Week 8: Integration by Parts and Substitution

Goals:

- The guess-and-check method for anti-differentiation.
- The substitution method for anti-differentiation.
- Learning integration by parts.
- Applying a slicing approach to construct integrals for areas.

We now return to the challenge of finding a *formula* for an anti-derivative function. We saw simple cases last week, and now we will extend our methods to handle more complex integrals.

Anti-differentiation by Inspection: The Guess-and-Check Method

Often, even if we do not see an anti-derivative immediately, we can make an educated guess and eventually arrive at the correct answer.

Example: *Based on your knowledge of derivatives, what should the anti-derivative of* $\cos(3x)$, $\int \cos(3x)\ dx$, *look most like?*

(a) $\cos(\cancel{x}) + C$

(b) $\sin\cancel{(x})+C$

(c) $\cos(3x) + C$

(d) $\sin(3x) + C$

$$\frac{d}{dx}\left(\sin(3x)\right) = \cos(3x)\cdot 3$$

good guess, but needs tweaking.

Evaluate $\int \cos(3x)\,dx$ *by guessing and checking your antiderivative.*

$$\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + C$$

Check:

$$\cos(3x) = \frac{\cos(3x)\cdot\cancel{3}}{\cancel{3}} = \frac{d}{dx}\left(\frac{\sin(3x)}{3} + C\right)$$

Example: *Find* $\int e^{7x-2}\,dx$.

$$\int e^{7x-2}\,dx = \frac{1}{7}e^{7x-2} + C$$

Check:

$$e^{7x-2} = \frac{e^{7x-2}\cdot \cancel{7}}{\cancel{7}} = \frac{d}{dx}\left(\frac{e^{7x-2}}{7} + C\right)$$

$\Leftarrow$

Example: *Both of our previous examples had* linear *'inside' functions. Here is an integral with a* quadratic *'inside' function:*

$$\int xe^{-x^2}\,dx$$

Evaluate the integral.

$$\int xe^{-x^2}\,dx = \frac{-1}{2}e^{-x^2} + C$$

Check:

$$xe^{-x^2} = \frac{e^{-x^2}\cdot(\cancel{-2}x)}{\cancel{-2}} = \frac{d}{dx}\left(\frac{e^{-x^2}}{-2} + C\right)$$

Why was it important that there be a factor x in front of e^{-x^2} in this integral?

$$\int e^{-x^2}\,dx = e^{-x^2} + C$$

check

$$\frac{-2xe^{-x^2}}{-2x} \neq \frac{d}{dx}\left(\frac{e^{-x^2}}{-2x} + C\right)$$

quotient rule

Integration by Substitution

We can formalize the guess-and-check method by defining an *intermediate variable* the represents the "inside" function.

Example: *Show that* $\int \underline{x^3\sqrt{x^4+5}}\, dx = \frac{1}{6}(x^4+5)^{3/2} + C.$

Check: $\frac{d}{dx}\left(\frac{1}{6}(x^4+5)^{3/2} + C\right)$

$= \frac{1}{6}\left(\frac{3}{2}(x^4+5)^{1/2} \cdot (4x^3)\right)$

$= \sqrt{x^4 + 5}\ \frac{1}{4}(4x^3)$

$= x^3\sqrt{x^4+5}$ = original integrand ✓

$$\int x^3\sqrt{x^4+5}\,dx = \frac{1}{6}(x^4+5)^{3/2} + C$$

Relate this result to the **chain rule**.

x^3 from $\frac{d}{dx}(\underbrace{x^4+5}_{\text{inside}})$

$$\underset{g'(x)}{x^3}\sqrt{\underbrace{x^4+5}_{g(x)}}$$

Now use the **method of substitution** *to evaluate* $\int x^3\sqrt{x^4+5}\,dx$

let $w = x^4+5$

so $\frac{\Delta w}{\Delta x} \approx \frac{dw}{dx} = 4x^3$

$dw = 4x^3\,dx$

$\frac{1}{4x^3}\,dw = dx$

Replace x's w/ w's and dx w/ dw

$$\int x^3\sqrt{x^4+5}\,dx$$

$$= \int \cancel{x^3}\sqrt{w}\left(\frac{1}{4\cancel{x^3}}\,dw\right)$$

$$= \frac{1}{4}\int \sqrt{w}\,dw \quad \leftarrow \text{simpler!}$$

$$= \frac{1}{4}\int w^{1/2}\,dw \quad \text{integrate}$$

$$= \frac{1}{4}\,\frac{w^{3/2}}{3/2} + C$$

$$= \frac{2}{12}\left(x^4+5\right)^{3/2} + C = \frac{1}{6}\left(x^4+5\right)^{3/2} + C$$

Steps in the Method Of Substitution

$\int$ ___ dx

1. Select a simple function $w(x)$ that appears in the integral.
   - Typically, you will also see w' as a **factor** in the integrand as well.
2. Find $\dfrac{dw}{dx}$ by differentiating. Write it in the form $\ldots dw = \; \underline{\underline{dx}}$
3. Rewrite the integral using only w and dw (no $\underline{x \text{ nor } dx}$).
   - If you can now evaluate the integral, the substitution was effective.
   - If you cannot remove all the x's, or the integral became harder instead of easier, then either try a different substitution, or a different integration method.

