Week 5: Implicit Derivatives, Related Rates and Optimization

Goals:

- Introduce implicit differentiation.
- Study problems involving related rates.
- Formalize the first derivative test and the second derivative test for identifying local maxima and minima.
- Distinguish global vs. local extrema.
- Practice optimization word problems.

Implicit Differentiation

If we define a graph by the relationship $y = f(x)$, then we have a formula for tangent lines.

Question: What function below is the tangent line to $y = e^{-x}$ at $x = 0$?

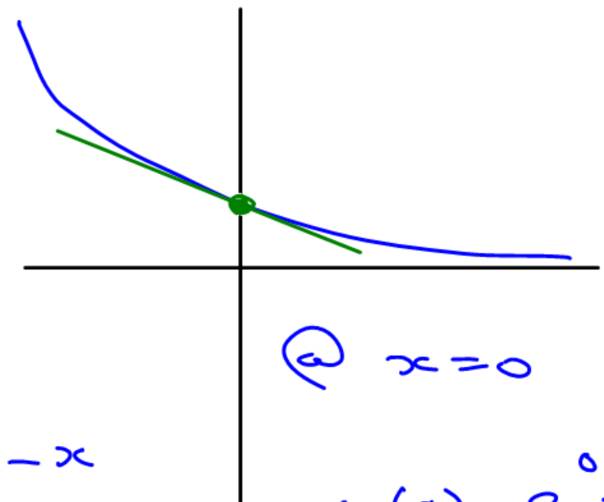
(a) $y = -x + 1$ ✓

(b) $y = -(x - 1) + 1$

(c) $y = -x + e$

(d) $y = x + 1$

(e) $y = -(x - 1) - 1$



$y = e^{-x}$

$y(0) = e^0 = 1$

value

$\frac{dy}{dx} = -e^{-x}$

$y'(0) = -e^0 = -1$

slope

$y = -1(x-0) + 1$ or $-x+1$

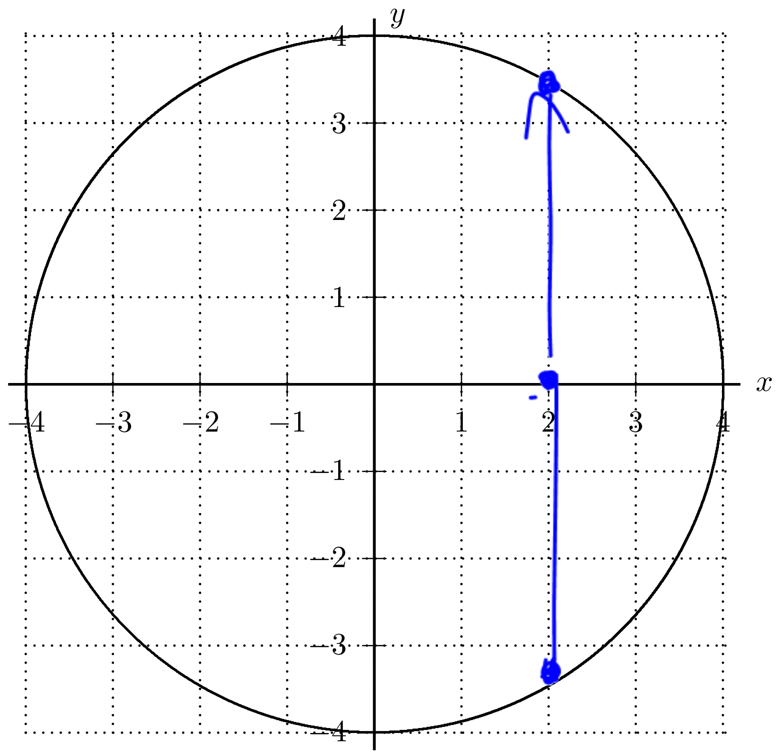
$= 1 - x$

Question: Can we define a function in the form $y = f(x)$ that describes the points on the circle below?

(a) Yes.

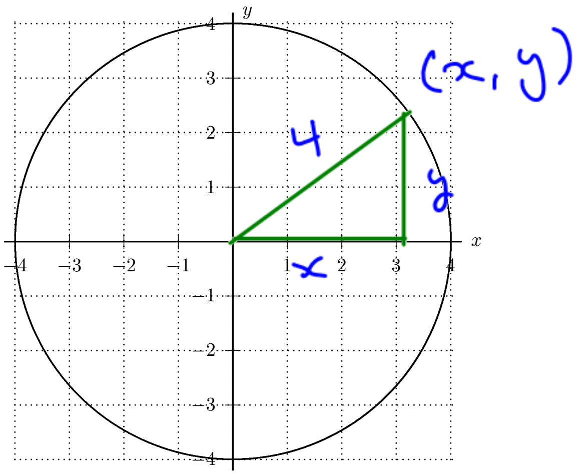
(b) No.

(c) Maybe.



fails vertical line test (for functions)

function has at most 1 y value for each x.

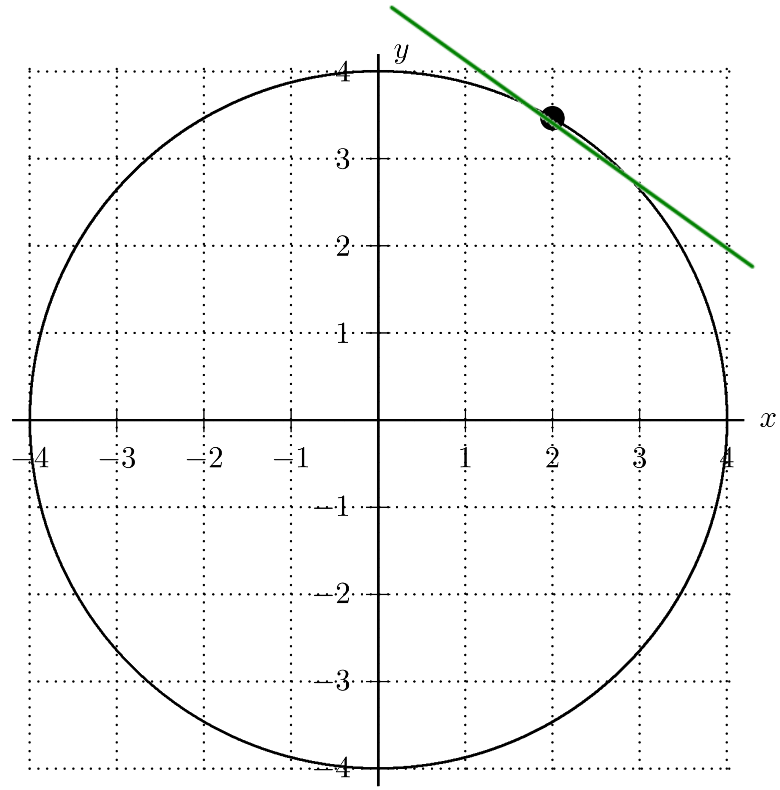


Write out a formula for the points in the circle shown.

$x^2 + y^2 = 4^2$

not in $y = \ldots$ form

Question: Can we, in principle, define a tangent line to the circle at the point $(2, \sqrt{12})$?



(a) Yes. 

(b) No.

(c) Maybe.

Implicit Differentiation - Example 1

4^2

$\shortparallel$

dy/dx

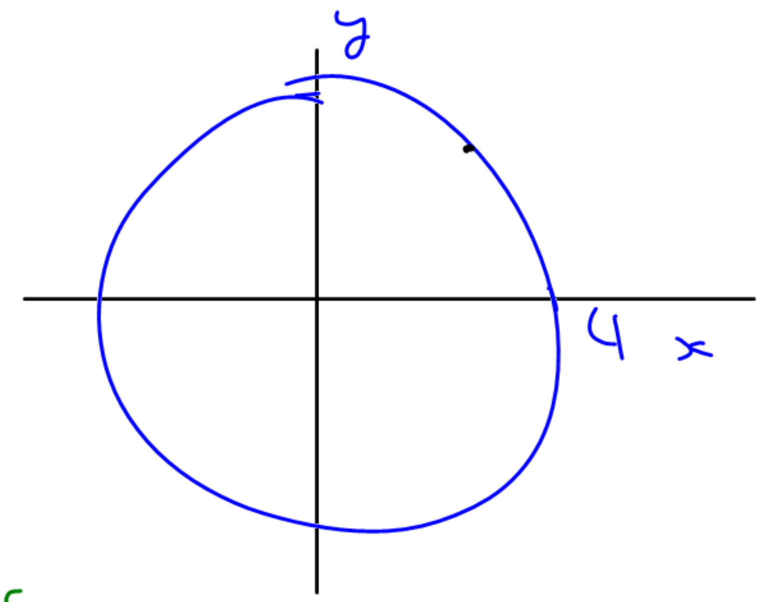
Find a formula for the slopes of tangent lines to the circle $x^2 + y^2 = 16$.

To find these slopes, weintroduce a technique called *implicit differentiation*. The name comes from treating y's in our formulas as implicit functions of x when we differentiate, even though we don't explicitly have y written as a function.

