)*?* $\longrightarrow$ $\sin(\theta) = 0.5 = \frac{1}{2}$ $\frac{o}{H}$

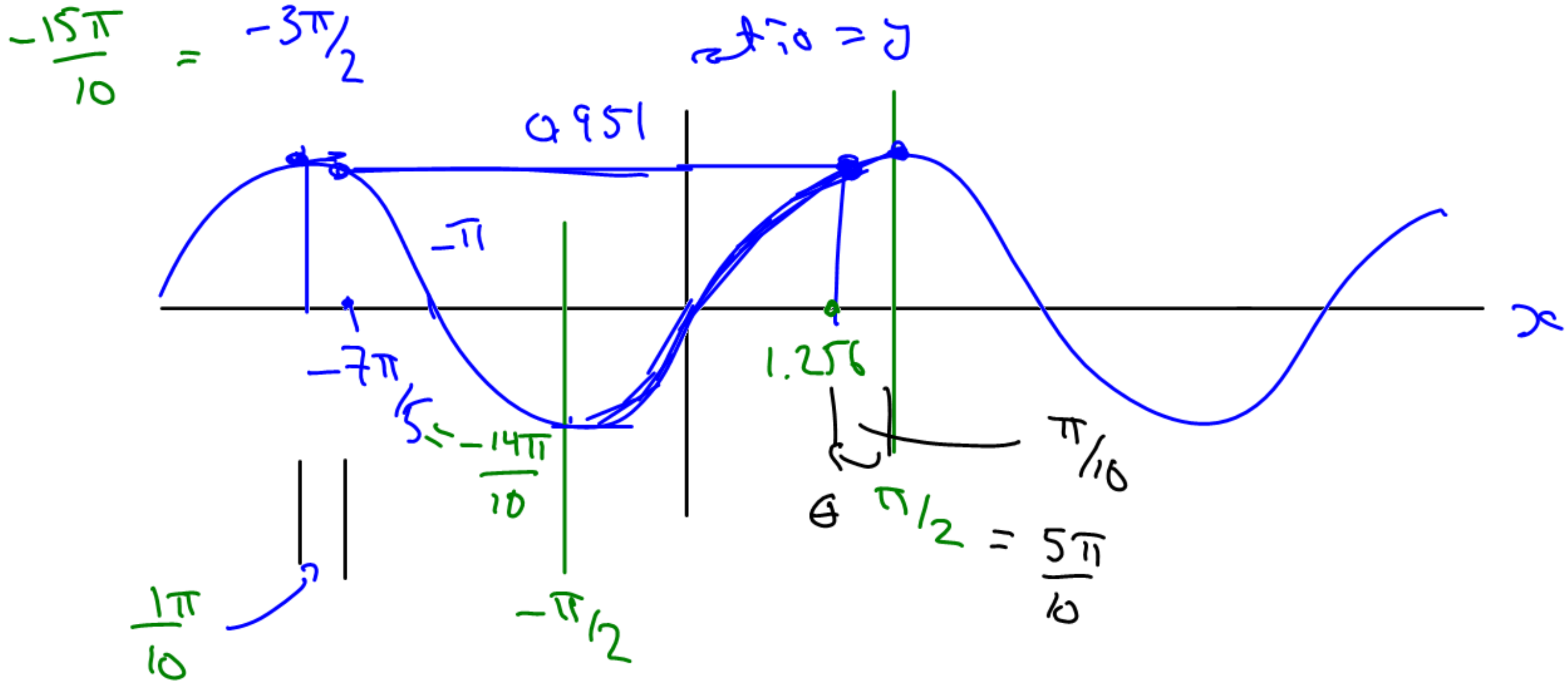
Calculator: $\arcsin(0.5) \simeq 0.524$



$\theta = \pi/6 = 0.524\ldots$

Example: $\sin\left(\dfrac{-7\pi}{5}\right) = 0.951$, *so what is the value of* $\arcsin(0.951)$*?*

calculator $\arcsin(0.951) = 1.256$ rad. $\left(\neq \dfrac{-7\pi}{5}\right)$



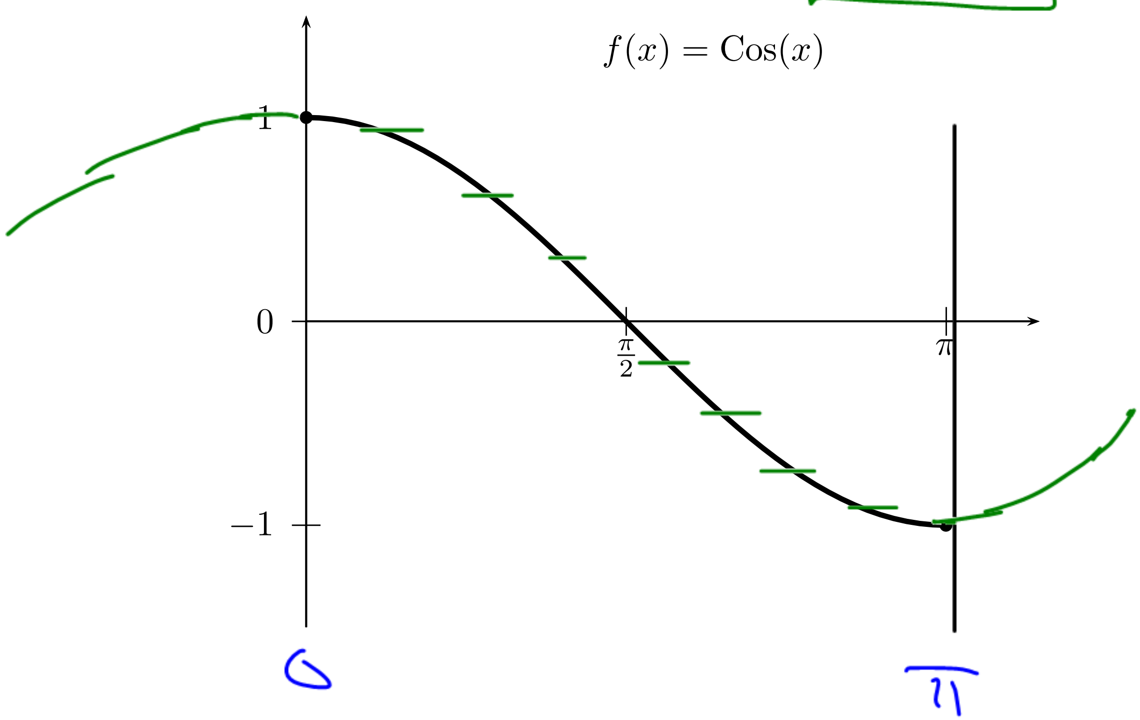
$$\theta = \frac{5\pi}{10} - \frac{\pi}{10} = \frac{4\pi}{10} = \frac{2\pi}{5}$$

$$\cong 1.256 \text{ rad.}$$

Cosine and arccosine

The inverse of *cosine* is obtained by a calculation similar to the way the inverse of *sine* was determined. We analyze *cosine* from 0 to π; this is shown in the graph on the right.
For convenience, we could call this new function $\mathrm{Cos}(x)$ where

$$\mathrm{Cos}(x) = \cos(x), \text{ provided } 0 \leq x \leq \pi.$$



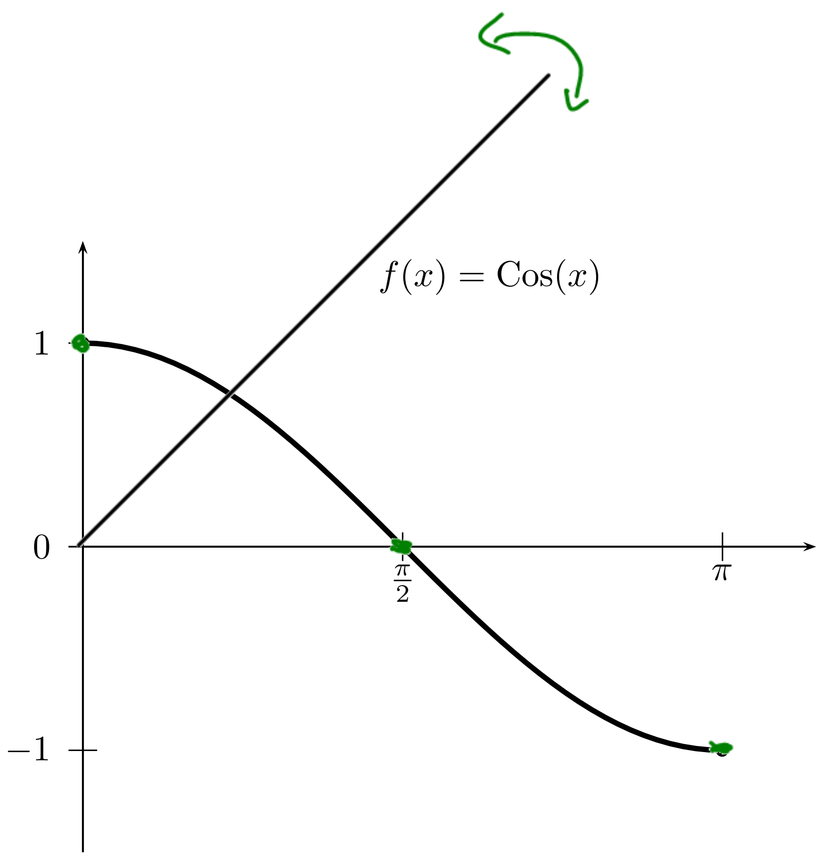
$\mathrm{Cos}(x)$ satisfies the horizontal line test and therefore has an inverse function which we call the **inverse cosine function** and denote it as

