Week 10: Differential Equations and Complex Numbers - Part 1

Goals:

- Modeling with Differential Equations
- Harmonic Motion
- Introduction to Complex Numbers

Differential Equations

gravitation

As pointed out on several other occasions in the course, most laws of nature and science, once they are translated into mathematics, take the form of a **differential equation**. $F = ma$ Maxwell's

A differential equation is quite unlike the other equations we have studied. For one thing, it involves derivatives, often even second or higher derivatives. But this is not the most important difference between a differential equation and the kinds of equations we have been used to and continue to study in our courses.

- Solutions to **algebraic** equations are **real values** (or sets of values)

$x^2 = 9$ sub into eq'n $\Leftarrow$ $x = \pm 3$ values

LHS $= 3^2 = 9$

RHS $= 9$

check LHS = RHS

- Solutions to **differential** equations are **functions** (or sets of functions)

= w/ derivative of an unknown function

$x'(t) = e^{2t}$ $\rightsquigarrow$ $x(t) = \frac{1}{2}e^{2t} + c$

function

Problem. Which of the following **functions** is a solution to the **differential** equation $x''(t) = -36\ x(t)$?

x = unknown

A. $x(t) = -6t^3$

B. $x(t) = \cos(6t)$

C. $x(t) = e^{-6t}$

D. $x(t) = -e^{-6t}$

Problem. Confirm your answer.

LHS = $-\cos(6t)\cdot 36$ are

RHS equal

$= -36\, x(t)$

$= -36\cos(6t)$

① $x''(t) = -36\, x(t)$

A. $x(t) = -6t^3$

$\boxed{\text{B. } x(t) = \cos(6t)}$

C. $x(t) = e^{-6t}$

D. $x(t) = -e^{-6t}$

If $x = -6t^3$

to sub into equation, we

need x'':

$\Rightarrow x' = -18t^2$

$\Rightarrow x'' = -36t$

So LHS of ① $= x'' = -36t$

and RHS of ① $= -36\, x(t)$

$\neq$

$\Rightarrow x''(t) = -36\, x(t)$

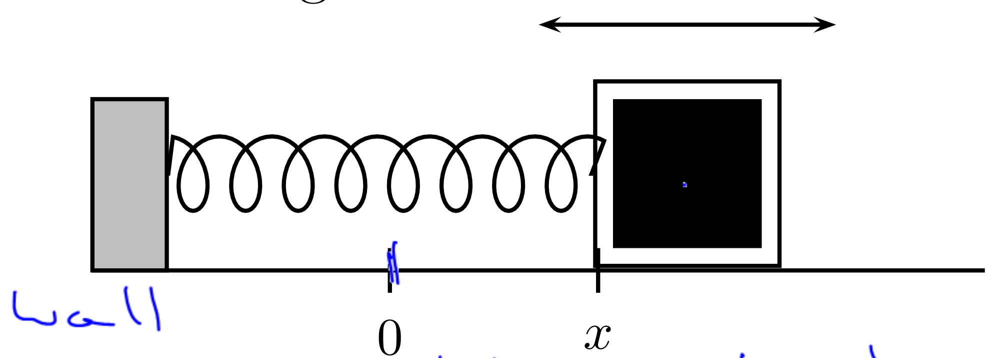
is not satisfied

by $x(t) = -6t^3$

Modeling with Differential Equations

Harmonic Motion

We will begin by studying the possible solutions of a particular differential equation that will be very important in your Physics course next term. Thinking about these solutions will help us understand what we mean when we say a function is a solution of a differential equation. The differential equation we saw in the previous concept question arises when we study the motion of a block at the end of a spring, as in the next diagram:



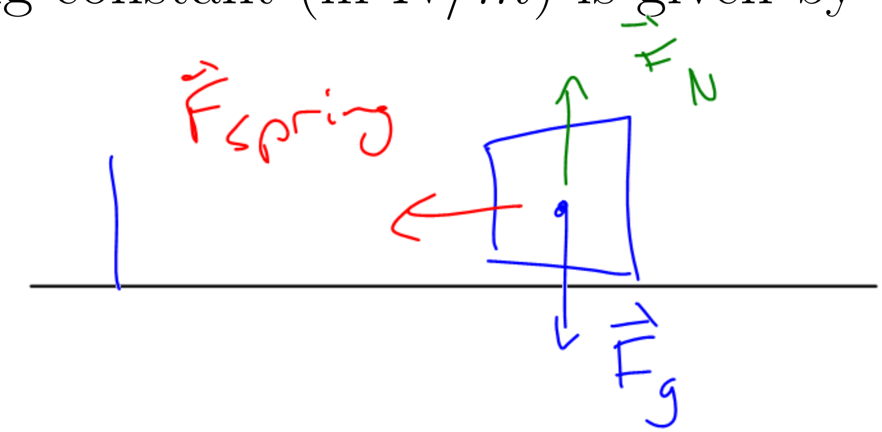
equilibrium location for spring

In this system, how would you describe x in words?