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(16)$$

$$2x\left(\frac{dx}{dx}\right) + 2y\,\frac{dy}{dx} = 0$$

$$2x(1) + 2y\,\frac{dy}{dx} = 0$$



Solve for $\frac{dy}{dx}$

$$2y\,\frac{dy}{dx} = -2x$$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}}$$

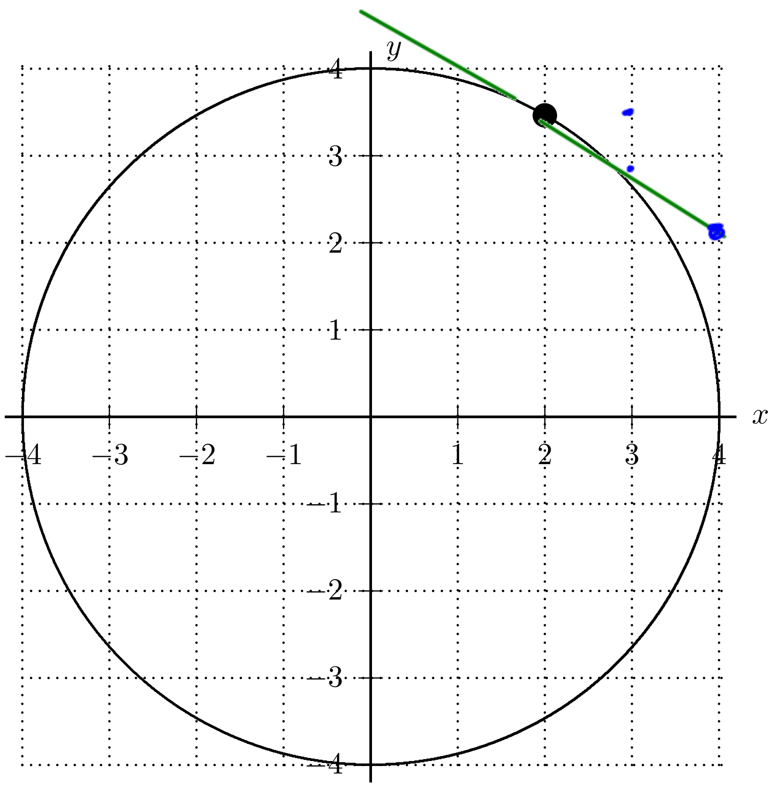
formula for slopes on circle.

$$\frac{d}{dx}x^2 = 2(x)\cdot\frac{dx}{dx} = 2x\cdot 1$$

but $\frac{d}{dx}y^2 = 2y\cdot \boxed{dy/dx}$ ~ slopes on curve

y is a function of x

For the circle $x^2 + y^2 = 16$*, find the slope at the point* $(2, \sqrt{12})$*, and sketch it on the graph.*



$$\frac{dy}{dx} = \frac{-x}{y}$$

$$@\ (2, \sqrt{12}),\quad \frac{dy}{dx} = \frac{-2}{\sqrt{12}}$$

$$= -0.578$$

Matches graph well!

What is different about the derivative found by implicit differentiation, compared to the derivative of a function?

Earlier

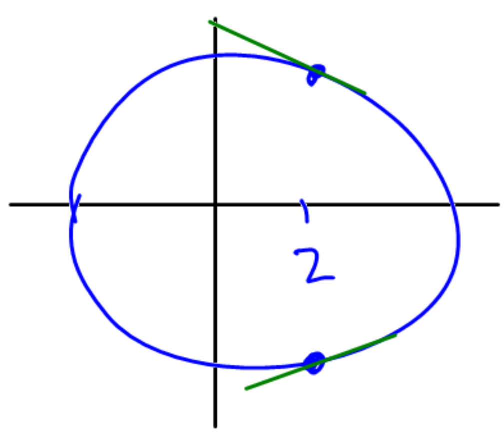
$y = x^2 \rightarrow \frac{dy}{dx} = 2x$

$y = e^{-x} \rightarrow \frac{dy}{dx} = -e^{-x}$

only x's

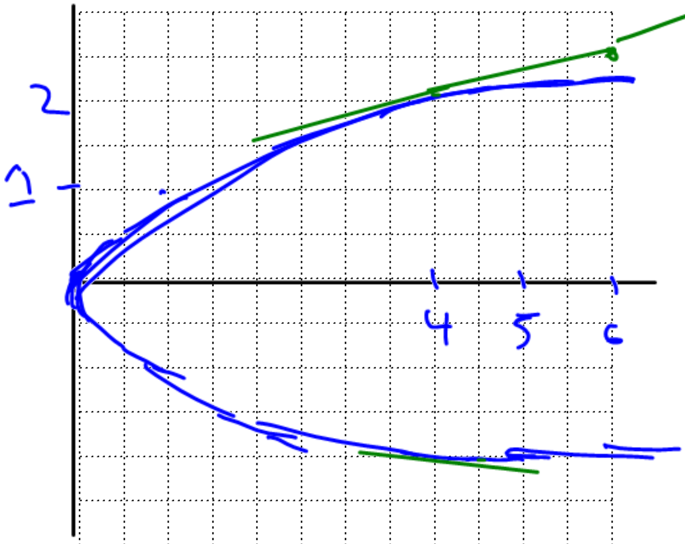
$$\frac{dy}{dx} = -\frac{x}{y}$$

x and y in slope formula.



Implicit Differentiation - Example 2

Example: *Sketch the curve defined by the relationship $x = y^2$.*



$x = y^2$

sideways parabola

tangent line @ (4, 2)

$y = \frac{1}{4}(x-4) + 2$

Find the equation of the tangent line to this graph at the point (4, 2).

Take d/dx both sides: $\frac{d}{dx}(x) = \frac{d}{dx}(y^2)$

$\frac{dx}{dx} = 2y^1 \cdot \frac{dy}{dx}$

$1 = 2y \boxed{\frac{dy}{dx}}$ formula for slopes.

$$\boxed{\frac{dy}{dx} = \frac{1}{2y}}$$

Tangent at (4,2) to $x = y^2$ (continued)

$$\frac{dy}{dx} = \frac{1}{2y}$$

@ $(4, 2)$ $\quad \frac{dy}{dx} = \frac{1}{2(2)} = \frac{1}{4}$

$\underset{\substack{a \\ \text{or } x_0}}{4}$, $\underset{\substack{c \\ \text{or } y_0}}{2}$

Tangent line $\quad y = (\text{slope})(x - \underset{x_0}{a}) + \underset{y_0}{c}$

$$\boxed{y = \frac{1}{4}(x-4) + 2}$$

Implicit Differentiation - Example 3

Note that implicit derivatives of even complicated relationships are straightforward to compute.

formula for slopes

Example: *Use implicit differentiation to calculate* $\dfrac{dy}{dx}$ *when*

$$x^3 + 2x^2y + \sin(xy) = 1.$$

Take $\frac{d}{dx}$ of both sides:

$$\frac{d}{dx}\left(x^3 + 2x^2 y + \sin(x \cdot y)\right) = \frac{d}{dx}(1)$$

$$3x^2\frac{dx}{dx} + 2\left[\left(2x\frac{dx}{dx}\right)(y) + (x^2)\left(\frac{dy}{dx}\right)\right] + \cos(xy)\left[\frac{dx}{dx}\cdot y + x\frac{dy}{dx}\right]$$

D 1st · 2nd   1st · D 2nd

$$3x^2 + 4xy + 2x^2\frac{dy}{dx} + \cos(xy)\cdot y + \cos(xy)\cdot x\frac{dy}{dx} = 0$$

Gather $\frac{dy}{dx}$ term

$$\frac{dy}{dx}\left[2x^2 + x\cos(xy)\right] + 3x^2 + 4xy + \cos(xy)\cdot y = 0$$

$$x^3 + 2x^2y + \sin(xy) = 1$$

Is the point (1, 0) on this curve?

$$\frac{dy}{dx}\left[2x^2 + x\cos(xy)\right] + 3x^2 + 4xy + \cos(xy)\cdot y = 0$$

$$\boxed{\frac{dy}{dx}\left(\cancel{2x^2 + x\cos(xy)}\right) = \frac{-3x^2 - 4xy - y\cos(xy)}{2x^2 + x\cos(xy)}}$$

check $(1, 0)$

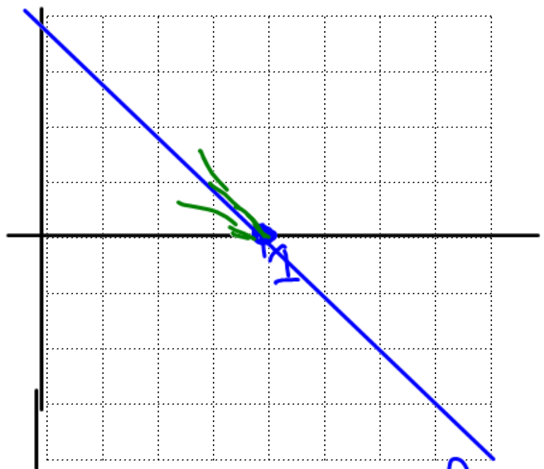
$\uparrow$ x $\uparrow$ y

$$\text{LHS} = (1)^3 + 2(1)^2\cdot 0 + \sin(1\cdot 0)$$
$$= 1 + 0 + 0$$
$$= 1$$
$$\text{RHS} = 1$$

equal, so $(1,0)$ is on graph.

$$x^3 + 2x^2y + \sin(xy) = 1$$

Sketch the curve locally around the point $(1, 0)$.



tangent line to graph

$$\frac{dy}{dx} = \frac{-3x^2 - 4xy - y\cos(xy)}{2x^2 + x\cos(xy)}$$

@ $(1, 0)$

$$\frac{dy}{dx} = \frac{-3 - 0 - 0}{2 + 1 \cdot \underbrace{\cancel{\cos(0)}}_{1}} = \frac{-3}{3} = -1$$

slope on graph at $(1, 0)$

$$x^3 + 2x^2y + \sin(xy) = 1$$

Estimate the y location of the nearby point on the graph at $x = 0.95$.