Example: *Find* $\int \tan(x)\, dx$.

$w = \tan(x)$

(No $\sec^2(x)$ in integral)

$$= \int \frac{\sin(x)}{\cos(x)}\, dx$$

let $w = \cos(x)$

so $\dfrac{dw}{\boxed{dx}} = \underline{-\sin(x)}$

$$= \int \frac{\cancel{\sin(x)}}{w} \left(\frac{-1}{\cancel{\sin(x)}}\, dw \right)$$

so $\left(\dfrac{-1}{\sin(x)}\, dw \right) = dx$

$$= -\int \frac{1}{w}\, dw$$

new integral, all w's, no x's ✓

simpler ✓

$$= -\ln|w| + C$$

back to x

$$\frac{d}{dw}\left(\ln|w|\right) = \frac{1}{w}$$

$$= -\ln|\cos(x)| + C$$

Though it is not required unless specifically requested, it can be reassuring to check the answer.

Verify that the anti-derivative you found is correct.

$$\frac{d}{dx}\left(-\ln|\cos(x)| + C\right)$$

$$= -\frac{1}{\cos(x)} \cdot \left(-\sin(x)\right)$$

$$= +\frac{\sin(x)}{\cos(x)}$$

$$= \tan(x)$$ = original integrand ✓

Example: *Find* $\int x^3 e^{\underline{x^4-3}} dx.$

let $w = x^4 - 3$ (dividing by $4x^3$)

so $\dfrac{dw}{dx} = 4x^3 \xrightarrow[\text{isolate } dx]{} \dfrac{1}{4x^3}\, dw = dx$

Rewrite integral

$$\int x^3 e^{x^4-3}\, dx = \int \cancel{x^3} e^{w} \left(\frac{1}{4\cancel{x^3}}\, dw \right)$$

$$= \frac{1}{4} \int e^{w}\, dw$$

simpler ✓
w's, w/no x's ✓
integrate

back to x's

$$= \frac{1}{4} e^{w} + C$$

$$= \frac{1}{4} e^{x^4-3} + C$$

Example: *For the integral,*

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2} dx$$

both $w = e^x - e^{-x}$ *and* $w = e^x + e^{-x}$ *are seemingly reasonable substitutions.*

Question: Which substitution will change the integral into the simpler form?

(a) $w = e^x - e^{-x}$

(b) $w = e^x + e^{-x}$

Compare both substitutions in practice.

$$I = \int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\,dx$$

with $w = e^x - e^{-x}$

so $\frac{dw}{dx} = e^x - (e^{-x}(-1))$

$\frac{dw}{dx} = e^x + e^{-x}$

Isolate dx $\quad \frac{1}{e^x + e^{-x}}\,dw = dx$

$$I = \int \frac{w}{(e^x + e^{-x})^2}\,\frac{1}{(e^x + e^{-x})}\,dw$$

$$= \int \frac{w}{(e^x + e^{-x})^3}\,dw$$

X mix of w's, x's

not simpler

with $w = e^x + e^{-x}$

so $\frac{dw}{dx} = e^x - e^{-x}$

Isolate dx

$\frac{1}{e^x - e^{-x}}\,dw = dx$

$$I = \int \frac{\cancel{e^x - e^{-x}}}{w^2}\left(\frac{1}{\cancel{e^x - e^{-x}}}\,dw\right)$$

$$= \int \frac{1}{w^2}\,dw$$ ✓ no x's ✓ simpler ✓

Integrate

$$= \int w^{-2}\,dw$$ back to x's

$$= \frac{w^{-1}}{-1} + C = -\frac{1}{(e^x + e^{-x})} + C$$

Example: *What substitution is most likely to be helpful for evaluating the integral*

$$\int \frac{\underline{\sin(x)}}{1+\cos^2(x)}dx \text{ ?}$$

(a) $w = \sin(x)$

(b) $\underline{w = \cos(x)}$ -> divide by $-\sin(x)$

(c) $w = 1 + \cos^2(x)$ -> divide by $2\cos(x) \cdot \sin(x)$

$$\textit{Evaluate} \int \frac{\sin(x)}{1+\cos^2(x)}dx. = I$$

let $w = \cos(x)$

Isolate dx

so $\dfrac{dw}{dx} = -\sin(x) \rightarrow \dfrac{-1}{\sin(x)}dw = dx$

Rewrite integral

$$I = \int \frac{\cancel{\sin(x)}}{1+w^2}\left(\frac{-1}{\cancel{\sin(x)}}dw\right)$$

$$= -\int \frac{1}{1+w^2}dw$$

simpler ✓

no x's ✓

$$\frac{d}{dw}\left(\qquad \right) = \frac{1}{1+w^2}$$

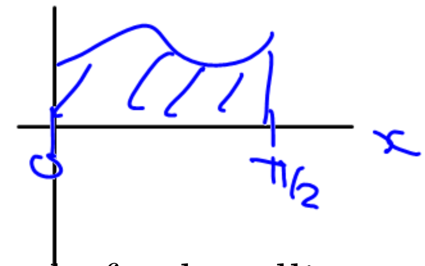
$$= -\arctan(w) + C$$

back to x's

$$= -\arctan(\cos(x)) + C$$

Using the Method of Substitution for Definite Integrals

If we are asked to evaluate a **definite** integral such as



$$\int_{x=0}^{x=\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx \quad ,$$

where a substitution will ease the integration, we have two methods for handling the limits of integration ($x = 0$ and $x = \pi/2$).