$x(t)$ = prediction of pos'n of the mass over time.

Problem. Draw a free-body diagram for the mass. Indicate the magnitude of the forces, assuming

- the mass of the block is m kg, and
- the spring constant (in N/m) is given by the constant k.



$$F_{spring} = -K \cdot x$$

↑ spring force in N/m.

↖ displacement from equil'm

x pos → spring stretched, → right spring pull left

x neg → " compressed, " push right

Let us work with our intuition about this system before beginning the mathematics.

even $\Rightarrow$ for small x, get large F

Problem. If the spring is very stiff, is k large or small?

(a) k will be **large**.

$F = -k \cdot x$

large

(b) k will be **small**.

Problem. If we exchange a soft spring for a stiff spring, do you expect the *period* of the oscillations to increase or decrease?

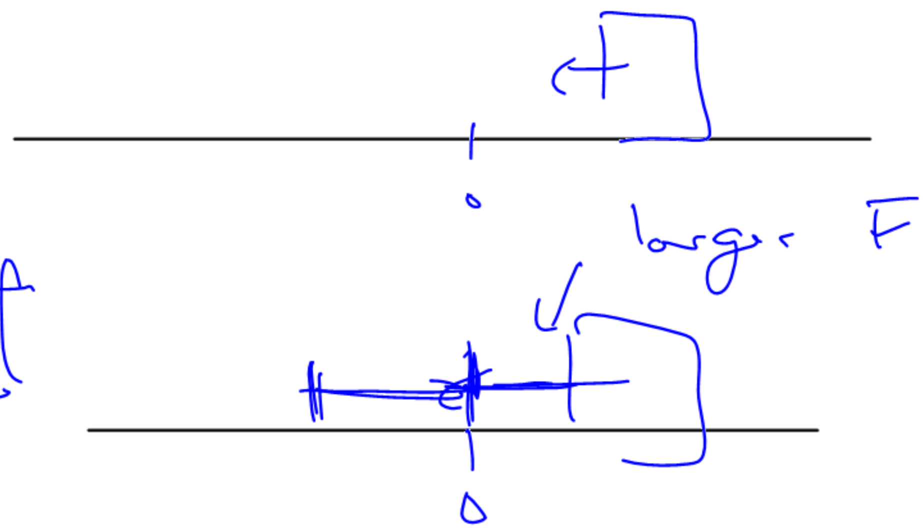
(a) The period will be **longer** if the spring is stiffer.

(b) The period will be **shorter** if the spring is stiffer.

Follow-up: why?

smaller F

soft spring



stiff spring

Problem. If we replace a **small mass** with a **heavier mass**, do you expect the *period* of the oscillations to increase or decrease?

(a) The period will will be **longer** if the mass is heavier.

(b) The period will will be **shorter** if the mass is heavier.

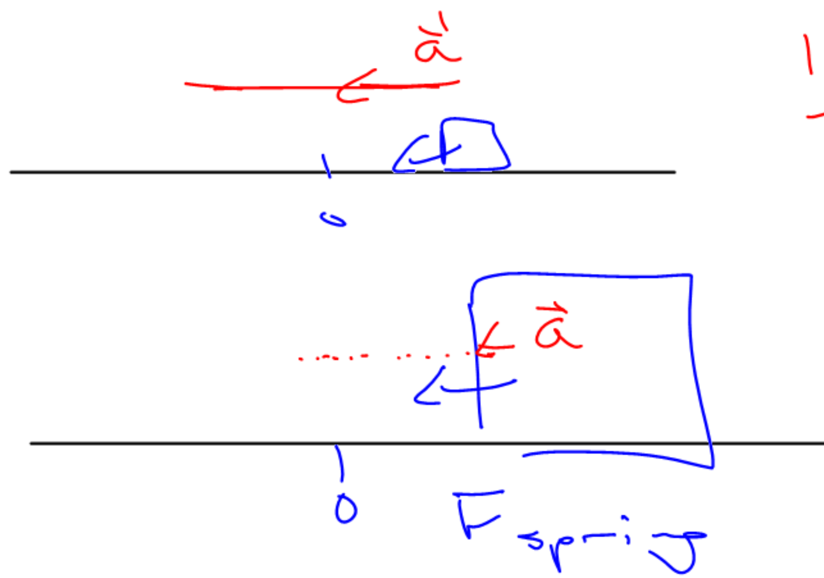
Follow-up: why?