Tangent line: $y = \underbrace{(-1)}_{\text{slope}}(x - \underbrace{1}_{x_0 \text{ or } a}) + \underbrace{0}_{y_0 \text{ or } c}$

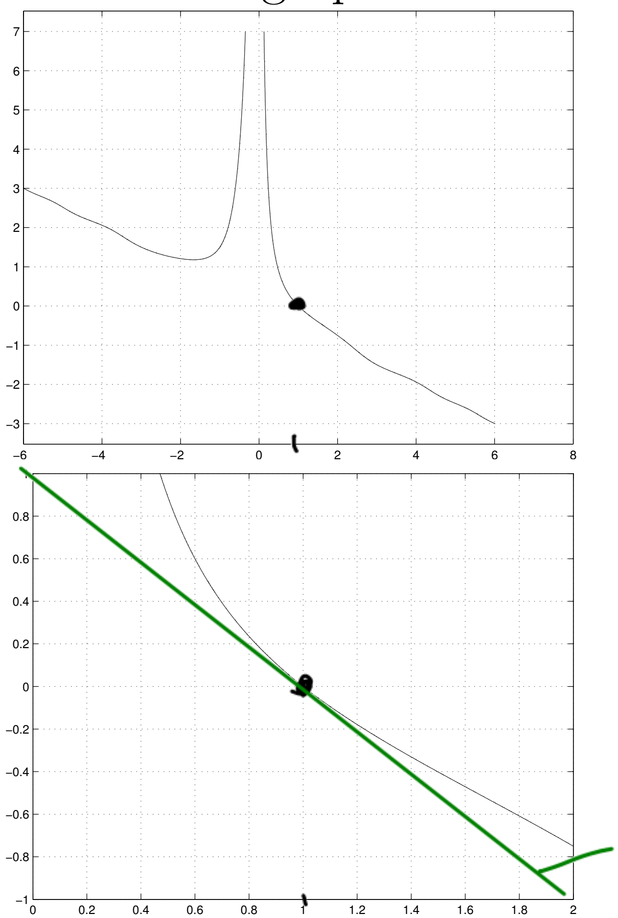
or $y = -x + 1$ or $1 - x$

Tangent line y value @ $x = 0.95$: $y = -0.95 + 1$

$$\boxed{y = 0.05}$$

on graph $y \approx 0.05$ at $x = 0.95$.

Here is the graph of the relation, shown at two different zoom levels.



slope of -1

Sketch the tangent line found in the previous question.

$$y = -1(x-1) + 0$$

Related Rates

derivatives

We can use the Chain Rule and Implicit Differentiation to solve problems involving **related rates**. As the name suggests, we use the rate of change (i.e., the derivative) of one function to calculate the rate of change (derivative) of a second function.

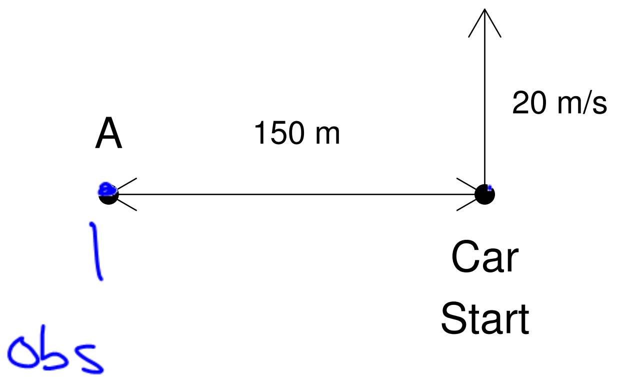
rate $\frac{dy}{dt}$

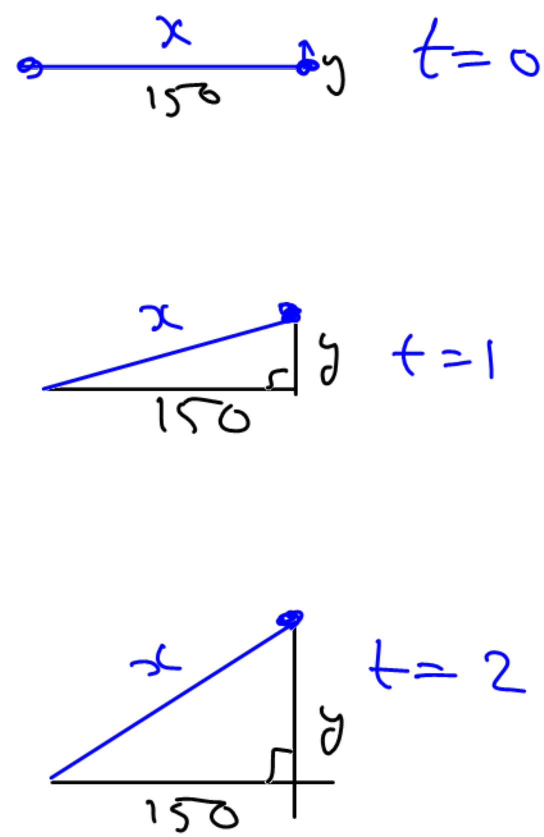
Example: *A car starts driving north at a point 150 meters east of an observer at point A. The car is traveling at a constant speed of 20 meters per second. How fast is the distance between the observer and car changing after 10 seconds?*

↳ other rate

want $\frac{dx}{dt}$

Always true $\Rightarrow 150^2 + y^2 = x^2$





Important: start with quantities, not rates (rates will come later)

x = dist. from observer to car; y = dist north of start point.

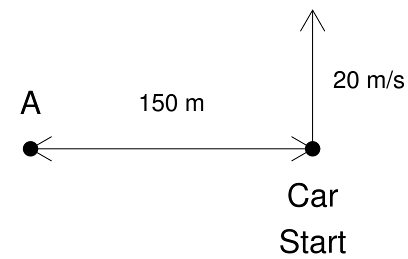
(Continued)

$150^2 + y^2(t) = x^2(t)$ ①

x and y both change over time

Take $\frac{d}{dt}$ of both sides:

$$\frac{d}{dt}\left(150^2 + y^2\right) = \frac{d}{dt}\left(x^2\right)$$



$$0 + 2y^1 \cdot \frac{dy}{dt} = 2x^1 \cdot \frac{dx}{dt}$$ ← have related rates now

Want $\frac{dx}{dt}$:

$$\boxed{\frac{dx}{dt} = \frac{2y}{2x}\frac{dy}{dt} = \frac{y}{x}\frac{dy}{dt}}$$

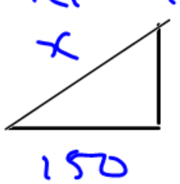
rate of change of dist to obs from car

speed of car $= 20$ m/s

at $t = 10$ s,

$$\frac{dx}{dt} = \frac{200}{250}(20) = 16 \text{ m/s}$$

After 10 s,

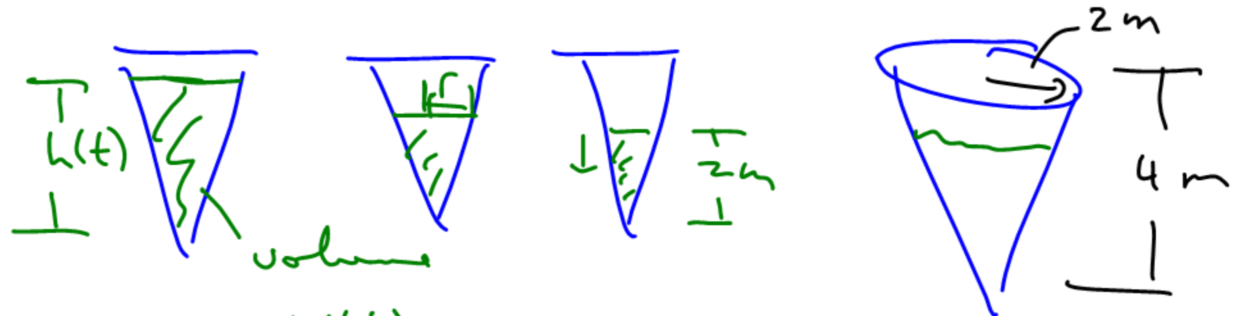


$y = (20)(10) = 200$, and $x^2 = 200^2 + 150^2 \Rightarrow x = 250$ m

①

By using implicit differentiation, we can solve related rates problems even if we do not have an explicit formula for the function in terms of the independent variable (usually time).

Example: *A conical water tank with a top radius of 2 meters and height 4 meters is leaking water at 0.5 liters per second. How fast is the height of the water in the tank changing when the remaining water in the tank is at a height of 2 meters?*

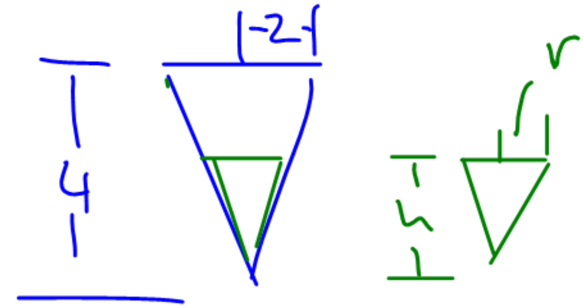


vol decr @ 0.5 l/s

Know $\frac{dV}{dt} = -0.5\,\frac{l}{s}$, want $\frac{dh}{dt}$

Find relationship b/w quantities V and h.

Vol of cone: $V = \frac{1}{3}\pi r^2 h$



$\frac{r}{h} = \frac{2}{4}$

or $r = \frac{1}{2}h$ (1)

(1)

$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 \cdot h$

$V = \frac{\pi}{12} h^3$ always true

(Continued)

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{12} h^3\right)$$

$$\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \frac{dh}{dt} \quad \leftarrow \text{related rates}$$

want

Want $\frac{dh}{dt}$: $\quad \frac{dh}{dt} = \frac{dV}{dt} \cdot \left(\frac{4}{\pi} \frac{1}{h^2}\right)$ $\quad$ @ $h=2$

$$\left[\frac{m}{s}\right] \quad \left[\frac{\cancel{l}\ m^3}{s}\right]\left[\frac{1}{m^2}\right]$$

Know $\frac{dV}{dt} = -0.5\ l/s$

$$= -0.5 \frac{\cancel{l}}{s}\left(\frac{1\ m^3}{1000\ \cancel{l}}\right)$$

$$= -0.0005\ m^3/s$$

$$\frac{dh}{dt} = (-0.0005)\left(\frac{4}{\pi} \cdot \frac{1}{2^2}\right)$$

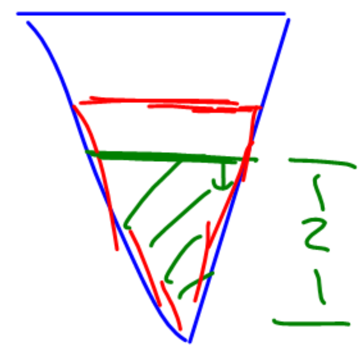
$$\doteq -1.59 \times 10^{-4}\ m/s$$

Water level is dropping at -1.59×10^{-1} mm/s

Question: When the water height is **above** 2 m, is the water height changing more **quickly** or more **slowly** than the value we just found?