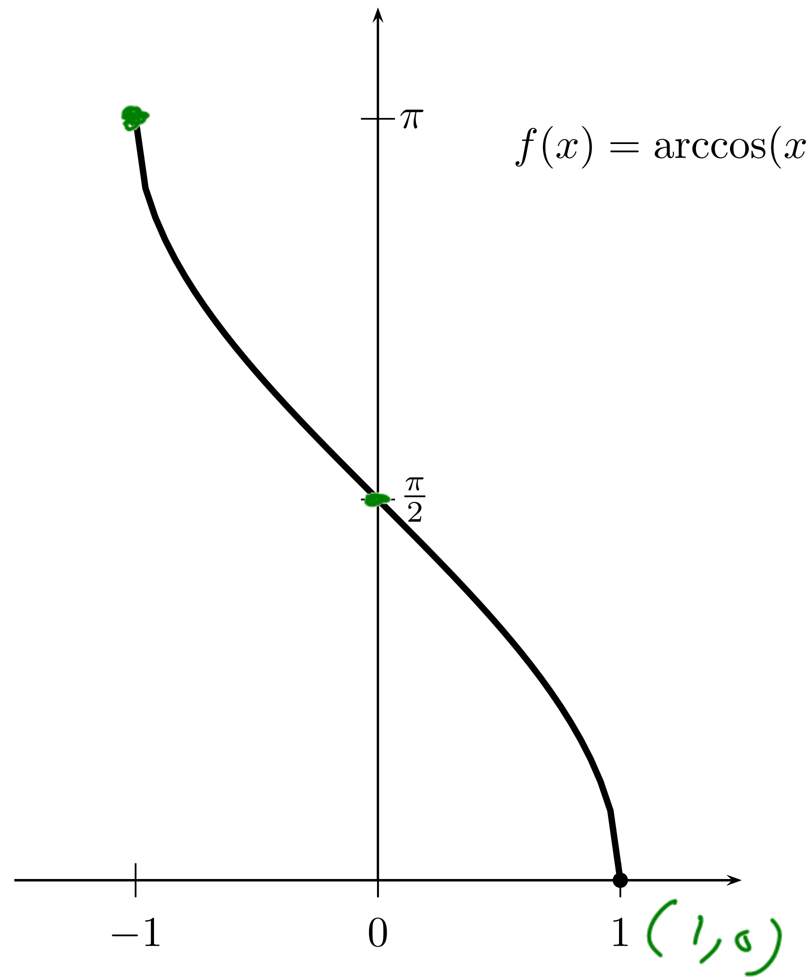
$$\cos^{-1}(x) \text{ or } \arccos(x)$$

inverse

noting that

$$\cos^{-1} x \neq \frac{1}{\cos x}.$$

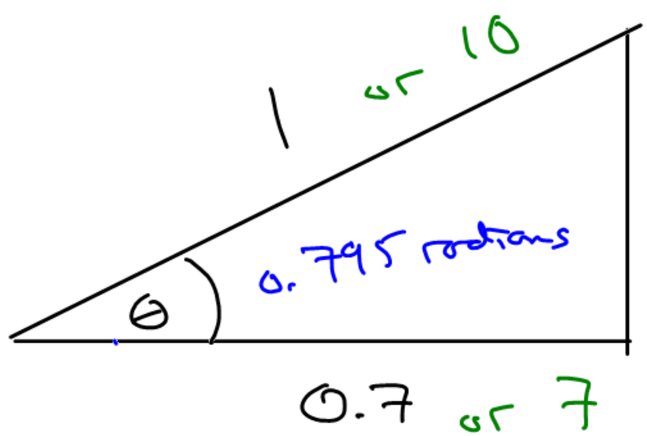




Example: *Compute the value of* $\arccos(0.7)$ *using your calculator.*

$$\underbrace{\arccos(0.7)}_{\text{ratio}} = \underbrace{0.795}_{\text{angle}} \text{ radians}$$

Draw a triangle that would capture a relationship based on what you just computed.



Example: *When you enter* arccos(2) *(via the* "$\cos^{-1}$" *button) on your calculator, it objects. Why is that?*

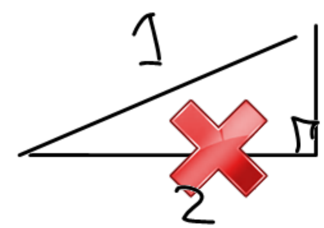
A. The numbers involved are too large for the calculator to handle. X

B. The calculator does not understand this business of taking the inverse using only part of the cosine function. X

C. The cosine function does not really have an inverse. (it does if we limit domain)

D. The number 2 is outside the domain of the function arccos.

ratio of $\dfrac{\text{adj}}{\text{hyp}} = 2$



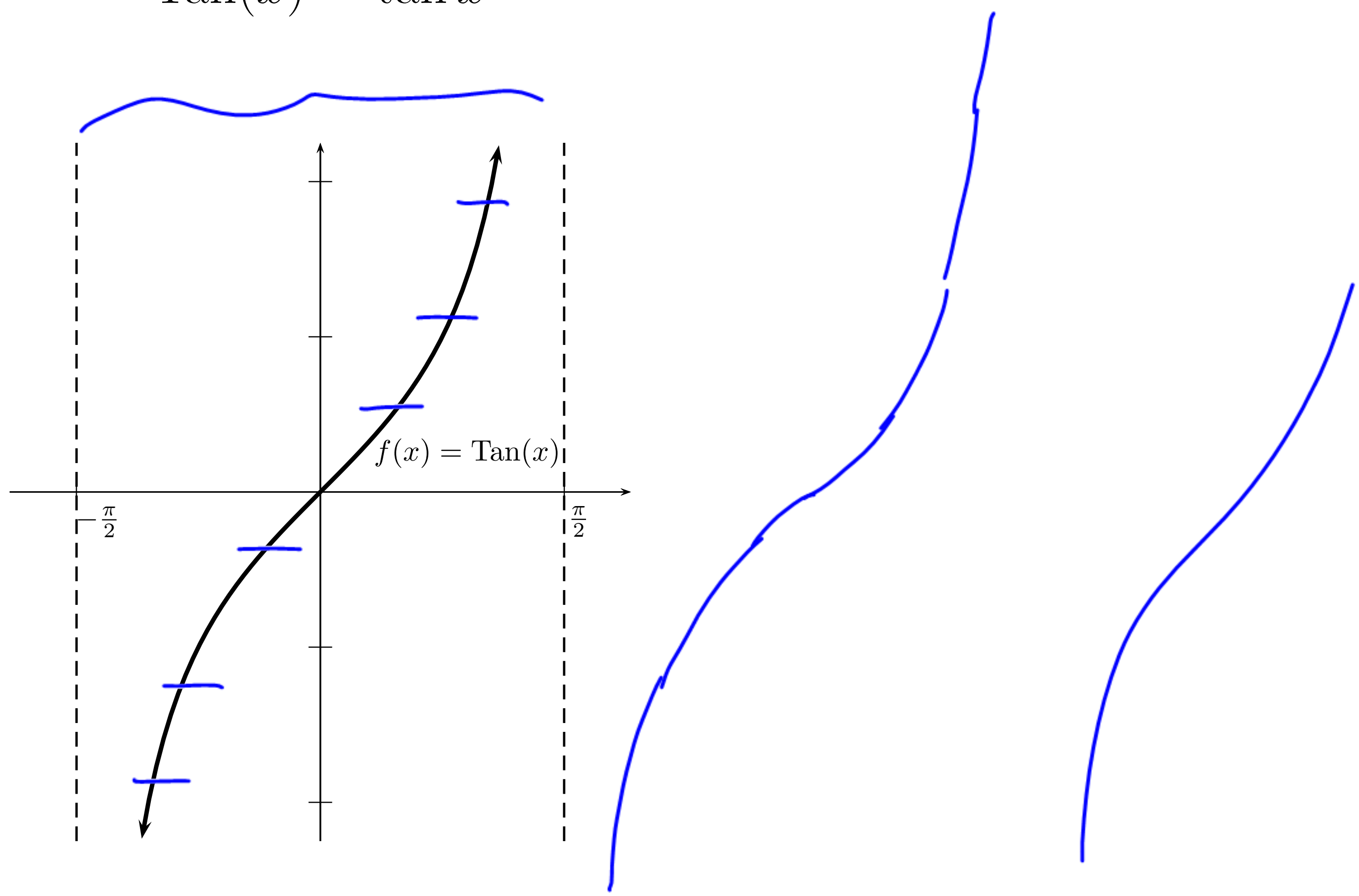
domain for arccos $[-1, 1]$

Tan and arctan

The inverse of tan is determined in the same way, only analyzing it from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.
This is shown in the graph on the next page:
As done before, we name this portion of the tan function $\mathrm{Tan}(x)$, where

$$\mathrm{Tan}(x) = \tan x$$

provided $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



Tan(x) satisfies the horizontal line test and therefore has an inverse, which we call the **inverse tangent function** and denote it as