$w = \ldots$

a) When we make our substitution, convert both the *variables* x and the *limits* (in x) to the new variable; or

b) do the integration while keeping the limits explicitly in terms of x, writing the final integral back in terms of the original x variable as well, and *then* evaluating.

Example: *Use method a), converting the bounds to the new variable, to evaluate the integral*

$$\int_{x=0}^{x=\pi/2} \frac{\sin x}{1+\cos^2 x}dx$$

let $w = \cos(x)$

$x = \pi/2 \rightarrow w = \cos(\pi/2) = 0$

so $\frac{dw}{dx} = -\sin(x)$

$x = 0 \rightarrow w = \cos(0) = 1$

or $\frac{-1}{\sin(x)}dw = dx$

so $I = \int_{w=1}^{w=0} \frac{\cancel{\sin(x)}}{1+w^2}\left(\frac{-1}{\cancel{\sin(x)}}dw\right) = -\int_{w=1}^{w=0} \frac{1}{1+w^2}dw$

$= +\int_0^1 \frac{1}{1+w^2}dw$

integrate

$= \arctan(w)\Big|_{w=0}^{w=1}$

$= \arctan(1) - \arctan(0)$

$= \pi/4$

Example: *Use method b) method, keeping the bounds in terms of x, and converting back to x's, to evaluate*

$$\int_{x=9}^{x=64} \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx \quad .$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2}\frac{1}{\sqrt{x}}$$

let $w = 1+\sqrt{x}$

so $\frac{dw}{dx} = \frac{1}{2}\frac{1}{\sqrt{x}}$ or $2\sqrt{x}\,dw = dx$

$$I = \int_{x=9}^{x=64} \frac{\sqrt{w}}{\cancel{\sqrt{x}}}\left(2\cancel{\sqrt{x}}\,dw\right)$$

$$= 2\int_{x=9}^{x=64} w^{1/2}\,dw$$

integrate

$$= 2\,\frac{w^{3/2}}{3/2}\Bigg|_{x=9}^{x=64}$$

back to x's

$$= \frac{4}{3}\left(1+\sqrt{x}\right)^{3/2}\Bigg|_{x=9}^{x=64}$$

sub in bounds

$$= 4/3\left(1+\sqrt{64}\right)^{3/2} - \left(4/3\left(1+\sqrt{9}\right)^{3/2}\right)$$

$$= 4/3\,(\cancel{1+8})^{3/2} - \frac{4}{3}(\cancel{1+3})^{3/2}$$

$$= 4/3\,(27) - \frac{4}{3}(8) = \frac{76}{3}$$

Integration by Parts

$\int x^2\,dx = \frac{x^3}{3} + C$

So far in studying integrals we have used

- direct anti-differentiation, for relatively simple functions, and
- integration by substitution, for some more complex integrals.

However, there are many integrals that can't be evaluated with these techniques.

Try to find $\int xe^{4x}\,dx$.

Substitution

$w = x$

$w = e^{4x}$ → neither leads to a simpler integral

This particular integral can be evaluated with a different integration technique, **integration by parts.** This rule is related to the **product rule** for derivatives.

Expand

$$\int \frac{d}{dx}(uv)\,dx = \int \frac{du}{dx}\cdot v\,dx + \int u\cdot\frac{dv}{dx}\,dx$$

Integrate both sides with respect to x and simplify.

$$uv = \int\left(\frac{du}{dx}\cdot v\right)dx + \underbrace{\int\left(u\,\frac{dv}{dx}\right)dx}$$

Express $\displaystyle\int u\frac{dv}{dx}\,dx$ relative to the other terms.

$$\int u\,\frac{dv}{dx}\,dx = uv - \int\left(\frac{du}{dx}\cdot v\right)dx$$

first integral $\longrightarrow$ trade for a second (hopefully simpler) integral

Integration by Parts

For short, we can remember this formula as

$$\underbrace{\int u\,dv}_{\text{given integral}} = uv - \int v\,du$$

Integration by parts:

- Choose a part of the integral to be u, and the remaining part to be dv.
- **Differentiate** u to get du.
- **Integrate** dv to get v.
- Replace $\int u\,dv$ with $uv - \int v\,du$.
- Hope/check that the new integral is easier to evaluate.

$$\int u\,dv = uv - \int v\,du$$

$$\int x e^{4x}\,dx$$

Example: *Use integration by parts to evaluate* $\int xe^{4x}\,dx$.

let $u = x$ $\qquad \int 1\,dv = \int e^{4x}\,dx$

$\downarrow d/dx \qquad\qquad \downarrow \int$

so $\dfrac{du}{dx} = 1 \qquad v = \dfrac{e^{4x}}{4}$ (skip $+C$)

or $du = dx$

simpler? ✓

$$\int x e^{4x}\,dx = \underbrace{(x)}_{u}\underbrace{\left(\frac{e^{4x}}{4}\right)}_{v} - \int \underbrace{\left(\frac{e^{4x}}{4}\right)}_{v} \underbrace{dx}_{du}$$

tidy

$$= \frac{x}{4}e^{4x} - \frac{1}{4}\int e^{4x}\,dx$$

integrate

$$= \frac{x}{4}e^{4x} - \frac{1}{4}\,\frac{e^{4x}}{4} + C$$

Verify that your anti-derivative is correct.