(a) more quickly

(b) more slowly ✓



$h > 2$

$$\frac{dh}{dt} = \underbrace{\frac{dV}{dt}}_{\substack{\text{const} \\ -0.5\ \ell/s}} \left(\underbrace{\frac{4}{\pi}}_{\text{const}} \frac{1}{h^2} \right)$$

larger h → smaller $\frac{1}{h^2}$

→ smaller $\frac{dh}{dt}$

General method for Related Rates problems

$\frac{dx}{dt}$ $\frac{dy}{dx}$

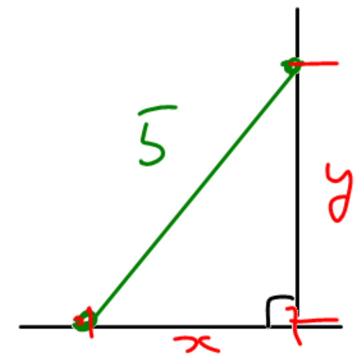
- Draw a picture (if possible) of the situation, and label all relevant variables.
- Identify which of the variables and their rate of change are known.
- Identify which rate you are trying to determine.
- Write an equation involving the changing variables, including the function whose rate you are trying to find.
  ↳ (no rates to start)
- Apply implicit differentiation to the equation, *before* substituting any known variables.
- Substitute known variables and rates.
- Solve for desired rate.

No calculus course is complete without a related rates ladder problem.

Example: *A 5 m ladder is propped against the wall. You climb to top, but then it starts to slip; the tip of the ladder is moving downwards at 1 m/s. Find the rate at which the* bottom *end of the ladder is sliding along the ground, when the bottom of the ladder is 3 m away from the wall.*

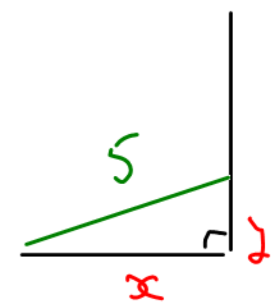
x = dist from wall to bottom of ladder

y = " " ground to top of ladder

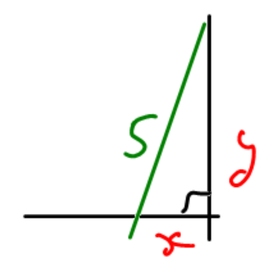


Always true: $x^2 + y^2 = 5^2$

Know $\frac{dy}{dt} = -1$ m/s



want $\frac{dx}{dt}$



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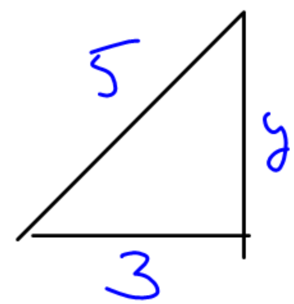
$$\frac{d}{dt}\left(x^2+y^2\right)=\frac{d}{dt}\left(5^2\right)$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

Want $\frac{dx}{dt}$: $2x\frac{dx}{dt}=-2y\frac{dy}{dt}$ tip of ladder velocity

or $\frac{dx}{dt}=-\frac{\cancel{2}y}{\cancel{2}x}\frac{dy}{dt}$

bottom of ladder velocity



@ $x=3 \rightarrow y=4$ so

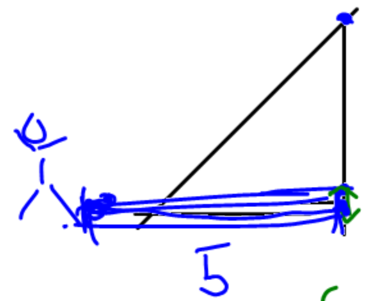
$$3^2+y^2=5^2$$
$$y^2=25-9$$
$$y^2=16$$
$$y=4$$

$$\frac{dx}{dt}=-\frac{4}{3}(-1)=+\frac{4}{3}\ m/s$$

Bottom of ladder is moving away from the wall at 1.333 m/s.

Repeat your calculations, but for the rate at which the bottom end of the ladder is slide when the top of the ladder is just about to hit the ground.

$$\frac{dx}{dt} = -\frac{\overset{0}{\cancel{y}}}{x}\underbrace{\frac{dy}{dt}}_{\text{fixed}}$$



$y = 0$ (ladder hits ground)

$$\frac{dx}{dt} = 0$$

$\rightarrow$ bottom of the ladder does not move.

Example: *The ideal gas law states that*

volume

number of moles

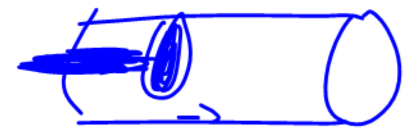
pressure $PV = nRT$ temp (in °K)

ideal gas const

0.1 moles of gas are held in a piston, and the temperature is held constant at 273° K. The piston moves to decrease the volume from 3 l to 1 l over 60 seconds. What is the pressure half-way through this process, and at what rate is the pressure changing at this time?

$P = ?$

$\frac{dP}{dt} = ?$



3 l → 1 l

over 60 s

T @ 273°K

$$\frac{dV}{dt} = \frac{\Delta V}{\Delta t} = \frac{-2\,l}{60\,s} = \frac{-1}{30}\,\frac{l}{s}$$

Always true $P \cdot V = nRT = 0.1$ (const)

const

Take $\frac{d}{dt}$ of both sides

$$\frac{d}{dt}(P \cdot V) = \frac{d}{dt}(nRT)$$

here const @ 273K

$$\frac{dP}{dt} \cdot V + P\,\frac{dV}{dt} = 0$$

(Continued)

$$PV = nRT$$

$$n = 0.1 \text{ mol}, \quad T = 273\,^o\text{K}$$

$$\frac{d\cancel{l}V}{dt} = \frac{-1}{30} \text{ l/s}$$

$$\frac{dP}{dt} \cdot V + P\frac{dV}{dt} = 0$$

$$\frac{dP}{dt} = -\frac{P}{V}\frac{dV}{dt}$$

$$= -\frac{113.43}{2}\left(\frac{-1}{30}\right)$$

$$= 1.89 \ \frac{\text{kPa}}{\text{s}}$$

Pressure is increasing @ $1.89 \ \frac{\text{kPa}}{\text{s}}$ at halfway point

@ halfway of piston movement

$3\ell \to 1\ell$

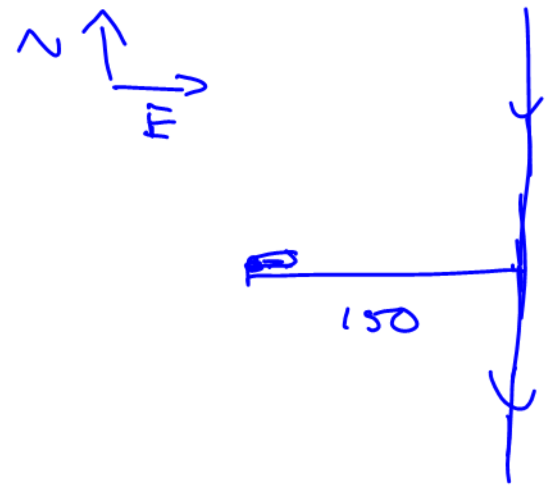
@ halfway, $V = 2\ell$

$$P = \frac{nRT}{V} = \frac{(0.1)(8.31)(273)}{(2)}$$

kPa . ℓ

$$P = 113.43 \text{ kPa}$$

Example: *A radar tracking site is following a plane flying in a straight line. At its closest approach, the plane will be 150 km away from the radar site. If the plane is traveling at 600 km/h, how quickly is the radar rotating to track the plane when the plane is closest?*



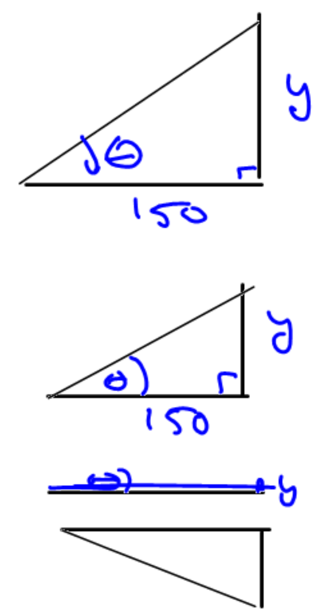
Want $\frac{d\theta}{dt}$

y = dist of plane north of radar station.