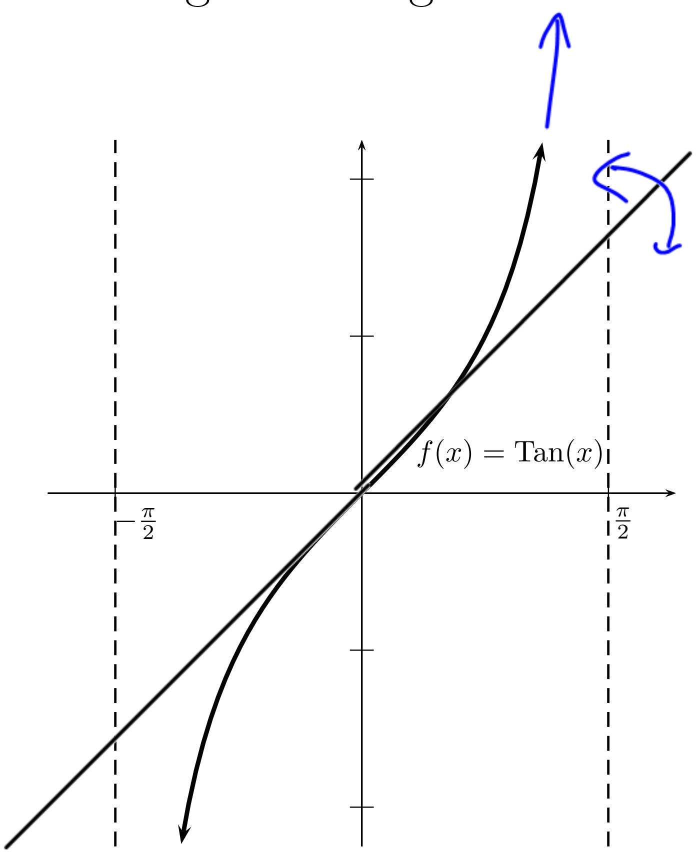
$$\tan^{-1} x \text{ or } \arctan(x)$$

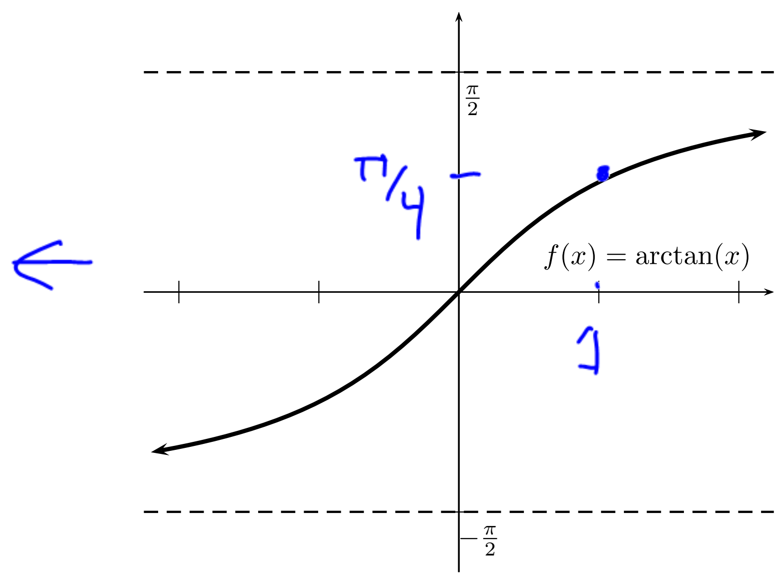
$$\lim_{x \to \infty} \arctan(x) = \pi/2$$

once again noting that

$$\tan^{-1} x \neq \frac{1}{\tan x}.$$

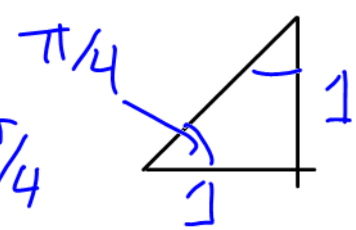
$$\lim_{x \to -\infty} \arctan(x) = -\pi/2$$





Example: *What is the value of* arctan(1)*?* $= 0.785 = \pi/4$

ratio ↑ $\quad \frac{\text{opp}}{\text{adj}}$



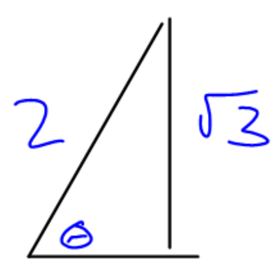
Identities with Inverse Trig

Computing values of inverse trigonometric functions by hand can often fairly quick, and also more enlightening than computations with a calculator.

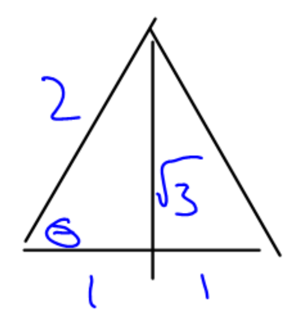
Problem. Find the exact value of $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \theta$

$$\sin\theta = \frac{\sqrt{3}}{2} = \frac{\text{opp}}{\text{hyp}}$$

↑ want



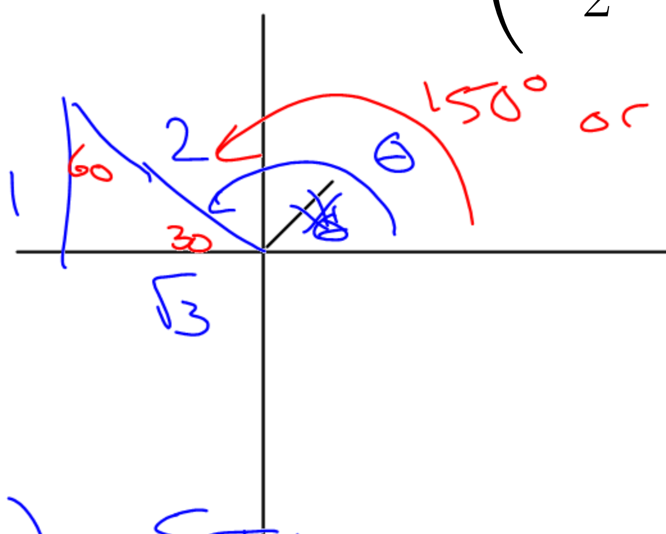
recall



$\theta = 60°$ or $\pi/3$ rad.

Problem. Find the exact value of $\arccos\left(\frac{-\sqrt{3}}{2}\right)$

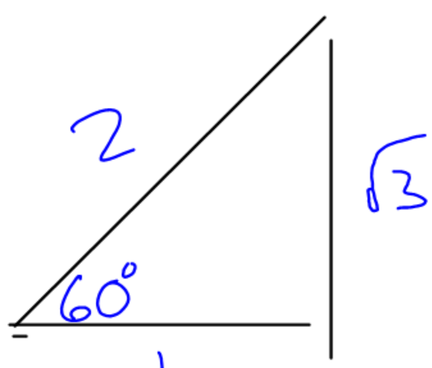
$\cos\theta = \frac{-\sqrt{3}}{2}\ \frac{\text{adj}}{\text{hyp}}$



$150°$ or $\frac{5\pi}{6}$ rad

$\arccos\left(-\frac{\sqrt{3}}{2}\right) = 5\pi/6$

Problem. Find the exact value of arcsec(2) $= \pi/3$



or $\pi/3$ rad

$\sec(\theta) = 2$

$\frac{1}{\cos(\theta)} = 2$

$\cos(\theta) = \frac{1}{2}$

Problem. Find the exact value of $\arcsin\left(\sin\left(\dfrac{8\pi}{3}\right)\right)$