$$\frac{d}{dx}\left(\frac{x}{4}\cdot e^{4x} - \frac{1}{4}\cdot\frac{e^{4x}}{4} + C\right)$$

(goal: $= xe^{4x}$)

$$= \left(\frac{1}{4}e^{4x} + \frac{x}{\cancel{4}}\cdot e^{4x}\cdot\cancel{4}\right) - \frac{1}{4}\,\frac{e^{4x}(\cancel{4})}{\cancel{4}}$$

$$= \cancel{\frac{1}{4}e^{4x}} + xe^{4x} - \cancel{\frac{1}{4}e^{4x}}$$

$$= xe^{4x} \checkmark$$

$$\int u dv = uv - \int v du$$

Guidelines for selecting u and dv

need to experiment!

- Try to select u and dv so that either
  - u' is simpler than u or
  - $\int dv$ is simpler than dv
- Ensure you can actually integrate the dv part by itself

Example: *Consider the integral* $\int x \cos x \; dx$.

Based on the guidelines, what choice of u and dv should you try first?

(a) $u = x\cos(x)$, $\int dv = \int dx$.

complicated du (try last)

$$u \xrightarrow{d/dx} du$$

$$\int$$

$$dv \longrightarrow \int dv$$

(b) $u = 1$, $dv = x\cos(x)\; dx$.

(c) $u = x$, $\int dv = \int \cos(x)\; dx$.

$\frac{du}{dx} = 1$ ← simpler

(d) $u = \cos(x)$, $\int dv = \int x \; dx$.

$\frac{du}{dx} = -\sin(x)$ $\quad v = \frac{x^2}{2}$ (try later)

Evaluate the integral $\int x \cos x \, dx$.

$$\int u\,dv = uv - \int v\,du$$

let $u = x$ $\qquad \int dv = \int \cos(x)\,dx$

$\downarrow d/dx \qquad \downarrow \int$

so $\frac{du}{dx} = 1 \qquad v = \sin(x)$

or $du = dx$

simpler? ✓

$$\int x\cos(x)\,dx = \underbrace{x}_{u} \cdot \underbrace{\sin(x)}_{v} - \int \underbrace{\sin(x)}_{v}\,\underbrace{dx}_{du}$$

tidy

$$= x\sin(x) - (-\cos(x)) + C$$

$$= x\sin(x) + \cos(x) + C$$

Verify that your anti-derivative is correct.

$$\int \underline{x \cos(x)}\, dx = x \sin(x) + \cos(x) + C$$

Check: $\frac{d}{dx}\left(x \cdot \sin(x) + \cos(x) + C \right)$

$$= 1 \cdot \cancel{\sin(x)} + x \cdot \cos(x) + \left(\cancel{-\sin(x)}\right)$$

$$= \underline{x \cos(x)} \quad \checkmark$$

Example: *Evaluate the slightly more challenging integral*

$$\int x^2 \cos x \; dx$$

Try subs? X

What choice of u and dv would be most likely to be helpful?

(a) $u = 1$, $\int dv = \int x^2 \cos(x)\; dx$. X

$u \to u'$ simpler

(b) $u = x$, $dv = \int x \cos(x)\; dx$.

$dv \longrightarrow$ have to be able to integrate

$\int dv$ simpler

(c) $u = x^2$, $dv = \int \cos(x) dx$. (promising)

$\frac{du}{dx} = 2x$

(d) $u = x^2 \cos(x)$, $dv = dx$.

$\frac{du}{dx}$ requires product rule

$$\int x^2 \cos x \; dx = I$$

let $u = x^2$ $\qquad \int dv = \int \cos(x)\,dx$

$\downarrow d/dx$

$\rightarrow \quad \frac{du}{dx} = 2x \quad \leftarrow \quad v = \sin(x)$

or $\quad du = (2x)\,dx$

$$I = \underbrace{x^2 \sin(x)}_{uv} - \underbrace{\int \sin(x)\,(2x\,dx)}_{\int v\,du}$$

tidy

$$= x^2 \sin(x) - 2 \underbrace{\int x \sin(x)\,dx}_{I_2}$$

simpler? ✓
Can we evaluate by simple anti-deriv? No...