$\frac{dy}{dt} = -600 \ \frac{km}{h}$

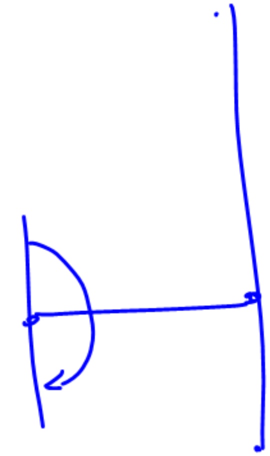
Relate quantities y and θ

$$\tan\theta = \frac{y}{150}$$



Take $\frac{d}{dt}$ of both sides

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y}{150}\right)$$



$$\left(\frac{1}{\cos^2\theta}\right)\cancel{\sec^2\theta}\cdot\frac{d\theta}{dt} = \frac{1}{150}\,\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{150}\cos^2\theta\,\frac{dy}{dt}$$

at point plane is closest

$\theta = 0$, $\frac{dy}{dt} = -600$ km/h

so $$\frac{d\theta}{dt} = \frac{1}{150}\underbrace{\cos^2(\cancel{0})}_{=1}(-600) = -4\,\frac{\text{rad}}{\text{hr}}$$

Does the angular rotation speed up or slow down as the plane moves away from the radar station?

$$\frac{d\theta}{dt} = \underset{\substack{\uparrow \\ \text{const}}}{\frac{1}{150}} \cos^2\theta \underset{\substack{\uparrow \\ \text{const}}}{\frac{dy}{dt}}$$

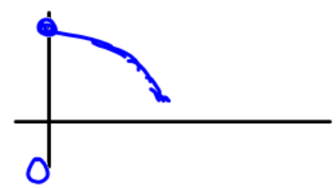
as θ incr from 0

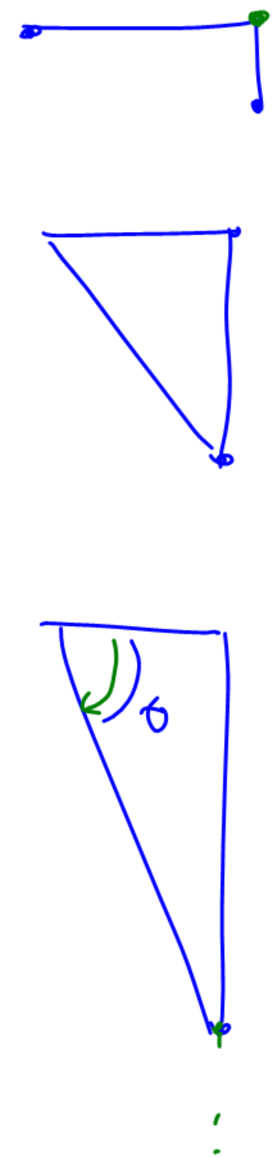
$\rightarrow$ $\cos\theta$ decr

$\rightarrow$ $\cos^2\theta$ decr

$\rightarrow$ $\frac{d\theta}{dt}$ decr as well

turn more slowly.



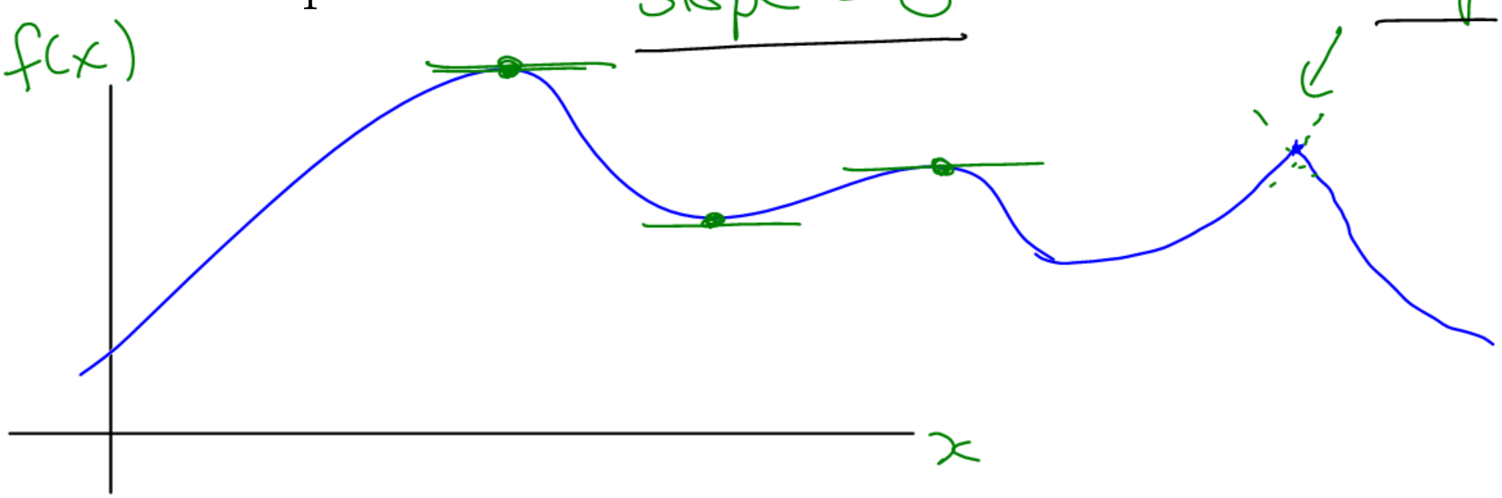


Sketch a graph of the angular rotation rate against the rotation angle.

Optimization - Classifying Critical Points

An important goal in many engineering applications is to **optimize** the system you are designing. E.g. choosing the angle in a truss that maximizes the strength of the structure.

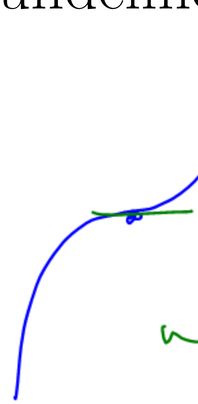
Problem. Sketch several continuous graphs, and identify characteristics of the points at which the function reaches a maximum or minimum output value.



We define these points of interest as **critical points**, and they satisfy either

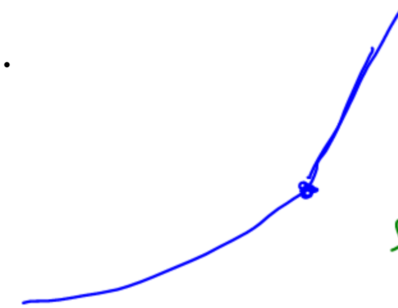
- $f'(x) = 0$, or
- $f'(x)$ is undefined (but $f(x)$ *is* defined).

$y = x^3$



$f'(x) = 0$

neither min nor max



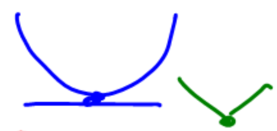
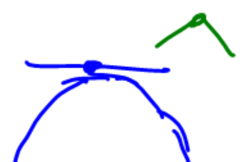
f' not defined

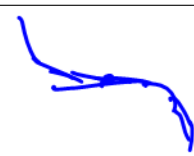
It is worth noting that all such points are critical points, and that the maximum and minimum values of the function will be a subset of these. Each individual critical point will be a local min, max, or neither, and we'll need to do further calculations to determine which.

First Derivative Test

One way to decide whether at a critical point there is a local maximum or minimum is to examine the sign of the derivative on opposite sides of the critical point. This method is called the **first derivative test**.

Complete this table:

	Sketch	f' sign left of c	f' sign right of c
local minimum at c	 neg slopes pos	$(-)$	$(+)$
local maximum at c	 pos slopes neg	$(+)$	$(-)$
neither local max nor min		$(+)$	$(+)$



$(-)$ $(-)$

Example: *Find the critical points of the function* $f(x) = 2x^3 - 9x^2 + 12x + 3$. *Use the first derivative test to show whether each critical point is a local maximum or a local minimum.*

Find critical points

Need $f'(x) = 6x^2 - 18x + 12$ Set $= 0$ for crit. pnts

$\quad = 6(x-2)(x-1)$

$\cancel{6}(x^2 - 3x + 2) = 0$

$(x-2)(x-1) = 0$

So $x = 2$, $x = 1$ are both crit. points for $f(x)$

Sketch of $f(x)$

$x = 1$ local max

$x = 2$ local min

by 1st deriv. test

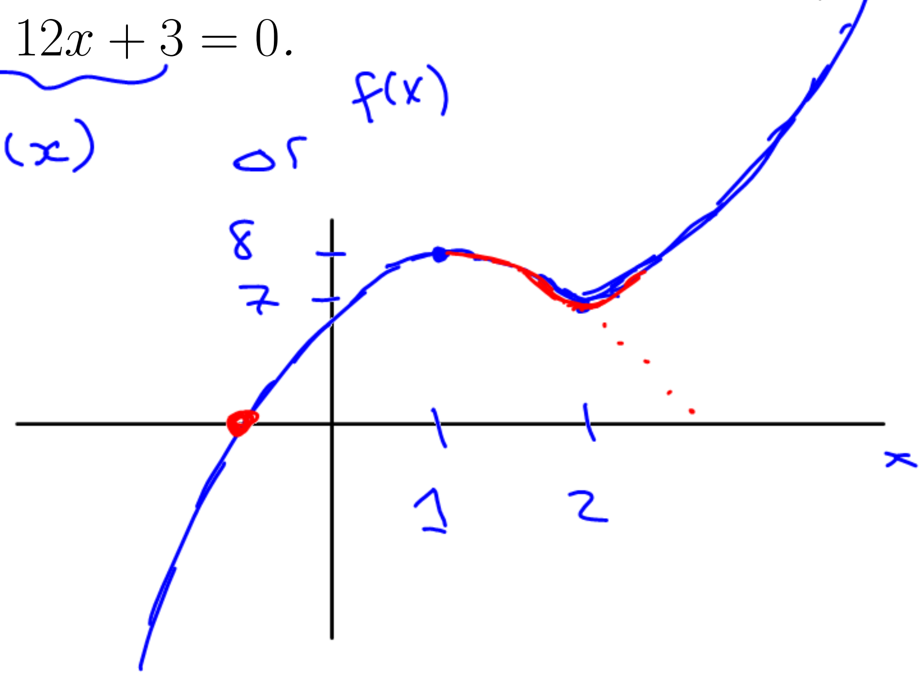
Sign chart for $f'(x)$

slopes $f'(x) = 6(x-2)(x-1)$	(+)	(−)	(+)
$(x-2)$	(−)	(−)	(+)
$(x-1)$	(−)	(+)	(+)

Using your answer to the preceding question, determine the number of real solutions of the equation $\underbrace{2x^3 - 9x^2 + 12x + 3}_{f(x)} = 0.$

Find y's at critical points

or $f(x)$



$f(1) = 2 - 9 + 12 + 3$

$= 8$

$f(2) = 16 - 36 + 24 + 3$

$= 7$

only 1 root/zero for $f(x)$.