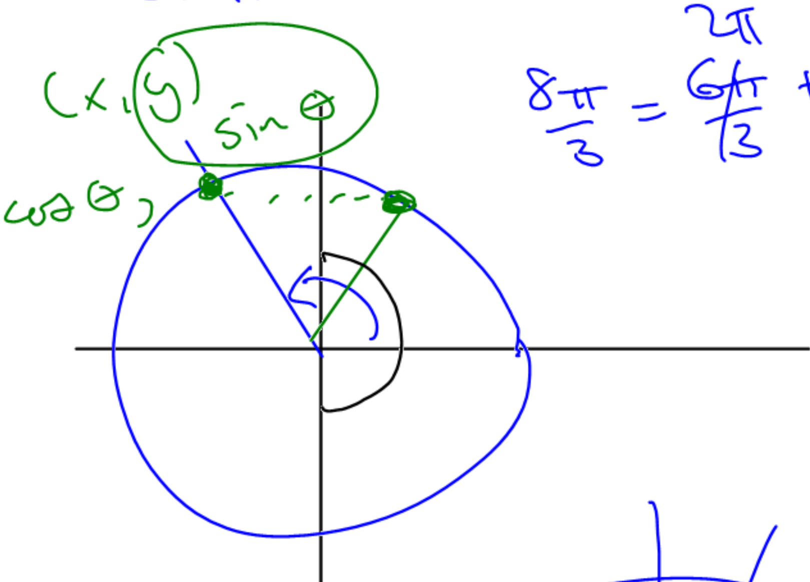
range arcsin is $\left[-\pi/2, \pi/2\right]$

angle

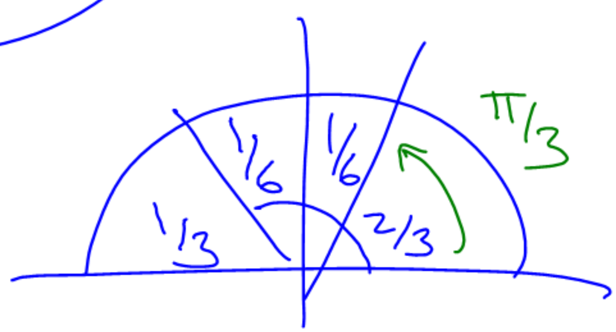
ratio

angle

Unit circle

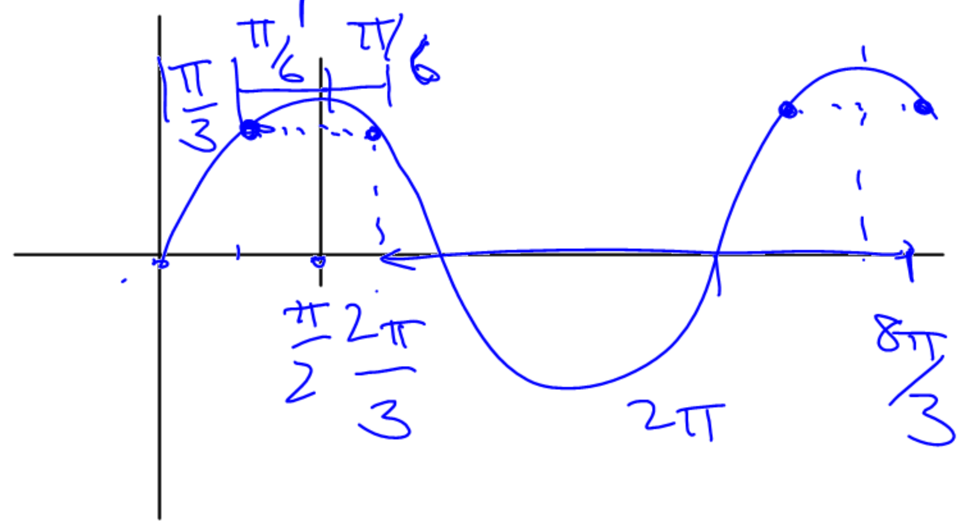


$$\frac{8\pi}{3} = \frac{\overset{2\pi}{\cancel{6\pi}}}{\cancel{3}} + \left(\frac{2\pi}{3}\right)$$



$$\boxed{\arcsin\left(\sin\left(\frac{8\pi}{3}\right)\right) = \frac{\pi}{3}}$$

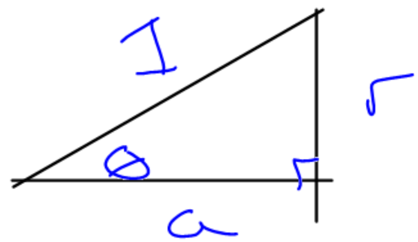
Graph of sin(θ)



Inverse trigonometric functions often appear in geometric applications in mechanical and mechatronic engineering, where angles of moving arms and 3D displacements are closely linked. It can be helpful to have tools to simplify expressions involving inverse trig formulas.

Problem. Simplify the expression $\cos(\underbrace{\arcsin(r)}_{\theta})$

$\sin\theta = \frac{r}{1} = \frac{opp}{hyp}$



$\cos\theta = \frac{adj}{hyp}$

$a^2 + r^2 = 1$

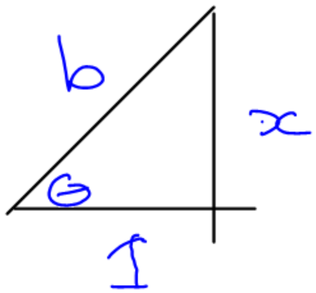
so $a^2 = 1 - r^2$

$a = \sqrt{1-r^2}$

so $\cos(\theta) = \frac{\sqrt{1-r^2}}{1}$

so $\cos(\arcsin(r)) = \sqrt{1-r^2}$

Problem. Simplify the expression $\cos(\underbrace{\arctan(x)}_{\theta})$



$$\tan\theta = \frac{x}{1} = \frac{opp}{adj}$$

$$\cos(\theta) = \frac{adj}{hyp}$$

Find hyp

$$1^2 + x^2 = b^2$$

$$b = \sqrt{1+x^2}$$

$$\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$$

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$$

Derivative of arcsin

Simplify $\sin(\arcsin x) = x$

inverses

Differentiate both sides of this equation, using the chain rule on the left. You should end up with an equation involving $\frac{d}{dx}\arcsin x$.

$$\frac{d}{dx}\left(\sin(\arcsin(x))\right) = \frac{d}{dx}(x)$$

$$\boxed{\cos(\arcsin(x))}\left[\frac{d}{dx}(\arcsin(x))\right] = 1$$

Solve for $\dfrac{d}{dx}\arcsin x$, *and simplify the resulting expression by means of the formula*

$$\cos\theta = \sqrt{1-\sin^2\theta},$$

$\cos^2(\theta) + \sin^2(\theta) = 1$

which is valid if $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

True, but not useful

$$\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\cos\left(\arcsin(x)\right)}$$

$\downarrow$ $\theta = \arcsin(x)$

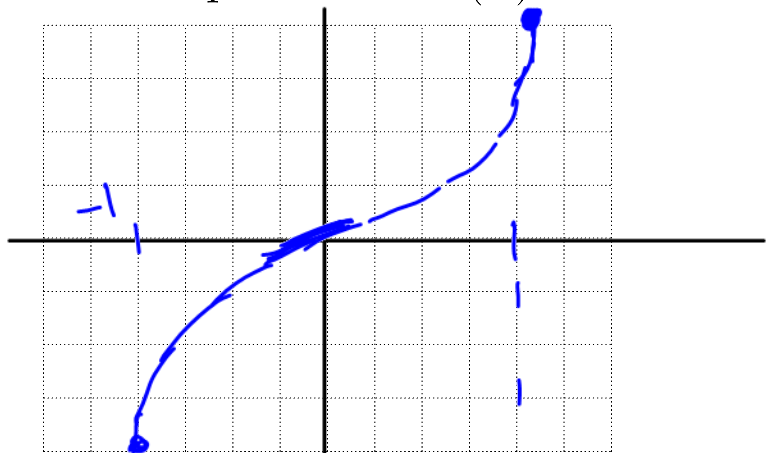
$$= \frac{1}{\sqrt{1-\left[\sin\left(\arcsin(x)\right)\right]^2}}$$

inverse

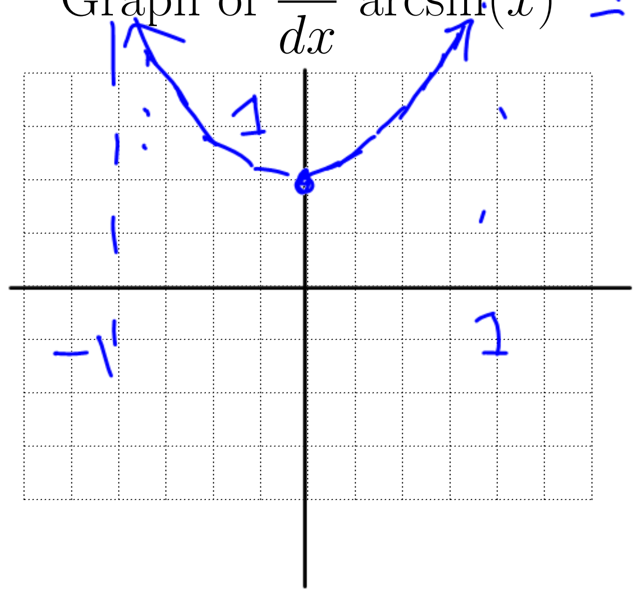
$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

True and useful.

Graph of $\arcsin(x)$



Graph of $\frac{d}{dx}\arcsin(x)$ $= \dfrac{1}{\sqrt{1-x^2}}$



as $x \to 1$, $\dfrac{1}{\sqrt{1-x^2}}$

$\to \dfrac{1}{0} \to \infty$

$x \to (-1)$, $\to \dfrac{1}{0}$

$\to \infty$

Interpreting the Derivative

Example: *Consider the statement* "I am walking at 1.2 m/s."
How far will you travel in the next second?

$$1.2 \text{ m}$$

How far will you travel in the two seconds?

$$2.4 \text{ m} = \left[v \cdot \text{time} = \left(\frac{1.2 \text{ m}}{\text{s}}\right)(2 \text{ s}) \right]$$

How far will you travel in the next $\frac{1}{3}$ *of a second?*

$$\text{dist} = \left(1.2 \frac{\text{m}}{\text{s}}\right)\left(\frac{1}{3} \text{ s}\right) = 0.4 \text{ m}$$

How far will you travel in the next 10 minutes?

$$\text{dist} = \left(1.2 \frac{\text{m}}{\text{s}}\right)(10 \cancel{\text{min}})\left(\frac{60 \text{ s}}{\cancel{\text{min}}}\right) = 720 \text{ m}$$

now @ 1.2 m/s

Note that all the values computed above are **estimates** *or* **predictions**. *Which of the estimates you just calculated will be the* **most accurate**?

smallest time interval

$\Downarrow$

most accuray.