I_2: let $u = x$ $\qquad \int dv = \int \sin(x)\,dx$

so $\quad du = 1\,dx \quad \leftarrow \quad v = (-\cos(x))$

$$\int x^2 \cos x \, dx$$

$$\int x^2 \cos(x)\,dx$$

$$= x^2\sin(x) - 2\int x\sin(x)\,dx$$

$$= x^2\sin(x) - 2\left[x(-\cos(x)) - \int(-\cos(x))\,dx\right]$$

↓ tidy

simpler ✓

$$= x^2\sin(x) - 2\left[-x\cos(x) + \int \cos(x)\,dx\right]$$

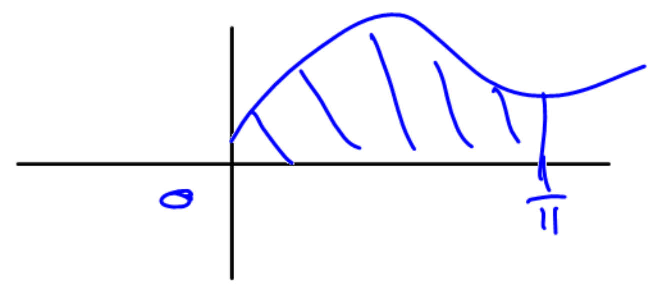
↓ integrate

$$= x^2\sin(x) - 2\left[-x\cos(x) + \sin(x) + C\right]$$

Integration By Parts - Definite Integrals

When using integration by parts to evaluate *definite* integrals, you need to apply the limits of integration to the **entire** anti-derivative that you find.

Example: *Evaluate* $\displaystyle\int_0^{\pi} x\sin(4x)\,dx = I$



let $u = x$ $\qquad \displaystyle\int dv = \int \sin(4x)\,dx$

so $du = 1\,dx \leftarrow v = \dfrac{-\cos(4x)}{4}$

$$\text{so } I = x\left(\frac{-\cos(4x)}{4}\right)\Bigg|_0^{\pi} - \int_0^{\pi} -\frac{\cos(4x)}{4}\,(dx)$$

tidy

$$= \frac{-x}{4}\cos(4x)\Bigg|_0^{\pi} + \frac{1}{4}\int_0^{\pi}\cos(4x)\,dx \qquad = -\frac{\pi}{4}$$

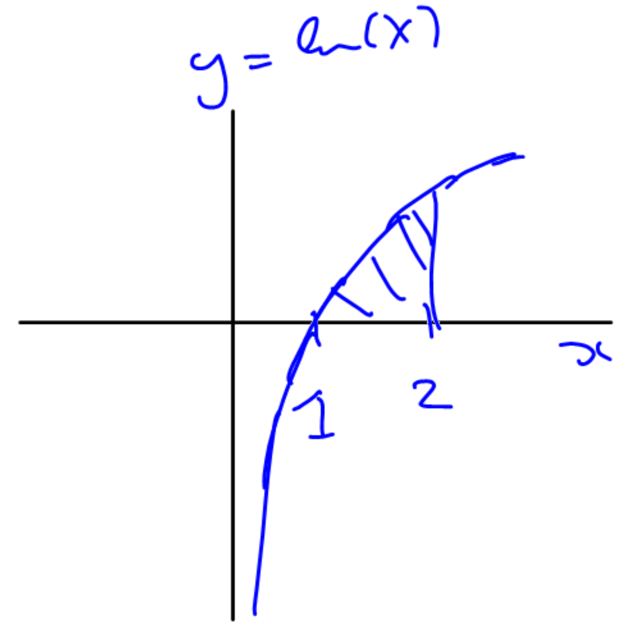
@ π

$$= \left[-\frac{x}{4}\cos(4x) + \frac{1}{4}\,\frac{\sin(4x)}{4} + C\right]\Bigg|_0^{\pi} = \left(\overbrace{-\frac{\pi}{4}\cdot 1 + \frac{1}{16}\cdot 0}^{@\,\pi}\right) - (0+0)$$

Don't forget that dv does not require any other factors besides dx. That can help when there is only a single factor in the integrand.

Example: *Find* $\int_1^2 \ln x \, dx = I$

$\frac{d}{dx} \ln(x) = \frac{1}{x}$



let $u = \ln(x)$ $\quad \int dv = \int dx$

so $du = \frac{1}{x} dx$ $\quad v = x$

so $I = \ln(x) \cdot x \Big|_1^2 - \int_1^2 x \cdot \left(\frac{1}{x} dx\right)$

tidy

$= x \ln(x) \Big|_1^2 - \int_1^2 1 \cdot dx$ $\quad$ simpler?

integrate

$= x \ln(x) \Big|_1^2 - x \Big|_1^2$

sub in

$= \left(2 \ln(2) - 1 \cancelto{0}{\ln(1)}\right) - (2 - 1)$

$= 2 \ln(2) - 1$

Integrals as Areas - Review

Example: *Write the integral that represents the area underneath the graph* $y = x^2$ *between* $x = 0$ *and* $x = 2$.