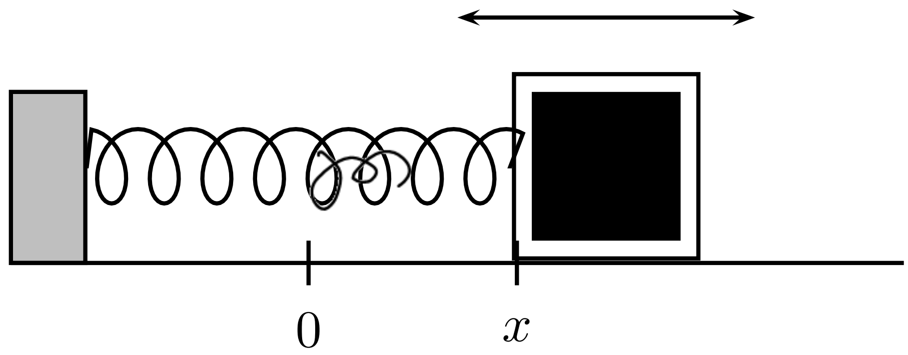
F same → accel $\frac{F}{m}$

larger

small mass

large mass





If we know k and m, and assume that friction is negligible, should we be able to determine the exact period of the oscillations?

↳ in seconds

Yes!

Does anyone know the formula for the period of oscillations of a spring system?

$$\text{period} = 2\pi\sqrt{\frac{m}{K}}$$

?????

The spring system is an excellent introduction to differential equations because

- we all know how it *should* work physically,
- the mathematics and physics are simple, **but**
- there's no obvious way to predict critical features (e.g. the period) from the given information.

We clearly need some new tools!

Analysis of Mass/Spring System

2nd Law

Problem. Use Newton's second law, $F = ma$, to construct an equation involving the position $x(t)$.

$-kx = ma$

$F_s = -kx$

$-kx = mx''$

a = 2nd deriv of posi'n

a differ'l eq'n → unknown function x, involves x'' deriv.

$$x'' = -\frac{k}{m}x$$

What are we solving for in this equation? I.e., what is the unknown?

$x(t)$ that satisfies that equation

$x(t)$ that predicts the motion of the mass subject to Newton's laws

Problem. To simplify matters temporarily, let us assume that both $k = 1$ N/m and $m = 1$ kg. Rewrite the previous differential equation.

$x'' = -x$ or $x''(t) = -x(t)$

This differential equation invites us to find a function $x(t)$ whose second derivative is its own negative. Try to think of such a function, or more than one!

$x(t) = \sin(t)$

or

$x(t) = \cos(t)$

$x(t) = e^{-t}$

$x' = -e^{-t}$ ✗

$x'' = -(-e^{-t})$

$= e^{-t} = x$

We have found a set of solutions for the differential equation

$$\frac{d^2x}{dt^2} = -x.$$

$x'' = -36x$

$\cos(6t)$

Does this help us discover solutions for the differential equation

$$\boldsymbol{m}\frac{d^2x}{dt^2} = -\boldsymbol{k}x?$$

Problem. Find one or more solutions to this second differential equation.

(1) $x'' = -\frac{k}{m}x$

$$\boxed{x(t) = \cos\left(\sqrt{\tfrac{k}{m}}\,t\right)}$$

Check: Sub in our proposed $x(t)$ into (1), check if LHS = RHS

Need x'': $x' = -\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)\cdot\sqrt{\tfrac{k}{m}}$

and $x'' = -\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)\sqrt{\tfrac{k}{m}}\sqrt{\tfrac{k}{m}}$

(1) LHS $= -\frac{k}{m}\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)$

(1) RHS $= -\frac{k}{m}x(t)$

$= -\frac{k}{m}\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)$

are equal!!

$$m\frac{d^2x}{dt^2} = -kx$$

or $x(t) = \sin\left(\sqrt{\frac{k}{m}}\,t\right)$ ✓

or $x(t) = -30\cos\left(\sqrt{\frac{k}{m}}\,t\right)$ ✓

Problem. Find **the most general family** of solutions to this differential equation.