What assumptions are you using to reach your answers?

on shorter time inverals,

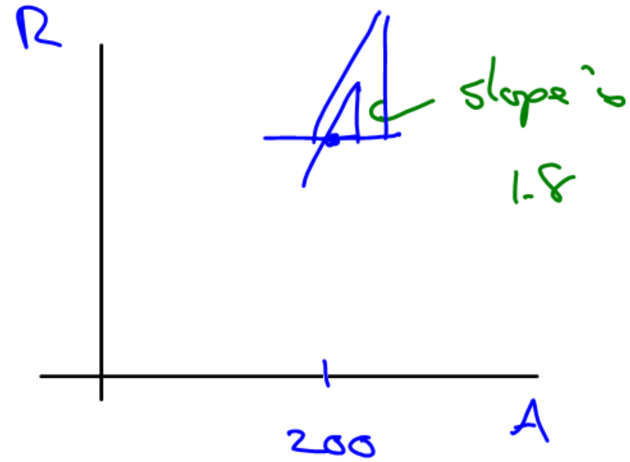
less change

Example: *Let $R = f(A)$ be the monthly revenue for a company, given advertising spending of A per month. Both variables are measured in thousands of dollars.*

slope

Interpret $f'(\underline{200}) = 1.8$ *in words.*

A



if we increase A, then R increases at $1.8 thousands revenue per $1 thousand extra spent on advertising. (from a base of $200 thousand current advertising)

If $A = 200$ currently, and you increased advertising spending by 2 thousand dollars, what would you expect your revenue increase to be?

revenue incr by $(1.8)(2) = 3.6$ thousand revenue.

If $A = 200$ currently, and you increased advertising spending by 1 million dollars, what would you expect your revenue increase to be? $= 1000$

revenue incr by $(1.8)(1000) = 1800$

(Suspect) large change in A, or 1.8 million incr in revenue.

If $f'(200) = 0.8$, and you are currently spending 200 thousand on advertising, should you spend more or less next month?

incr advertising $\Rightarrow$ incr of \$0.8 thousand in revenue for each \$1 thousand extra spent on advertising.

spend \$1 thousand to get \$800 ...

$\rightarrow$ we should <u>decrease</u> our ad spending.

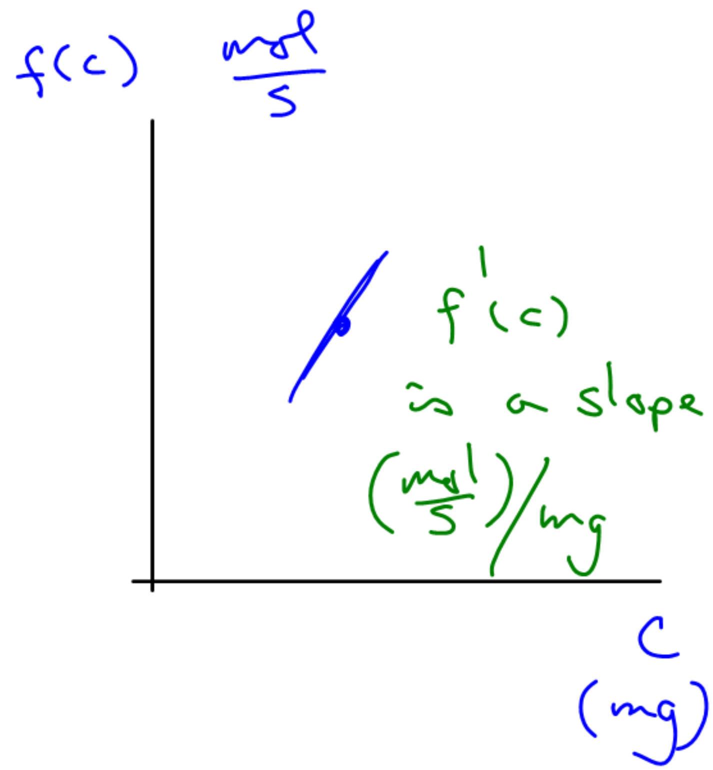
Question: A chemical reaction consumes reactant at a rate given by $f(c)$, where c is the amount (mg) of catalyst present. $f(c)$ is given in moles per second. The units of the derivative, $f'(c)$, are

(a) mg/s

(b) moles/s

(c) moles/(s mg)

(d) (mg moles)/s



c=10 mg

incr cat, see decr of 0.2 mol/s per mg of cat added

Question: If $f'(10) = -0.2$,

(a) Adding more catalyst to the 10 mg present will speed up the reaction. ✗

(b) Adding more catalyst to the 10 mg present will slow down the reaction. ✓

(c) Removing catalyst, from 10 mg present, will speed up the reaction. ✓

(d) Removing catalyst, from 10 mg present, will slow down the reaction. ✗

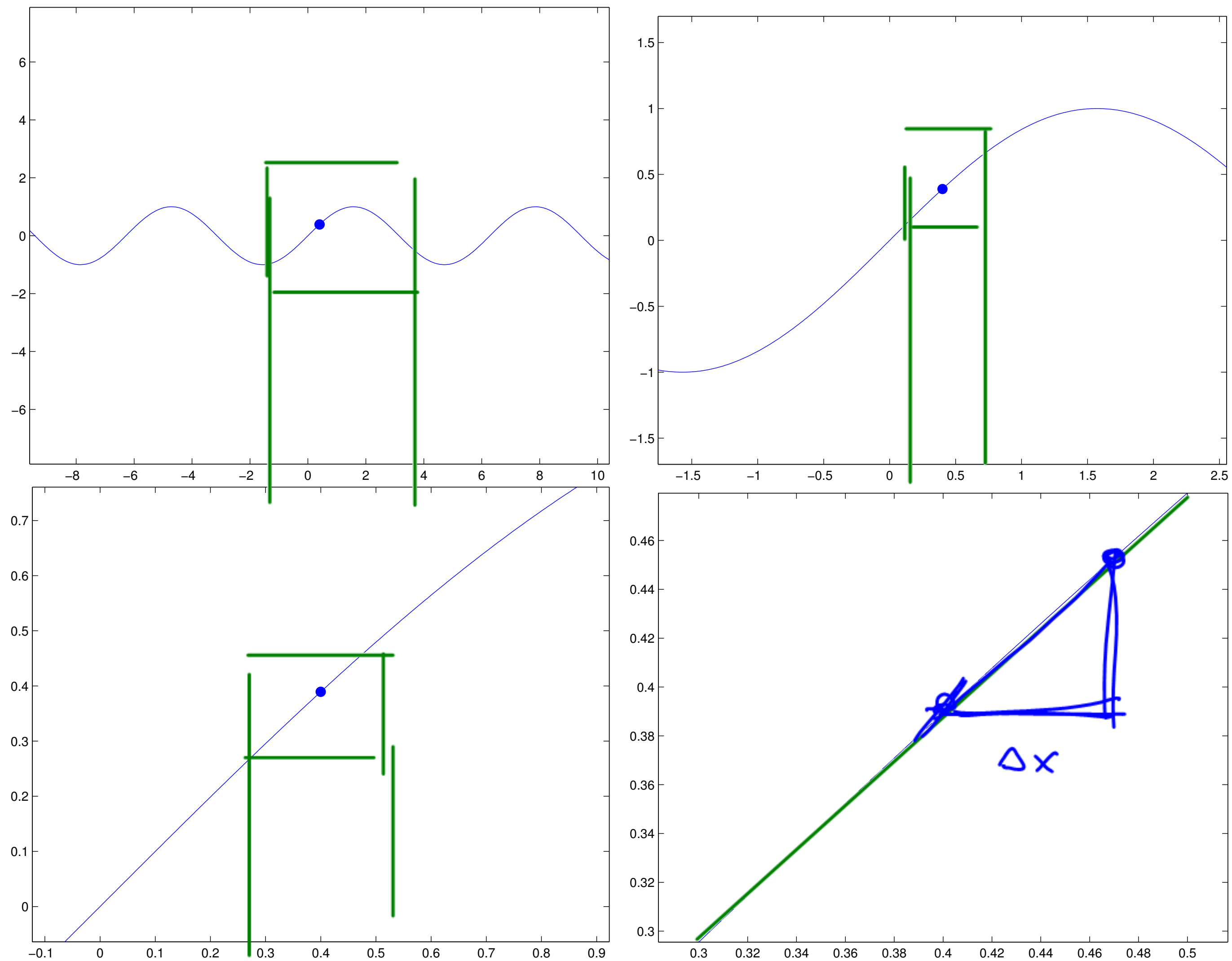
Local Linearity

In all these estimates we have been making, we have been relying on the **local linearity** of a differentiable function.