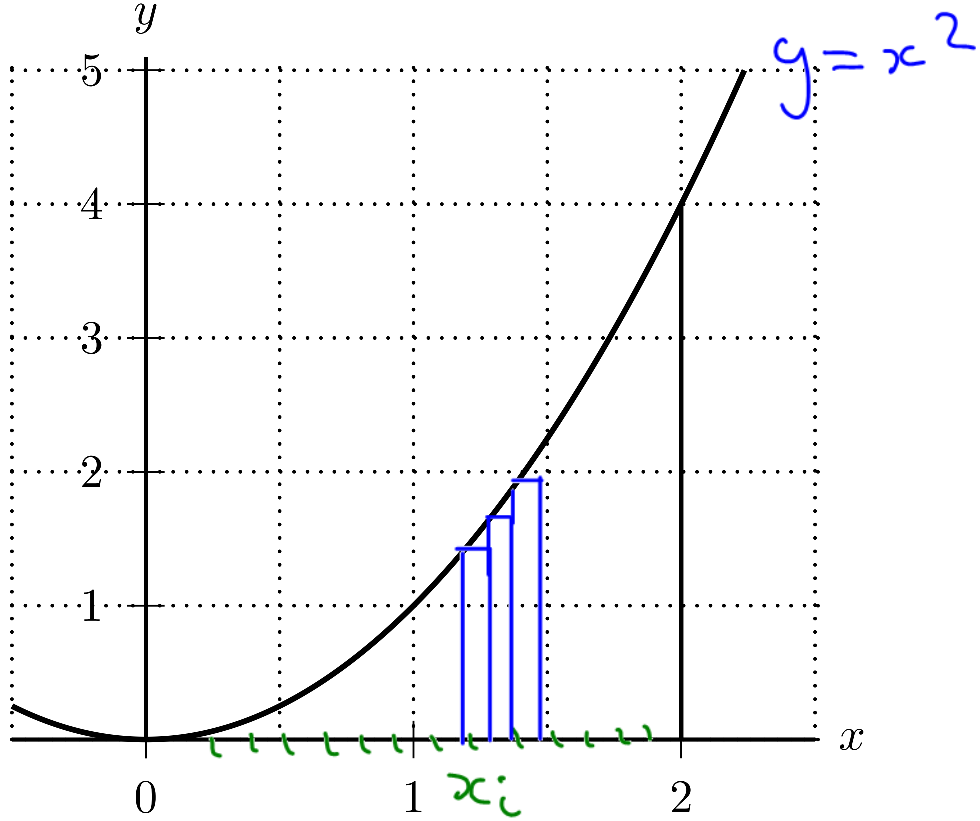
$$\text{area} = \int_0^2 x^2 \, dx$$

Evaluate the integral to find the area.

$$\int_0^2 x^2 \, dx = \frac{x^3}{3} \Big|_0^2$$

$$= \frac{8}{3} - \frac{0}{3} = \frac{8}{3} \text{ sq units.}$$

Illustrate how this region under the graph of $y = x^2$ can be constructed by accumulating small rectangles of varying heights.