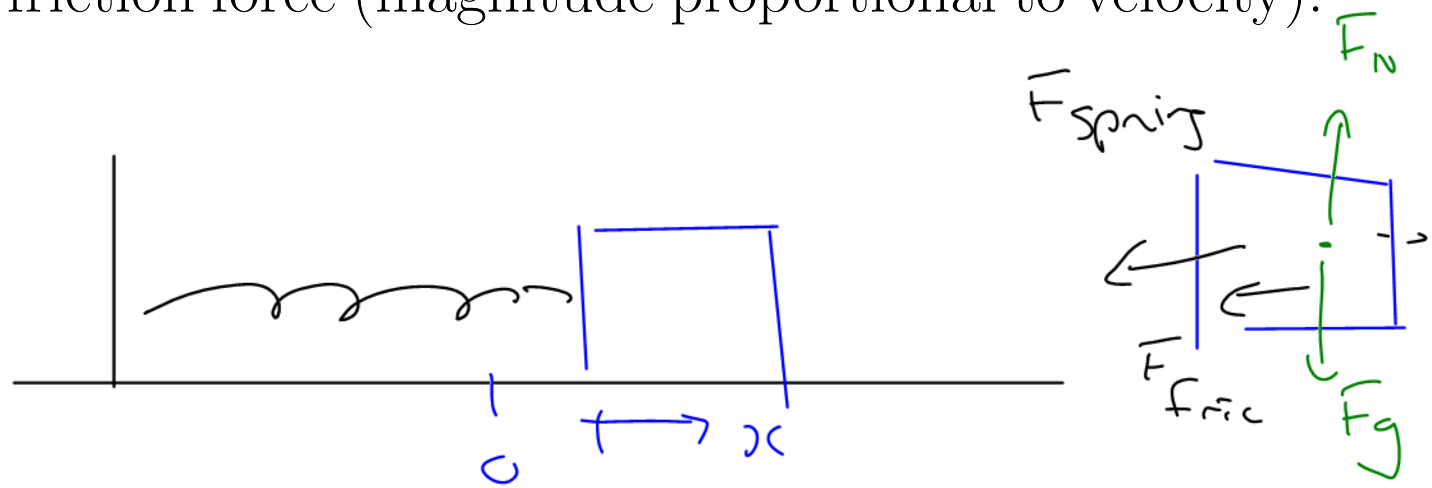
$$x(t) = A \cos\left(\sqrt{\frac{k}{m}}\, t\right) + B \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

some const.

Linear comb'n of $\left\{ \cos\left(\sqrt{\frac{k}{m}}\, t\right), \sin\left(\sqrt{\frac{k}{m}}\, t\right) \right\}$

Spring System - Adding Friction

Problem. Re-draw the free body diagram for the mass in the spring system, and add a friction force (magnitude proportional to velocity).



Starting with $\sum F = ma$, write an equation that the position function $x(t)$ must satisfy.