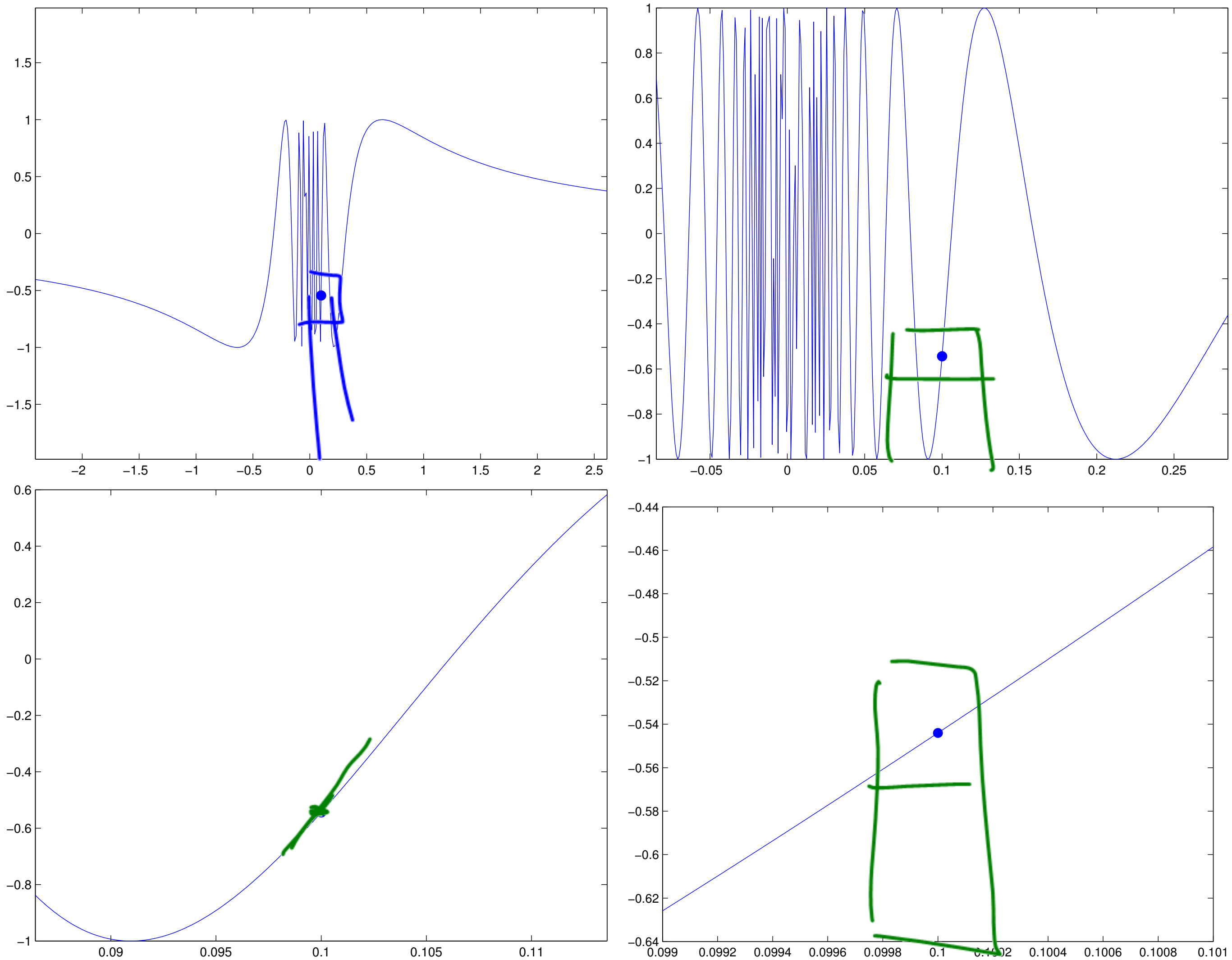
> If a function is differentiable at a point, then it behaves like a linear function for x sufficiently close to that point. ↳ have slope

Another interpretation of differentiability is that if we "zoom in" sufficiently on a point, the graph will eventually look like a straight line.

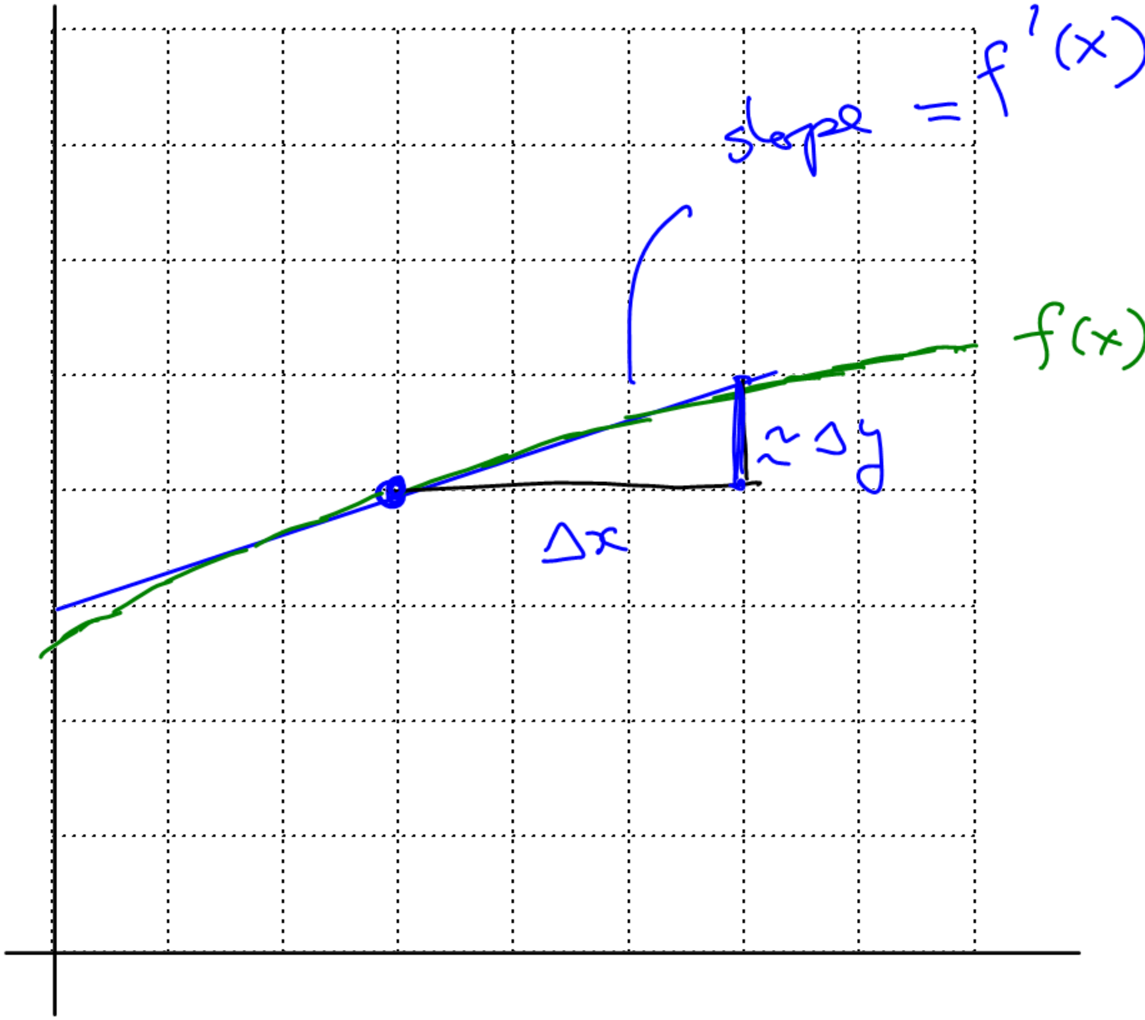
Consider the graph of $y = \sin(x)$ at different scales, around the point $x = 0.4$:



And the more exotic $y = \sin(1/x)$ at different scales, around $x = 0.1$:



Sketch a graph of a locally linear function $f(x)$. Add on the tangent line, and use the derivative to estimate Δy for a given change in x.



$$\frac{\Delta y}{\Delta x} \approx f'(x)$$

$$\Delta y \approx f'(x) \cdot \Delta x$$

$$(\text{vel}) \cdot (\Delta t)$$

Derivative as Approximation of Change instantaneous

avg. rate of change.

$$f'(x) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

so given a value of Δx,

$$\underline{\Delta y} \approx f'(x) \cdot \Delta x$$

assuming that Δx is "sufficiently" small.

The larger Δx is, the worse the approximation will generally be.

Let's return for a minute to an earlier example, and see how we can formalize our previous work.

Example: *Let $R = f(A)$ be the monthly revenue for a company, given advertising spending of A per month. Both variables are measured in thousands of dollars.*

If $A = 200$ currently, and you increased advertising spending by 2 thousand dollars, what would you expect your revenue increase to be?