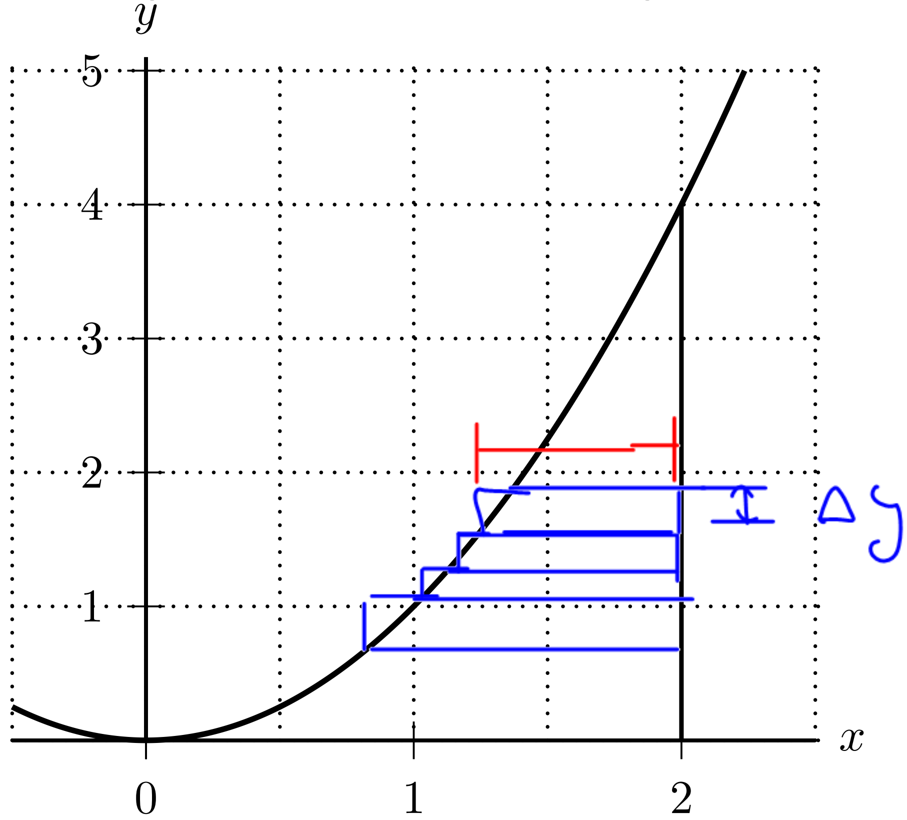
$$\int_0^2 x^2\,dx$$

$$\downarrow$$

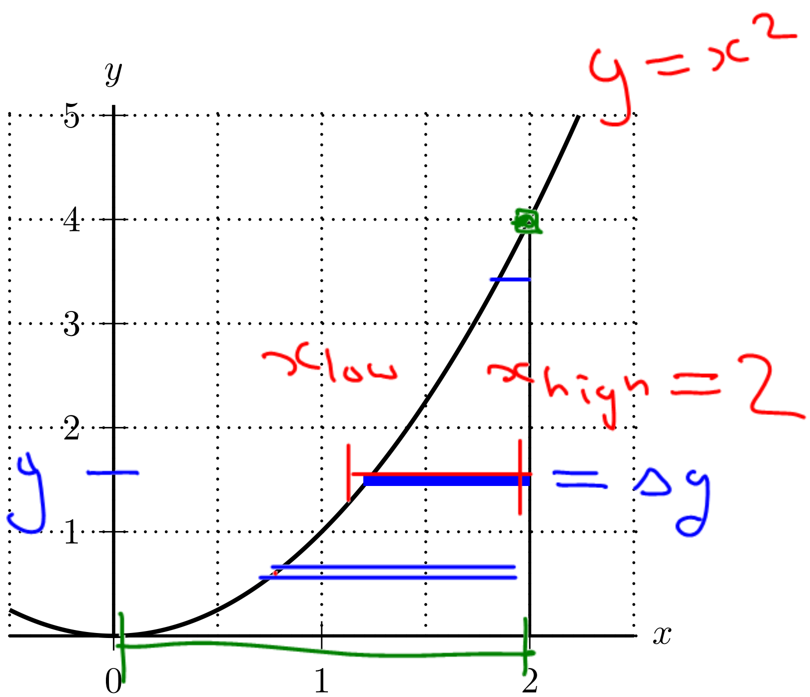
$$\sum_{i=1}^{n} \underbrace{(x_i)^2}_{\text{height}} \; \underbrace{\Delta x}_{\text{width}}$$

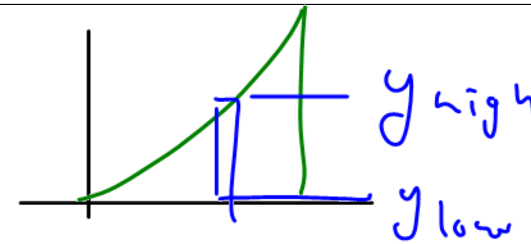
$\lim$ $\Delta x \to 0$ or $n \to \infty$

Now show how the exact same area can be constructed by using **horizontal** *rectangles. Write the analogous intervals, widths, etc. on the diagram.*



$$\sum_{i=1}^{n} (\text{width})(\text{height}) = \sum_{i=1}^{n} (\text{width?})(\Delta y)$$





height $= y_{high} - y_{low}$

x_{low} on $y = x^2$

$$\boxed{\sqrt{y} = x_{low}}$$

What is the **width** of each rectangle, given its y location?

(a) width $= \sqrt{y}$

(b) width $= \sqrt{y} - 2$

(c) width $= 2 - \sqrt{y}$

(d) width $= 4 - \sqrt{y}$

width $= x_{high} - x_{low}$

$= 2 - \sqrt{y}$

Write out first a sum, then a new integral, that would represent the exact area in the sketch. Evaluate the integral.

Total area $\cong \sum (\text{width})(\text{height})$

$\downarrow$

$$\cong \sum (2-\sqrt{y})\,\Delta y$$

$\lim_{\Delta y \to 0}$

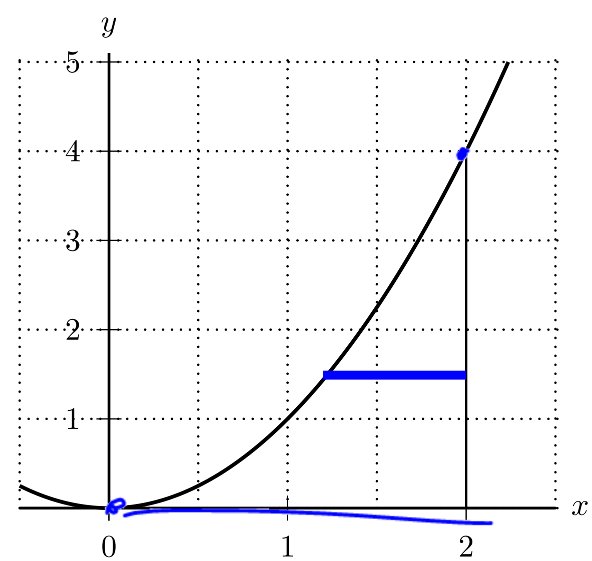
Exact Total Area $= \displaystyle\int_{y=0}^{y=4} (2-\sqrt{y})\,dy$