Second Derivative Test

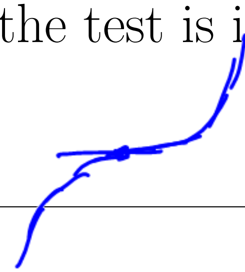
You may also use the Second Derivative Test to determine if a critical point is a local minimum or maximum.

- The first derivative test uses the **first** derivative **around** the critical point.
- The second derivative test uses the **second** derivative **at** the critical point.
- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.

concave up

- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

- If $f'(c) = 0$ and $f''(c) = 0$ then the test is inconclusive.

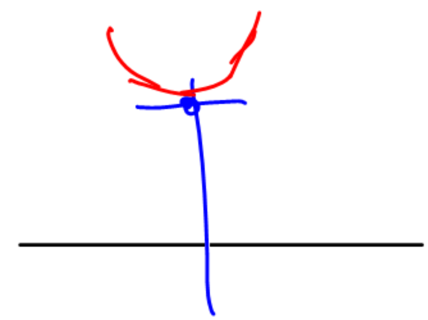


x^4

Example: *A function f has derivative $f'(x) = \cos(x^2) + 2x - 1$. Use the second derivative test to determine whether it has a local maximum, a local minimum, or neither at its critical point $x = 0$.*

Is $x=0$ a crit point?

$$f'(0) = \cos(0) + 0 - 1 = 0 \quad \checkmark$$



Second deriv test

Need $f''(x) = -\sin(x^2)\cdot 2x + 2$

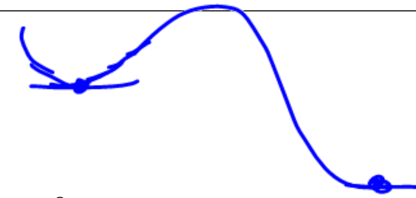
At $x=0$, $f''(0) = -\sin(0)\cdot 0 + 2$

$$= 2$$

$\Rightarrow$ $f(0)$ is concave up

$\Rightarrow$ $x=0$ is a local min for $f(x)$

Global vs. Local Optimization



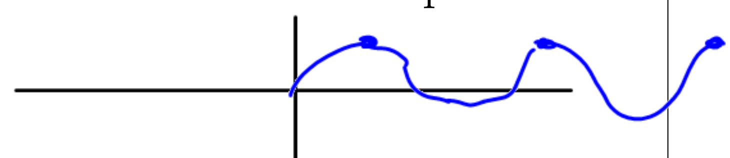
The first and second derivative tests only give us *local* information in most cases. However, if there are multiple local maxima or minima, we usually want the **global** max or min. The ease of determining when we have found the global max or min of a function depends strongly on the properties of the question.

Local vs Global Extrema

A **local max** occurs at $x = c$ when $f(c) > f(x)$ for x values **around** c.

A **global max** occurs at $x = c$ if $f(c) \geq f(x)$ for **all** values of x in the domain.

It is possible to have several global maxima if the function reaches its peak value at more than one point. ↪ sin(x)



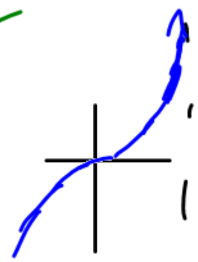
Corresponding definitions apply for local and global minima.

Example: *Give an example of a simple function with multiple global maxima.*

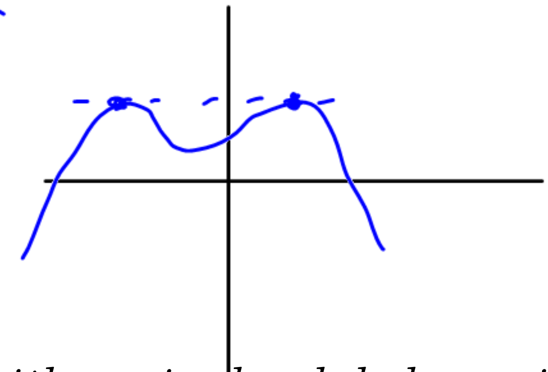
$\sin(x)$ ✓

$\cos(x)$ ✓

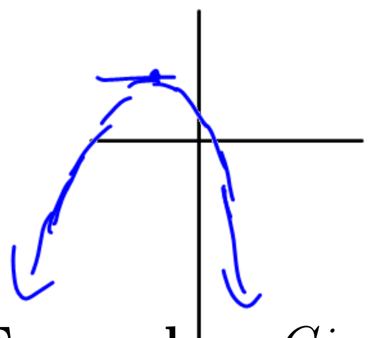
~~$\tan(x)$~~



poly'l



Example: *Give an example of a simple function with a single global maximum, but no global minimum.*



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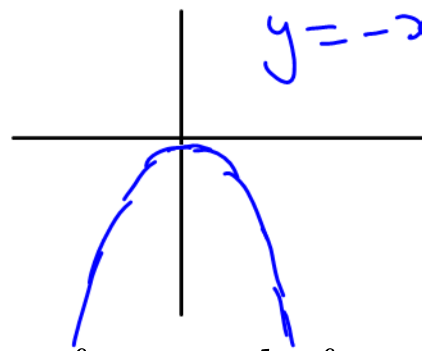
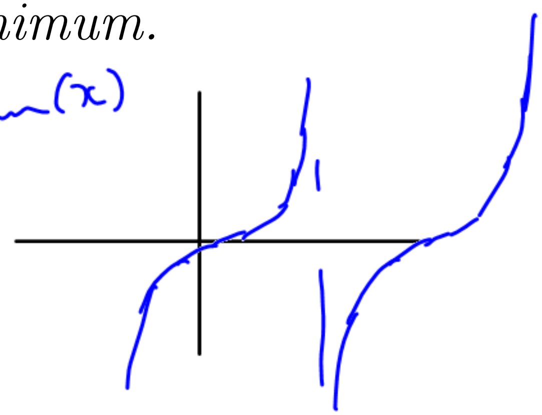


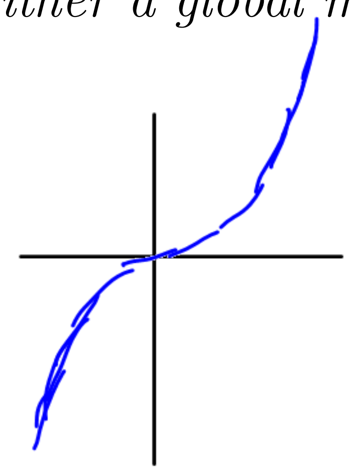
$y = -x^2$

Example: *Give an example of a simple function with neither a global maximum nor a global minimum.*

$y = \tan(x)$

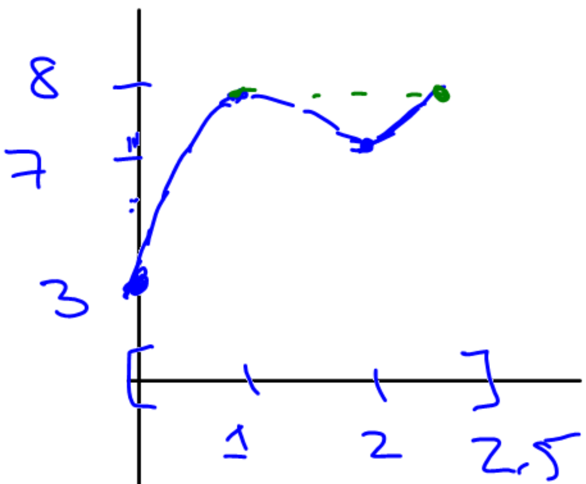


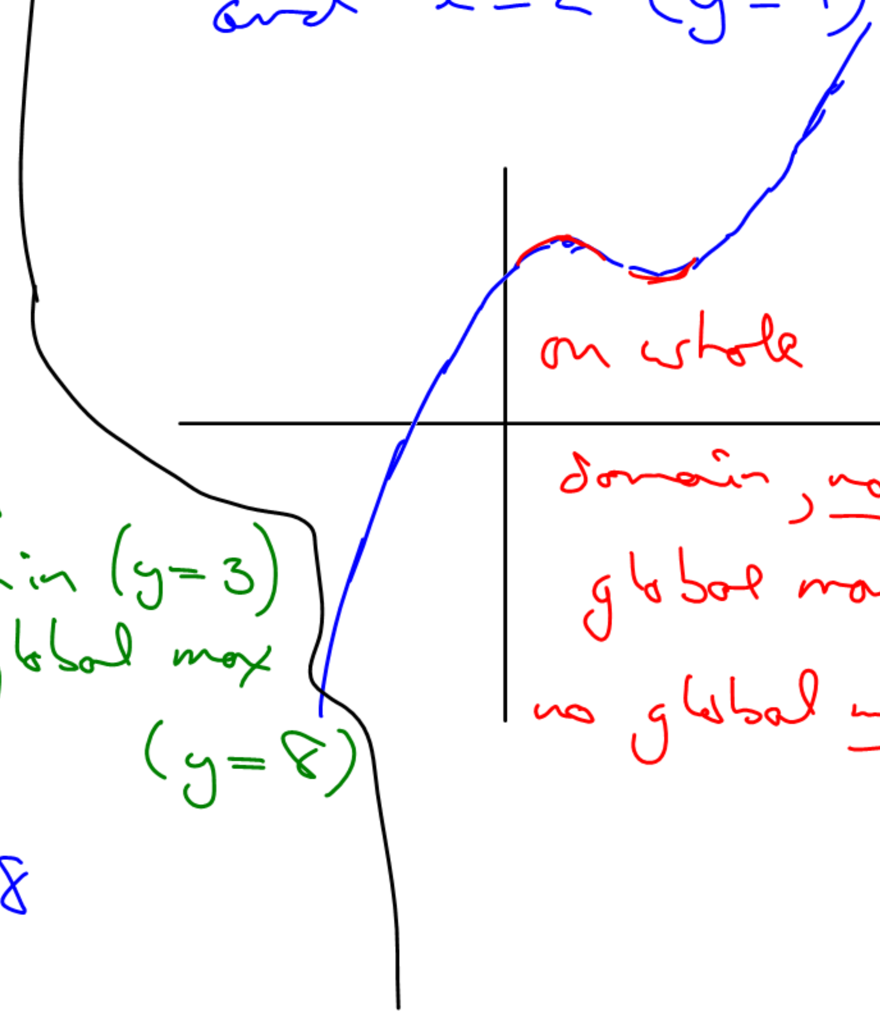
$y = x^3$



Example: *Earlier we worked with the function* $f(x) = 2x^3 - 9x^2 + 12x + 3$. *If we limit the function to the interval* $x \in [0, 2.5]$, *what are the* **global max** *and* **global minimum** *values on that interval?*