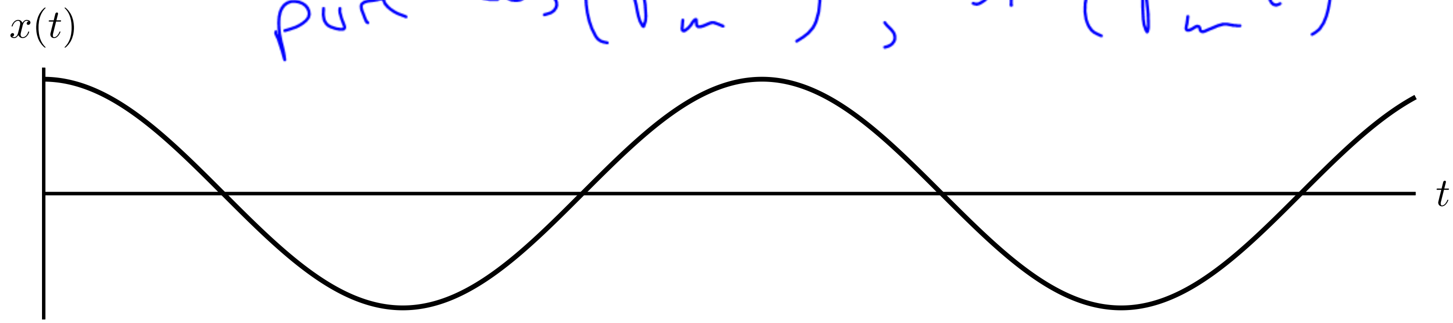
$$F_{spring} + F_{fric} = ma$$

$$-Kx \underbrace{-cx'}_{\text{prop'l to velocity}} = mx''$$

$$mx'' + cx' + kx = 0$$

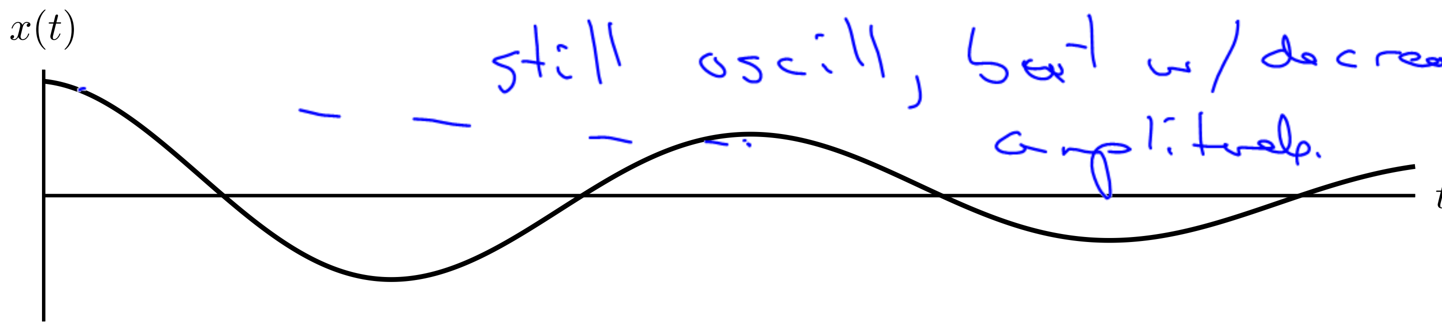
As we increase the friction coefficient c, the motion of the spring will be altered. Here are three plots of the position over time for a mass with increasing friction.

pure $\cos\left(\sqrt{\frac{k}{m}}\,t\right)$, $\sin\left(\sqrt{\frac{k}{m}}\,t\right)$

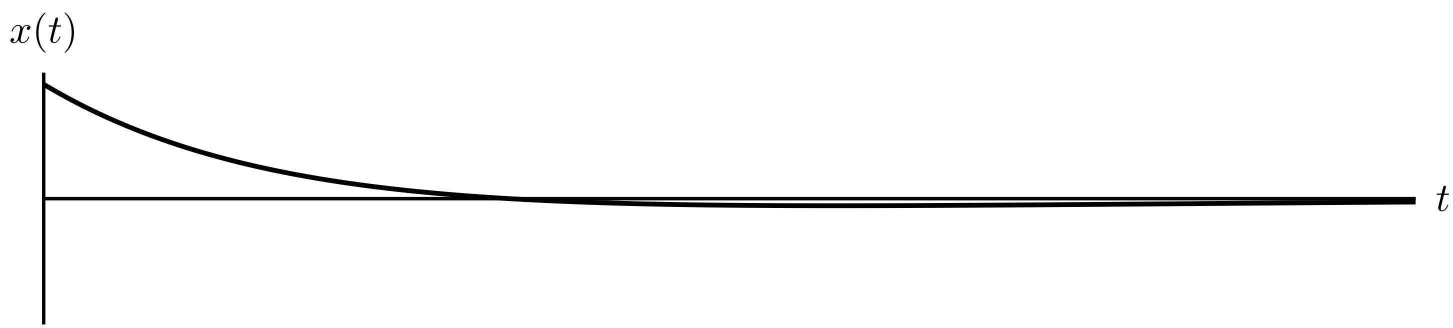


No friction

still oscill, but w/ decreasing amplitude.



Light Friction



High Friction

$$mx'' + cx' + kx = 0$$

The new fun fact is that all of these graphs can be made of the same building blocks, a simple exponential, if we add a secret sauce: the "imaginary" number, $\sqrt{-1}$ or i.

Problem. To connect e^{it} or $e^{\sqrt{-1}\ t}$, let us return to the simpler no-friction system, $\dfrac{d^2x}{dt^2} = -x(t)$. (1)

$c = 0$ no friction

$k = 1$, $m = 1$ $\quad i =$ const

$ma = -kx$

Show that $x(t) = \sin(t)$ satisfies this equation.

Need x'':

$x' = \cos(t)$

and

$x'' = -\sin(t)$

(1) $LHS = x'' = -\sin(t)$

(1) $RHS = -x = -\sin(t)$

are equal!

Show that $x(t) = e^{it}$ satisfies this equation, given that $i^2 = -1$. $\longleftrightarrow$ $i = \sqrt{-1}$

Need x''

$\rightarrow x' = i \cdot e^{it}$

so $x'' = i\left[i e^{it}\right] = i^2 e^{it} = -e^{it}$

(1) $LHS = x'' = -e^{it}$

(1) $RHS = -x = -e^{it}$

are equal!

This points to e^{it} somehow being related to $\sin(t)$: this is a major discovery! We need to investigate this value $i = \sqrt{-1}$ in more detail.