$$= \left(2y - \frac{y^{3/2}}{3/2}\right)\Bigg|_0^4$$

$$= \left[8 - \frac{2}{3}\left(4^{3/2}\right)\right] - [0-0]$$

$$= 8 - \frac{2(8)}{3} = \frac{24}{3} - \frac{16}{3} = 8/3 \text{ sq units}$$

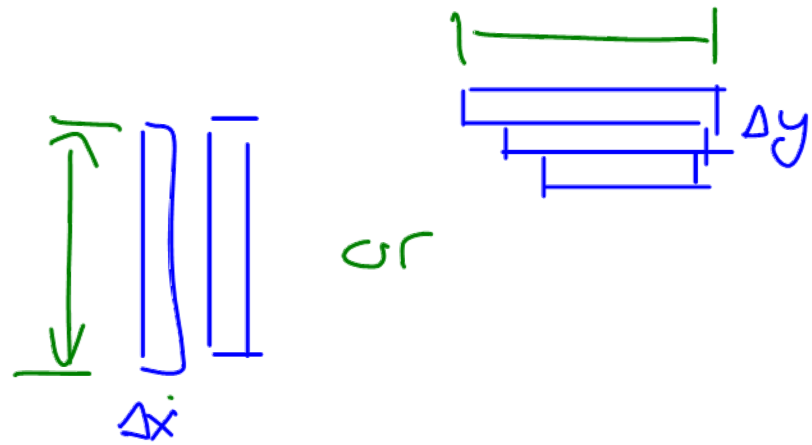
$$\frac{8}{3} = \int_{x=0}^{x=2} x^2\,dx$$



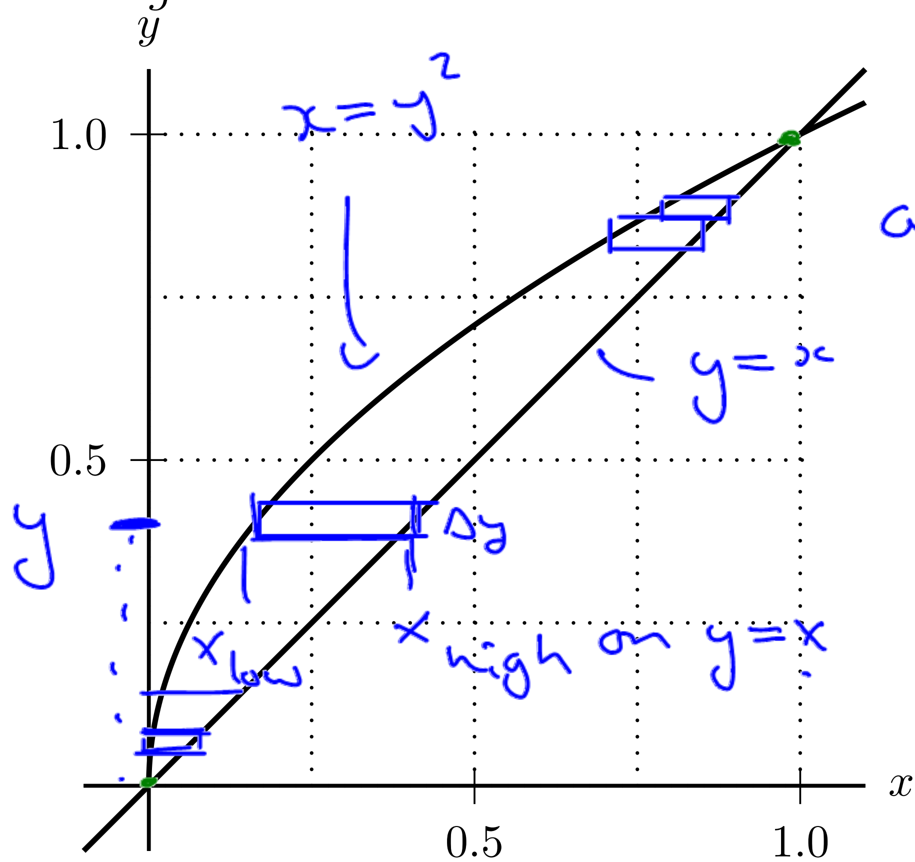
Integrals as Sums of Slices

Most people visualize this approach as *slicing* the shape into thin pieces. To find the total area, the process is:

- decide along which axis you want to slice (say slices perpendicular to x)
- find the size of a **generic** slice, as a function of the position x
- write out the sum that represents to the total you want
- transform the sum into an integral
- evaluate the integral



Example: *Find the area enclosed between the graphs of $x = y^2$ and $y = x$, using horizontal slices.*



area of a slice (@ location y)

$$= (\text{width})(\text{hight})$$

$$= (x_{high} - x_{low})(\Delta y)$$

y=x x=y^2

$$= (y - y^2)(\Delta y)$$

convert to

Sum (total area) $\approx \sum (y_i - y_i^2)\,\Delta y$

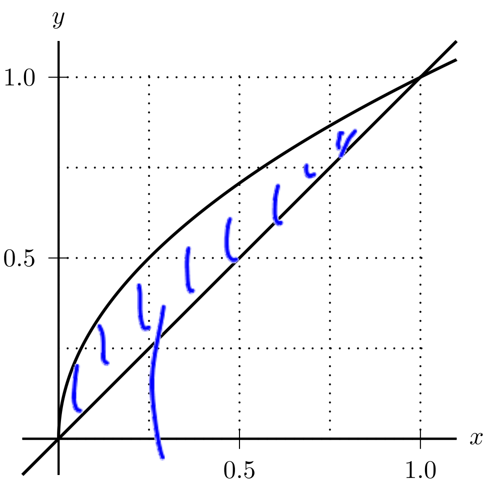
$$\text{Total area} = \int_{y=0}^{y=1} (y - y^2)\,dy$$

$$\text{area} = \int_0^1 (y - y^2)\,dy$$

$$= \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0)$$

$$= \frac{1}{6}$$



has area $= \frac{1}{6}$

sq unit.