ΔA

$$\left[\frac{dR}{dA} = \right] f'(A) \cong \frac{\Delta R}{\Delta A} \quad \text{or} \quad \Delta R \cong f'(A) \cdot \Delta A$$

$$= (1.8)(2) = \$3.6 \text{ thousand}$$

incr in revenue.

If $A = 200$ currently, and you increased advertising spending by 1 million dollars, what would you expect your revenue increase to be?

$\Delta A = 1000$

$$\Delta R \cong f'(A) \cdot \Delta A$$

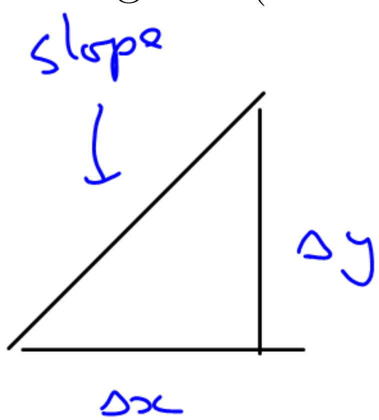
$$= (1.8)(1000) = \$1800 \text{ thousand}$$

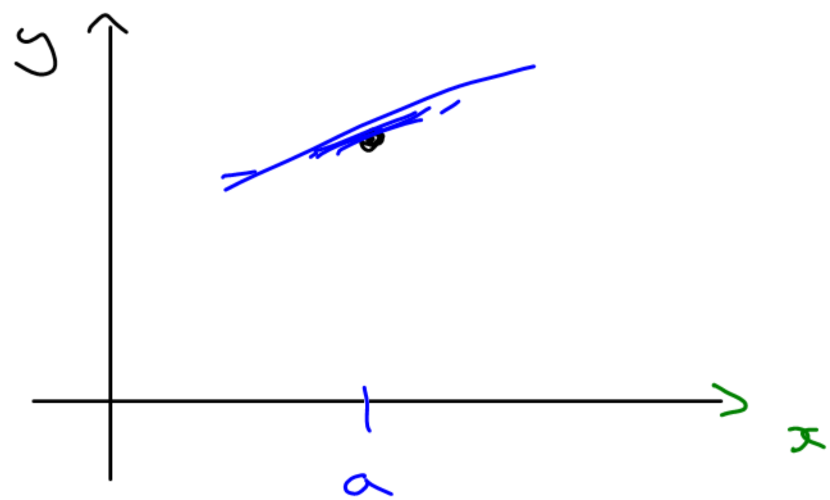
$$= \$1.8 \text{ million.}$$

The Tangent Line, or Local Linearization

$\Delta y \approxeq f'(x)\, \Delta x$

In the last few examples, we focused on the *change* in y (or f, or revenue, etc.), based on a set *change* in the input. Note that all these changes were relative to a given starting value. ($A = 200$, $c = 10$, etc.)
We can take the ideas one step further and create a *linear function* that approximates our given (usually non-linear) function.





Example: *Let us return to the advertising problem, where $R = f(A)$ represents the revenue of a company (in thousands of dollars), given the amount A spent on advertising (also in thousands of dollars).*
Suppose $f(200) = 1500$, and $f'(200) = 1.8$.

State the interpretation of both values in words.

revenue will incr by \$1.8 thousand per \$1 thousand incr in advertising.

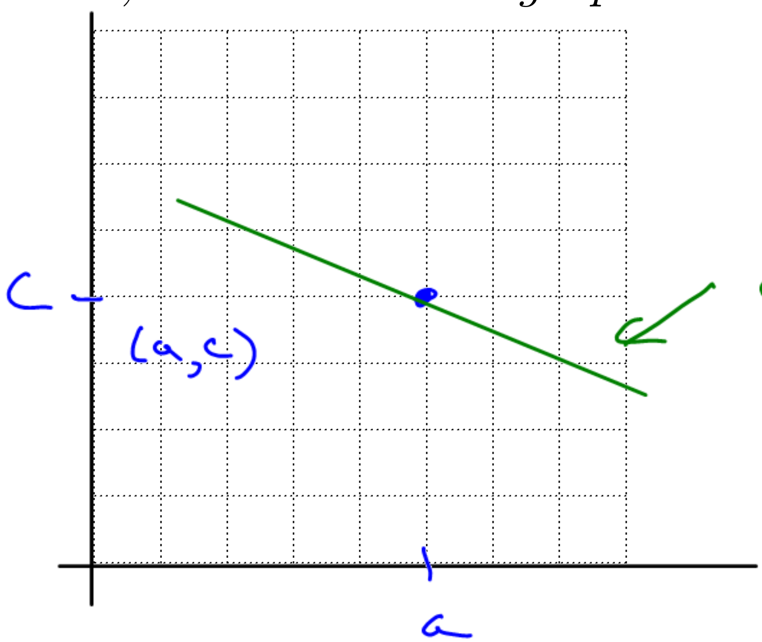
if we spend \$200 thousand on advertising, then revenue is \$1500 thousand (or \$1.5 million)

Recall the **point/slope** *form for a linear function:*

$$y = m(x-a) + c$$

slope

Sketch out the graph of this function, indicating the effect of the parameters m, a and c on the graph.



$y = m(x-a)+c$

(line

- slope m
- through (a,c).

point

Use the point/slope formula, and the information that $f(200) = 1500$ *and* $f'(200) = 1.8$*, to build a local linear approximation for the revenue function* R *for advertising budgets* A *around 200.*

slope m a c

approximate for R.

$$R(A) \approx 1.8\,(A - 200) + 1500$$

$$y = m\,(x - a) + c \qquad \text{linear}$$

What revenue would we expect if we reduced advertising to 190 thousand dollars?

$A = 190$

prediction: $R = 1.8(190 - 200) + 1500$

$= -18 + 1500$

$= \$1482$ thousand

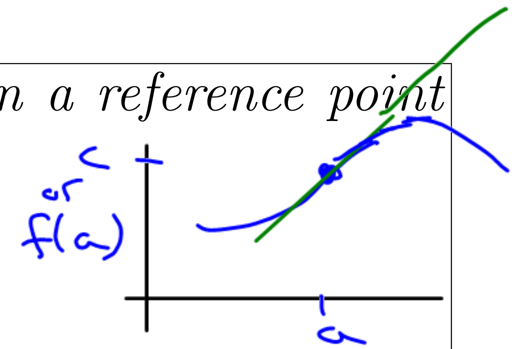
or $\$\underline{1.482}$ million.

Revenue decr to $1.482 million.

Linearization Formula

We can construct a linear approximation of a function, *given a reference point* $x = a$, using

$m(x - a) + c$



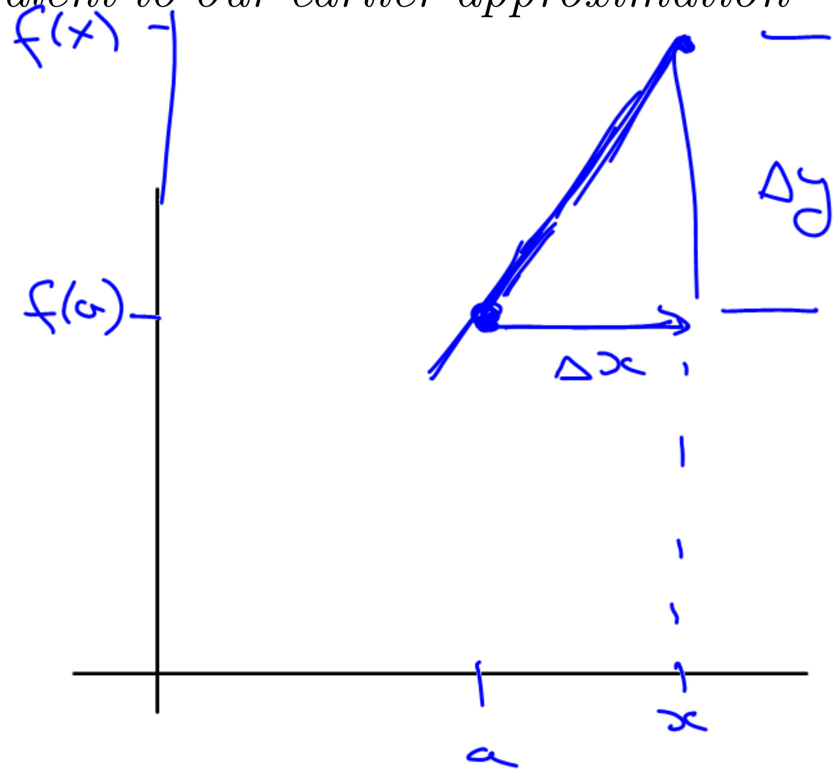
$$f(x) \approx f'(a)(x - a) + f(a)$$

This approximation is good assuming that the x values used are "sufficiently" close to the reference point $x = a$.

The larger $(x - a)$ (or Δx) is, the worse the approximation will generally be.