Introduction to Complex Numbers

A complex number is a number with two components, called the **real** and the **imaginary** components. They can be written in the form:

$$\text{or } z = x = a + bi$$

$z = 2 + 3i$

where $i = \sqrt{-1}$ or $i^2 = -1$.

$z = 2i$

a, b are both real

$z = -7\pi + 6i$

Problem. How are the *real numbers* you know already related to the set of all *complex numbers*?

real numbers $\subset$ complex numbers

subset of

$z = 2 + 0i$

$x = 2$

?? ← $z = 2 + 3i$

Addition of Complex Numbers

$i^2 = -1$

Adding complex numbers works in a way analogous to vectors.

Problem.

$\langle 3, 2\rangle + \langle -1, 5\rangle = \langle 2, 7\rangle$

(a) Compute $(3 + 2i) + (-1 + 5i)$. $= 3 - 1 + 2i + 5i$

$= 2 + 7i$

(b) Compute $(1 + 2i) - (2 + i)$.

$(1 + 2i) - 2 - i = -1 + i$

(c) Compute $3 + 7$. $= 10 + 0i = 10$

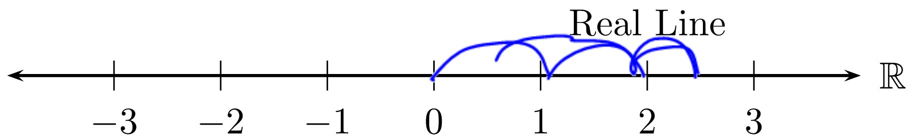
$+0i \quad +0i$

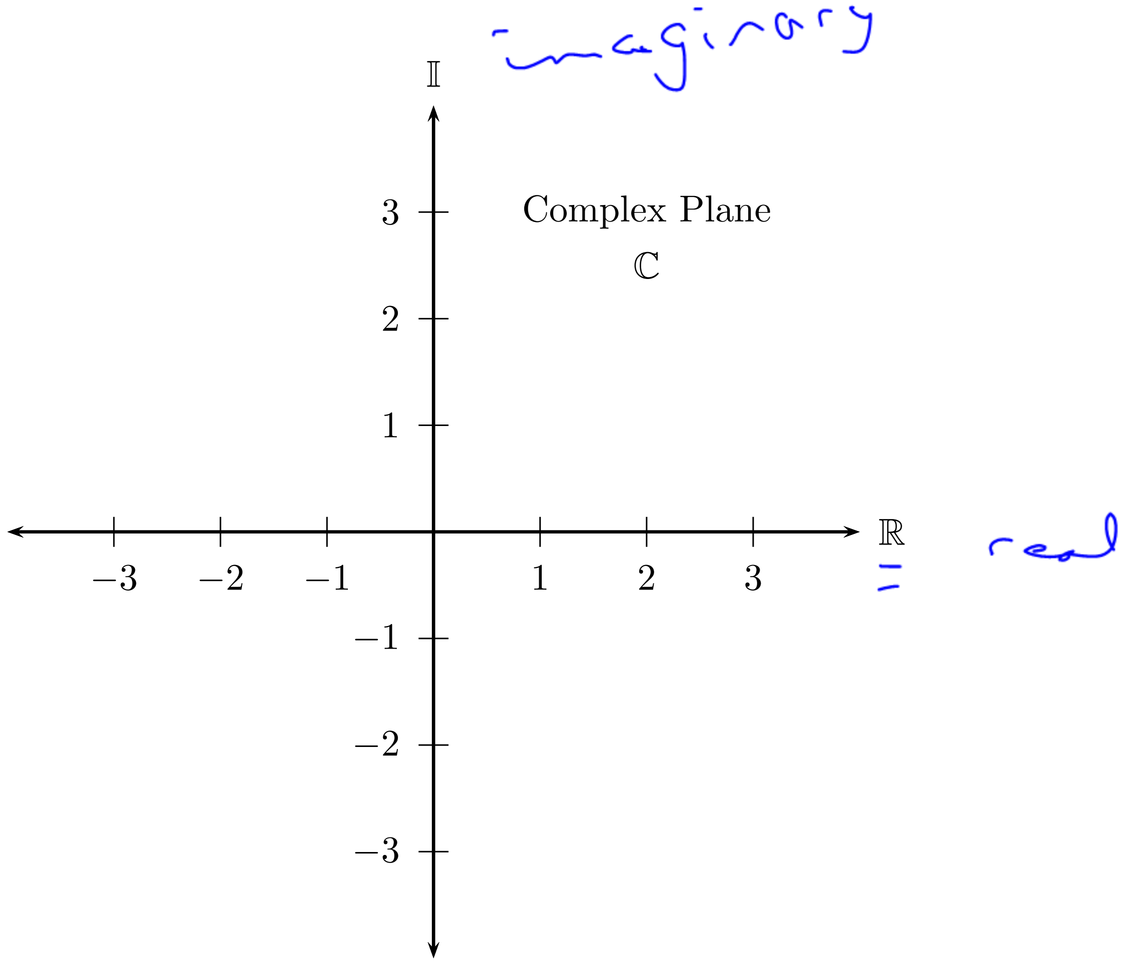
(d) Compute $6i + (-2 - 4i)$. $= -2 + 2i$