↳ crit points at $x = 1$ $(y = 8)$ and $x = 2$ $(y = 7)$





on whole domain, no global max
no global min

On domain $[0, 2.5]$
$x = 0$ is the global min $(y = 3)$
$x = 1, 2.5$ tie for the global max $(y = 8)$

$x = 0$, $f(0) = 3$

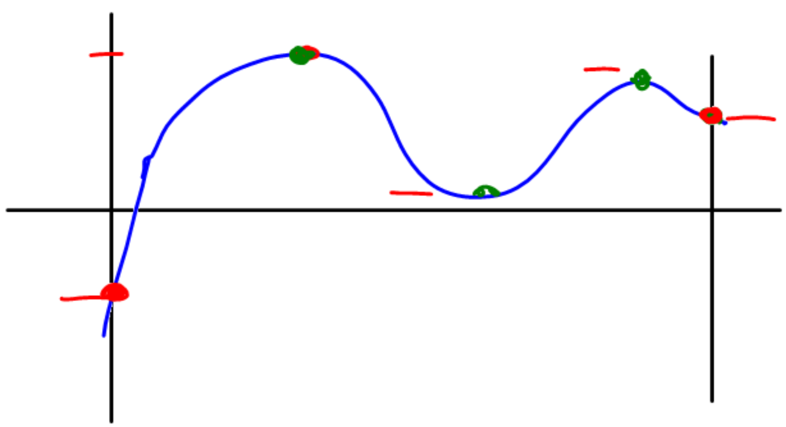
$x = 2.5$, $f(2.5) = 8$

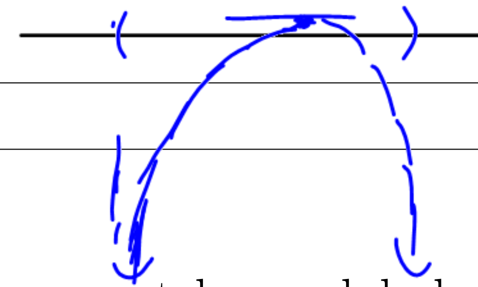
Global Extrema on Closed Intervals

A continuous function on a closed interval will **always** have a global max and a global min value. These values will occur at either

- a critical point *or*
- an end point of the interval.

To find which value is the global extrema, you can compute the original function's values at all the critical points and end points, and select the point with the highest/lowest value of the function.





Global Extrema on Open Intervals

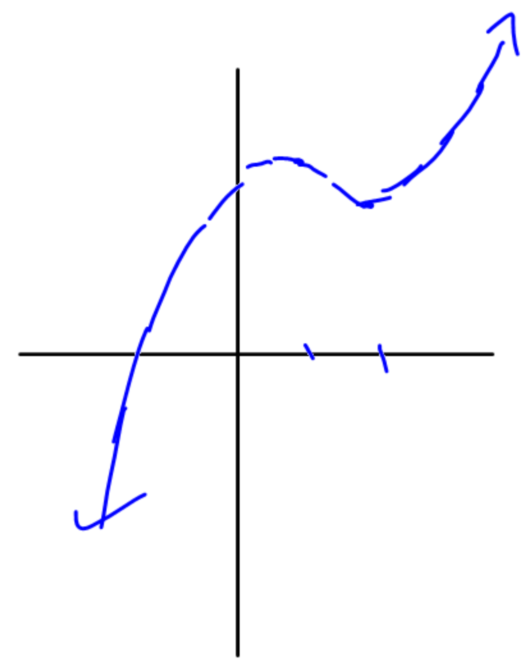
A function defined on an open interval may or may not have global maxima or minima.

If you are trying to demonstrate that a point is a global max or min, and you are working with an open interval, including the possible interval $(-\infty, \infty)$, proving that a particular point is a global max or min requires a careful argument. A recommendation is to look at either:

- values of f when x approaches the endpoints of the interval, or $\pm\infty$, as appropriate; or
- if there is only one critical point, look at the sign of f' on either side of the critical point.

With that information, you can often construct an argument about a particular point being a global max or min.

Example: *Consider the function* $f(x) = 2x^3 - 9x^2 + 12x + 3$. (Seen before)
Does $f(x)$ *have a global max?*



(a) Yes, $f(x)$ has a global maximum somewhere in its domain.

(b) No, $f(x)$ does not have a global maximum.

$$\lim_{x \to \infty} f(x) \to +\infty$$

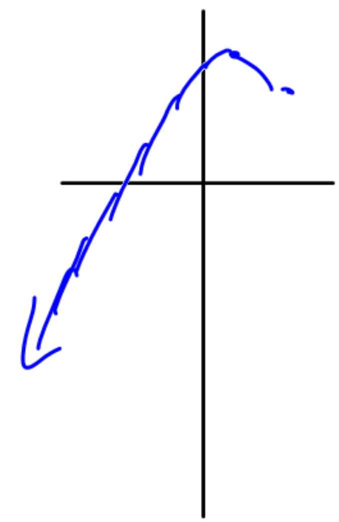
$\Rightarrow$ no global max

(no single highest point)

For the function $f(x) = 2x^3 - 9x^2 + 12x + 3$*, does* $f(x)$ *have a glocal minimum?*

(a) Yes, $f(x)$ has a global minimum somewhere in its domain.

(b) No, $f(x)$ does not have a global minimum. ✓



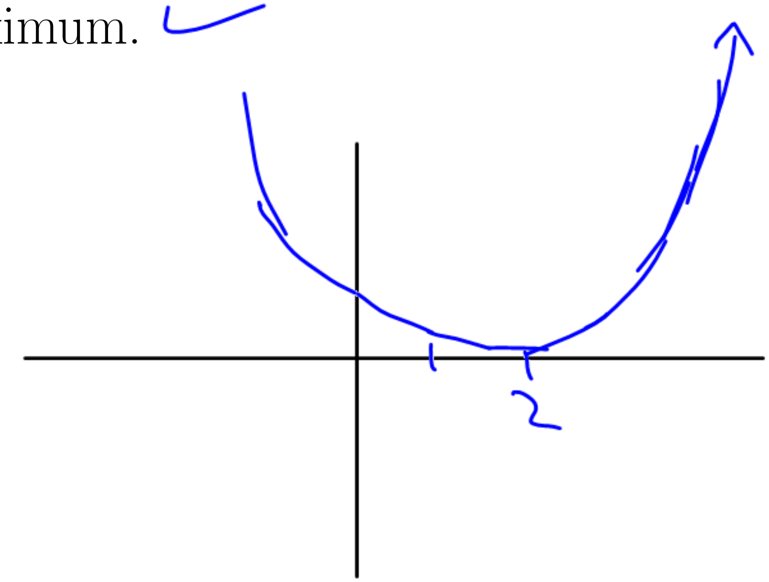
or $\lim_{x \to -\infty} f(x) \to -\infty$

(no single lowest point)

Example: *Does the function $g(x) = (x-2)^4$ have a global maximum?*

(a) Yes, $g(x)$ has a global maximum somewhere in its domain.

(b) No, $g(x)$ does not have a global maximum. ✓



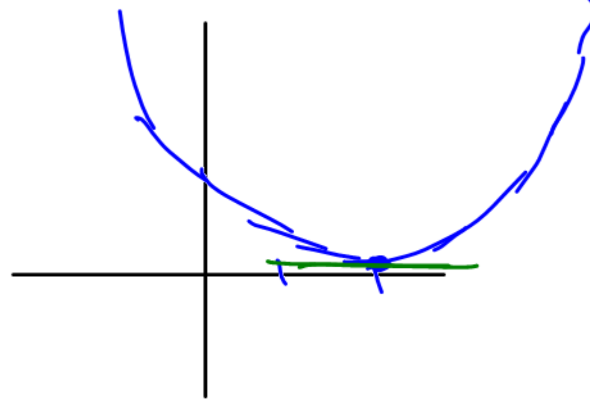
$\lim_{x \to \infty} g(x) \to \infty$

no single highest point

Example: *Does the function $g(x) = (x-2)^4$ have a global minimum?*

(a) Yes, $g(x)$ has a global minimum somewhere in its domain. ✓ (Need critical point analysis)

(b) No, $g(x)$ does not have a global minimum.



Optimization

An optimization problem is one in which we have to find the maximum or minimum value of some quantity. In principle, we already know how to find the maximum and minimum values of a function if we are given a formula for the function and the interval on which the maximum or minimum is sought. Usually the hard part in an optimization problem is interpreting the word problem in order to find the formula of the function to be optimized.

Example: *A farmer wants to build a rectangular enclosure to contain livestock. The farmer has 120 meters of wire fencing with which to build a fence, and one side of the enclosure will be part of the side an already existing building (so there is no need to put up fence on that side). What should the dimensions of each side be to maximize the area of the enclosure?*

What is the quantity to be maximized in this example?