Show that $f(x) \approx f'(a)(x-a)+f(a)$ *is equivalent to our earlier approximation*

$$\frac{dy}{dx} = f'(a) \approx \frac{\Delta y}{\Delta x}$$



$f(x) \approx f'(a)(x-a)+f(a)$

$f(x) \approx f'(a) \cdot \Delta x + f(a)$

$f(x) - f(a) \approx f'(a)\,\Delta x$

$\Delta y \approx f'(a) \cdot \Delta x$

$f'(a) \cong \dfrac{\Delta y}{\Delta x}$

Example: *Build a local linear approximation formula for the population of Canada, given it is currently 33 million, and the population is currently increasing at a rate 300,000 people per year.*

$P(0) = 33{,}000{,}000$

now : $t = 0$

$P'(0) = 300{,}000$

$$P(t) \cong 300{,}000\,(t - 0) + 33{,}000{,}000$$

(t = years from now)

Use your approximation to estimate the Canadian population two years from now.

@ $t = 2$

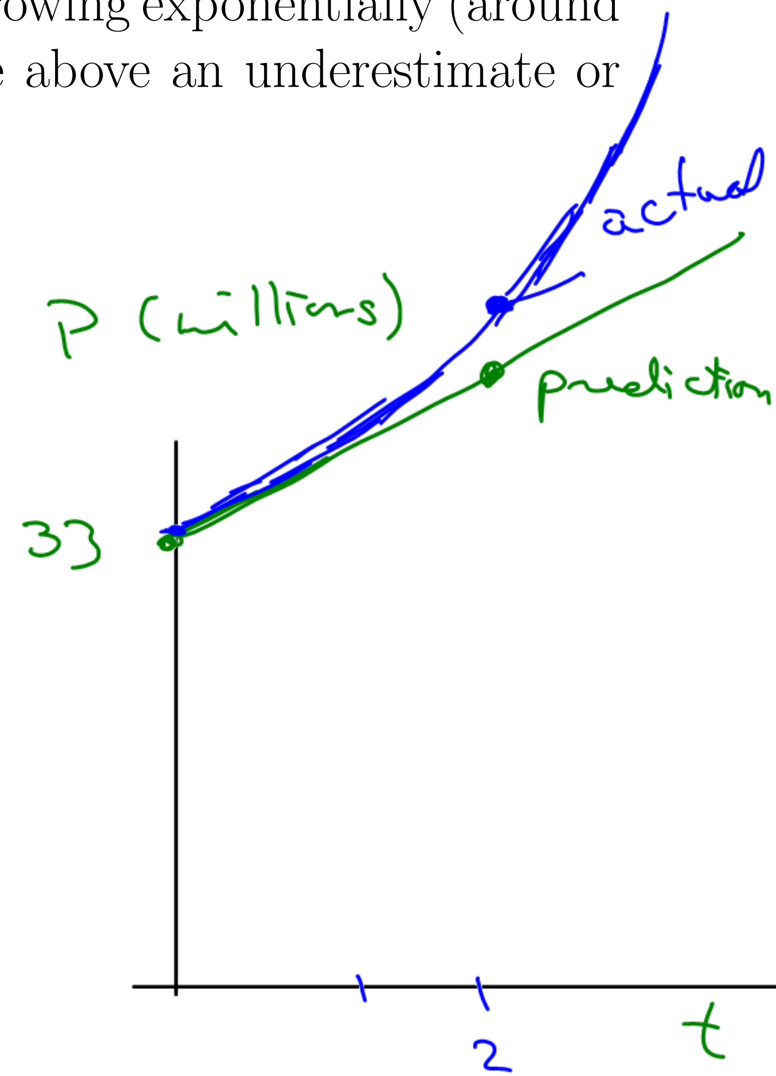
$$P(2) \approx 300{,}000\,(2 - 0) + 33{,}000{,}000$$

$$\cong 33{,}600{,}000$$

Question: Given that the Canadian population is growing exponentially (around 1% per year), will your previous population estimate above an underestimate or an overestimate of the real population in that year?

(a) **Over**estimate

(b) **Under**estimate



Support your answer with a sketch of the population curve, and the linear approximation.

Geometric Applications of Linearization

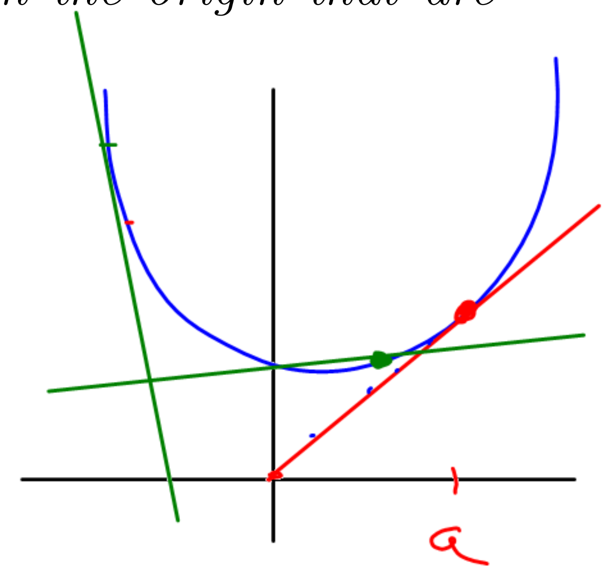
We can also construct and answer interesting geometric questions using tangent lines.

Example: *Find the equations of all the lines through the origin that are also tangent to the parabola.*

↑ linear approx

$$f(x) = y = \overbrace{x^2 - 2x + 4}^{f(x)}$$

parabola (opens up)



Equation of tangent line

$$y = f'(a)(x - a) + f(a)$$

based on point $x = a$ on parabola

We want a's such that line passes through $(0,0) \to x=0,\ y=0$ satisfies line equation

$$0 = f'(a)(0 - a) + f(a)$$

Continued. $y = x^2 - 2x + 4$

$f(x) = x^2 - 2x + 4 \rightarrow f(a) = a^2 - 2a + 4$

so $f'(x) = 2x - 2$ $\qquad f'(a) = (2a - 2)$

Sub into tangent line formula, also satisfying $(0,0)$

$$0 = (2a-2)(0-a) + (a^2 - 2a + 4)$$

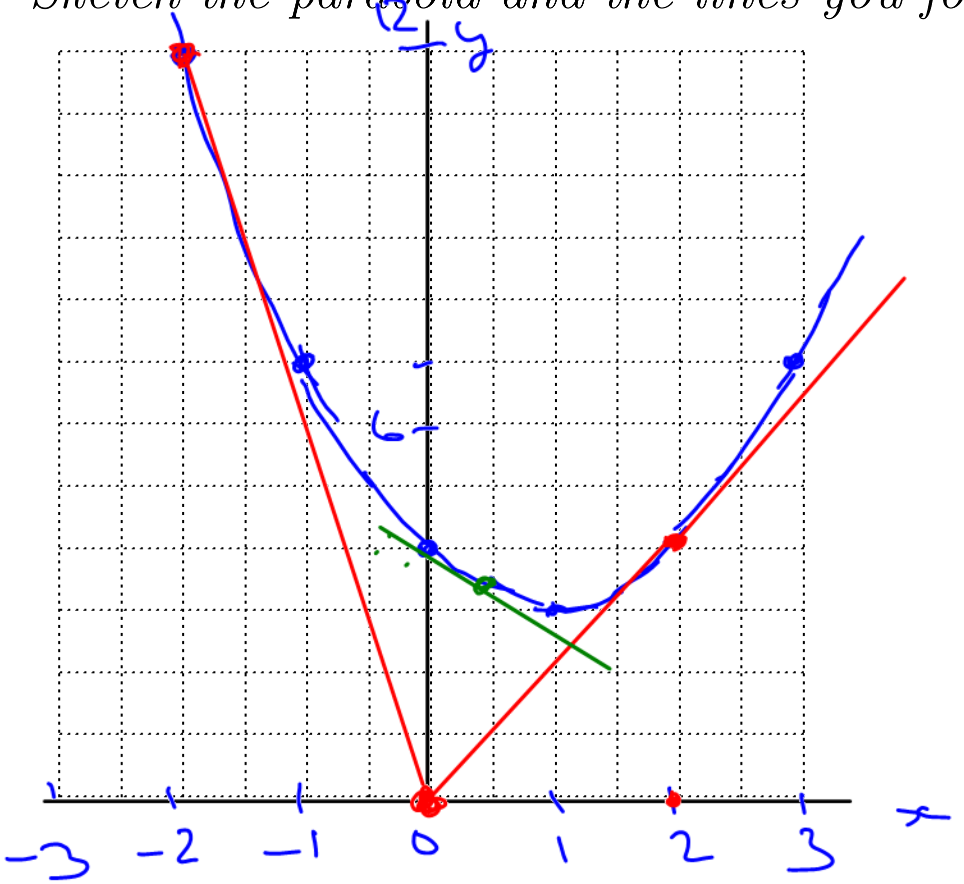
$$0 = -2a^2 + 2a + a^2 - 2a + 4$$

$$0 = -a^2 + 4$$

$a^2 = 4$ or $\boxed{a = 2, -2}$

points on parabola where tangent line will pass through the origin

Sketch the parabola and the lines you found.



$y = x^2 - 2x + 4$

x	y
-2	12
-1	7
0	4
1	3
2	4
3	7

confirmed $x = 2$, $x = -2$: tangent lines to parabola pass through origin