$(0 +$

Graphical Addition of Complex Numbers

To visualize complex numbers, we can extend our traditional **real line** into the two-dimensional **complex plane**.



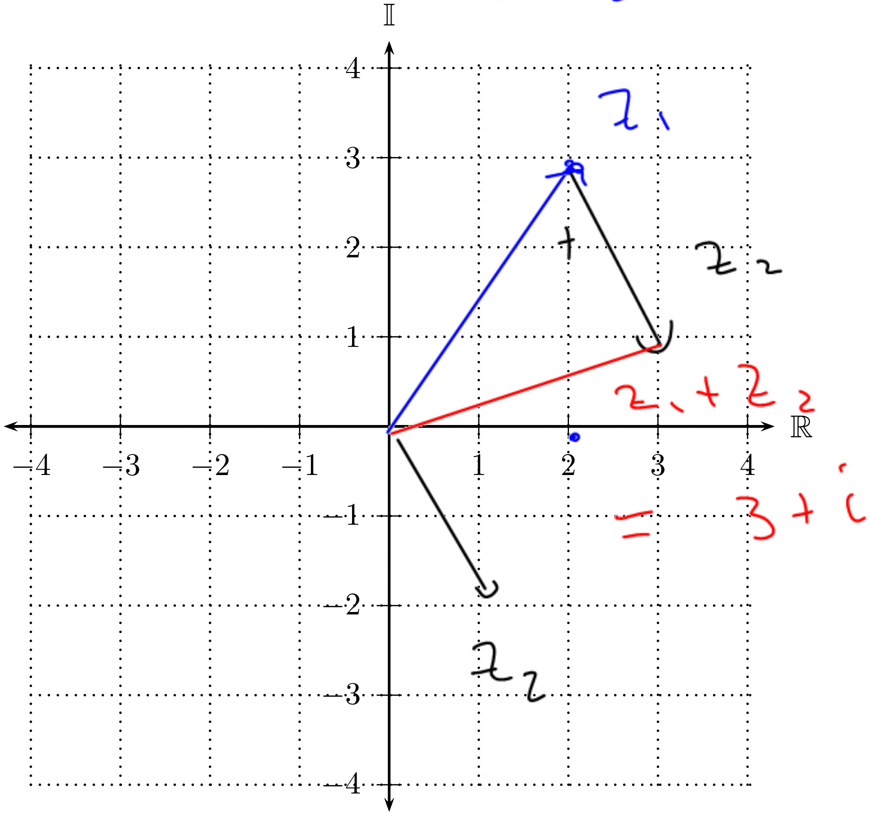


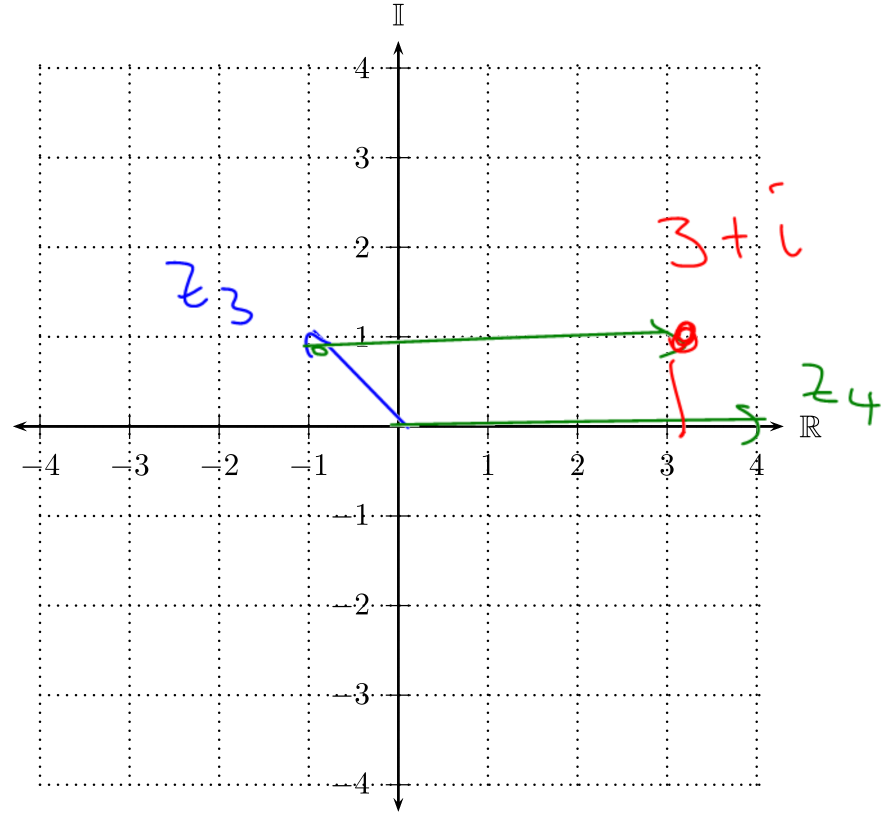
Problem. Draw the following complex numbers on the plane, then compute and draw their sum.