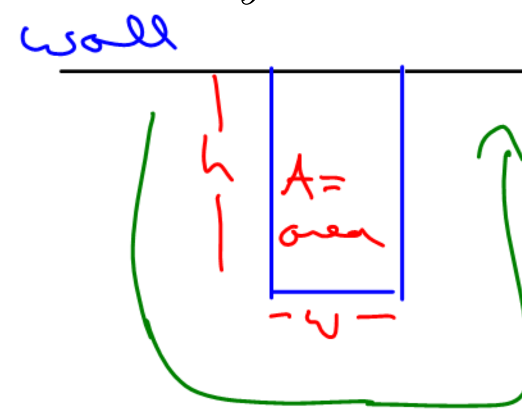
(a) Length of wire.

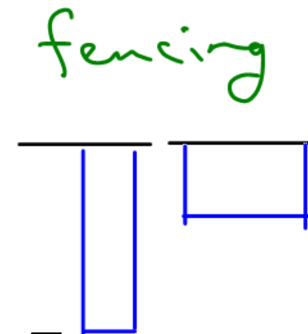
(b) Area enclosed.

(c) Side lengths beside barn.

$A =$

What are the variables in this question, and how are they related? You may want to draw a picture.





$$\boxed{A = h \cdot w}$$

Total fence length $= 120$ m

$= h + w + h$

$$\boxed{120 = 2h + w} \leftrightarrow 120 - 2h = w$$

Express the quantity to be optimized in terms of the variables above. Try to eliminate all but one of the variables.

$A = h \cdot w$ w/ $w = 120 - 2h$

so $A = h(120 - 2h)$ just 1 indep. variable

What is the domain on which the one remaining variable makes sense?

for h: $h \geq 0$, $h \leq 60$

Use the techniques learned earlier in the course to maximized the function on this domain. Give reasons explaining why the answer you found is the **global** *maximum.*

$A = h\cdot(120-2h)$ on $0 \le h \le 60$

closed domain, A is continuous

Look for critical points

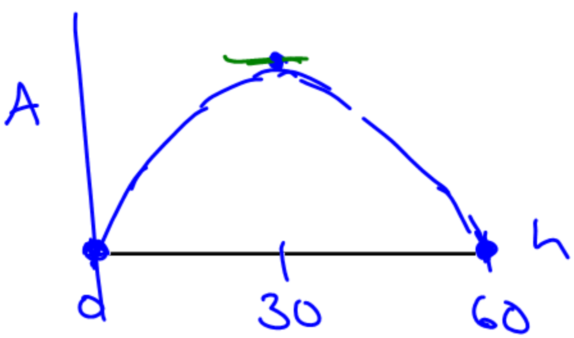
First rewrite $A = 120h - 2h^2$ (more derivative friendly)

set $= 0$ for c.p.

or $A' =$ so $\frac{dA}{dh} = 120 - 4h = 0$

$120 = 4h$

$\boxed{h = 30}$ one single critical point



Find $A(0) = 0\ m^2$

$A(30) = 30(120-60) = 1800\ m^2$

$A(60) = 60\cdot 0 = 0\ m^2$

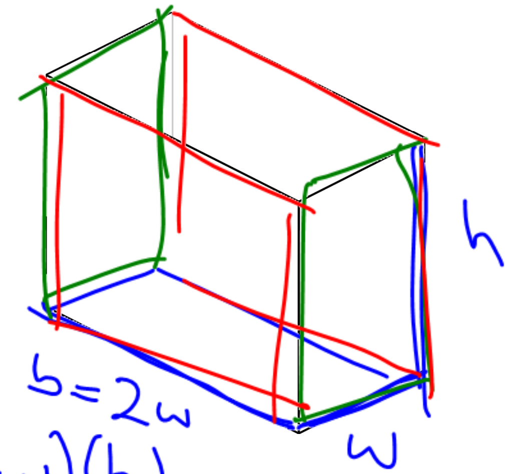
b/c A is cont's on a closed & bounded domain, A(30) is the max area

$h = 30\ m$
so $w = 60\ m$

(Fencing example continued)

Example: *(Storage Container)*
A rectangular storage container with an open top is to have a volume of 10 m^3*. The length of its base is to be twice its width. Material for the base costs \$10.00 per m*^{2}*, and material for the sides costs \$6.00 per m*^{2}*. Determine the cost, of the material for the cheapest such container.*

dimensions



Goal: minimize cost

Need: cost = (base cost) + (sides cost)

$$= 10 \cdot \underbrace{w \cdot 2w} + 6 \cdot 2\underbrace{(w \cdot h)} + 6 \cdot 2\underbrace{(2w)(h)}$$

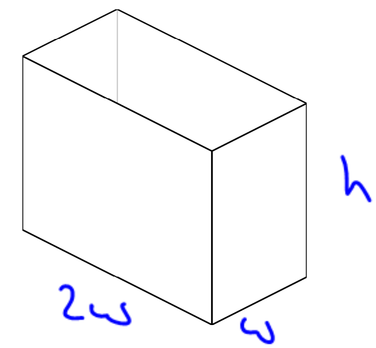
$\$/m^2 \cdot m^2$ $\quad$ $\$/m^2 \cdot m^2$ $\quad$ $\$/m^2 \cdot m^2$

$$= 20w^2 + 12h \cdot w + 24w \cdot h$$

$$\text{Cost} = 20w^2 + 36w \cdot h$$

require

$$\text{Vol} = (2w)(w)(h) = 10 \text{ m}^3$$



$$2w^2h = 10$$

$$h = \frac{5}{w^2}$$

and so $\text{Cost} = 20w^2 + 36\, w \cdot h$

$$= 20w^2 + 36\, w\left(\frac{5}{w^2}\right)$$

$$C(w) = 20w^2 + \frac{180}{w} = 20w^2 + 180w^{-1}$$

Look for critical points

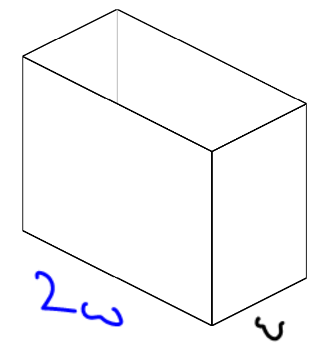
set $=0$ for c.p.

$$\frac{dC}{dw} = 40w - 180w^{-2} = 40w - \frac{180}{w^2} = 0$$

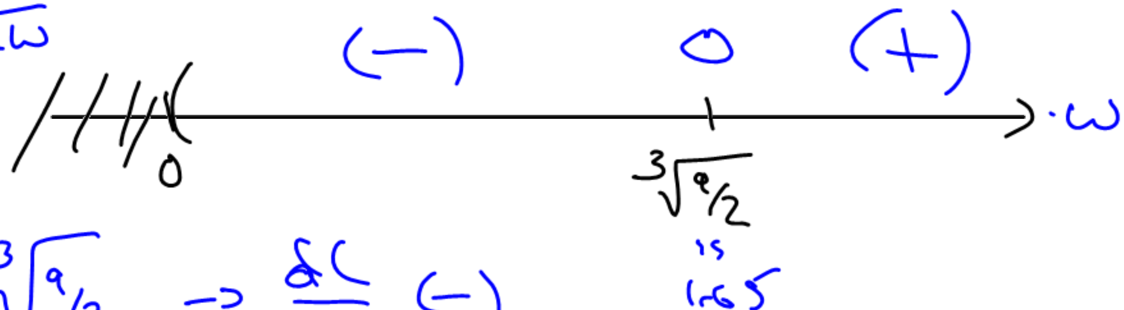
$$40w = \frac{180}{w^2}$$

$$w^3 = \frac{180}{40} = \frac{9}{2} \quad \text{so} \quad w = \sqrt[3]{9/2} \cong 1.65 \text{ m}$$

Sign chart for $\frac{dC}{dw} = 40w - \frac{180}{w^2}$



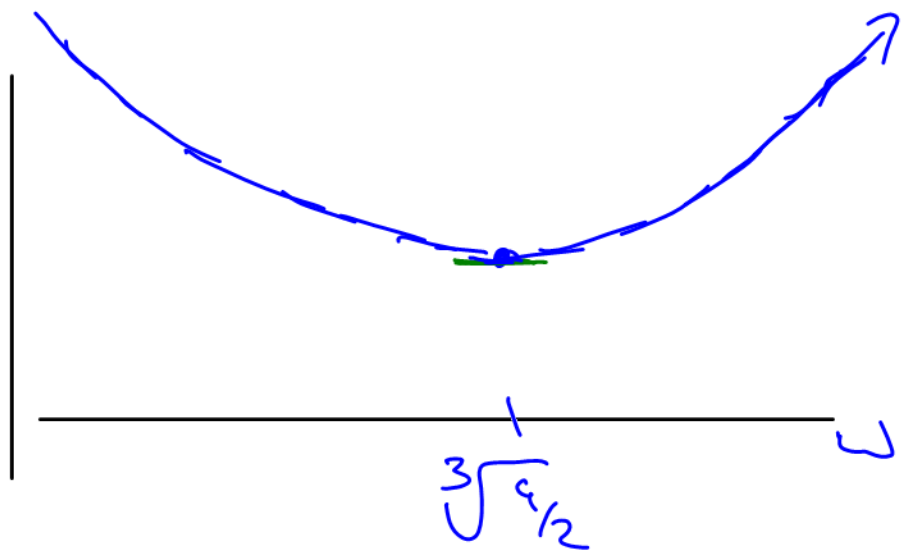
sign of $\frac{dC}{dw}$



$w < \sqrt[3]{9/2} \rightarrow \frac{dC}{dw}\ (-)$

$w > \sqrt[3]{9/2} \rightarrow \frac{dC}{dw}\ (+)$

Sketch of $C(w)$



$\Rightarrow w = \sqrt[3]{9/2}$

is a global min for cost.

$w \cong 1.65$ m

$2w \cong 3.3$ m

$h \cong 5/w = 1.8$ m

optimal Cost $= 20w^2 - \frac{180}{w} \cong \$163$