$z_3 + z_4 = 3 + i$

$z_1 = \underline{2 + 3i}$, $z_2 = 1 - 2i$ $\qquad$ $z_3 = -1 + i$, $z_4 = 4 + 0i$

$z_1 + z_2 = \underline{3 + i}$





Multiplication of Complex Numbers

× scalar × dot product × cross prod

Addition of complex numbers can make it feel like complex numbers are just another way to represent vectors. However, **multiplication** of complex numbers will show that we are in new territory.

$i^2 = -1$ or $i = \sqrt{-1}$

(a) Compute $(3 + 2i) \cdot (-1 + 5i)$.

$= (3)(-1) + (2i)(-1) + (3)(5i) + (2i)(5i)$

$= -3 - 2i + 15i + 10(\underbrace{i^2}_{-1})$

$= -13 + 13i$

sign of imag

complex conjugates

(b) Compute $(1 + 2i) \cdot (1 - 2i)$.

$= 1 + \cancel{2i} - \cancel{2i} + (-4(\underbrace{i^2}_{=-1}))$

$= 1 + 4$

$= 5 + 0i$

(c) Compute $(3+0i)\cdot(7+0i)$. all real

$= 21 + \cancel{0i + 0i + 0i^2}$

$= 21$

(d) Compute $6i \cdot (2+4i)$.

$= 12i + 24\cancel{i^2}$ -1

$= 12i - 24$

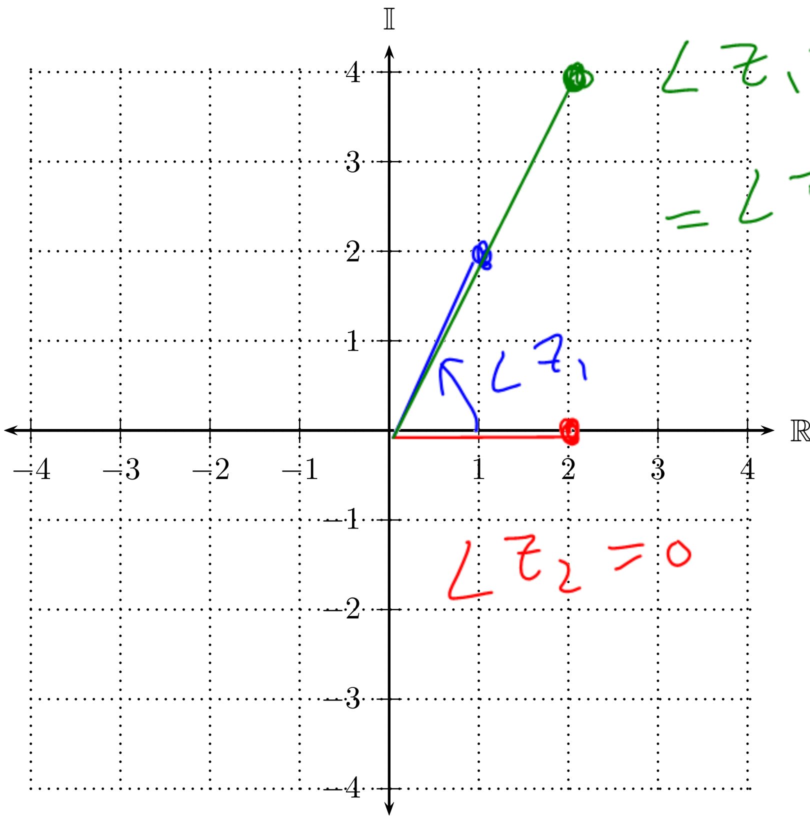
$= -24 + 12i$

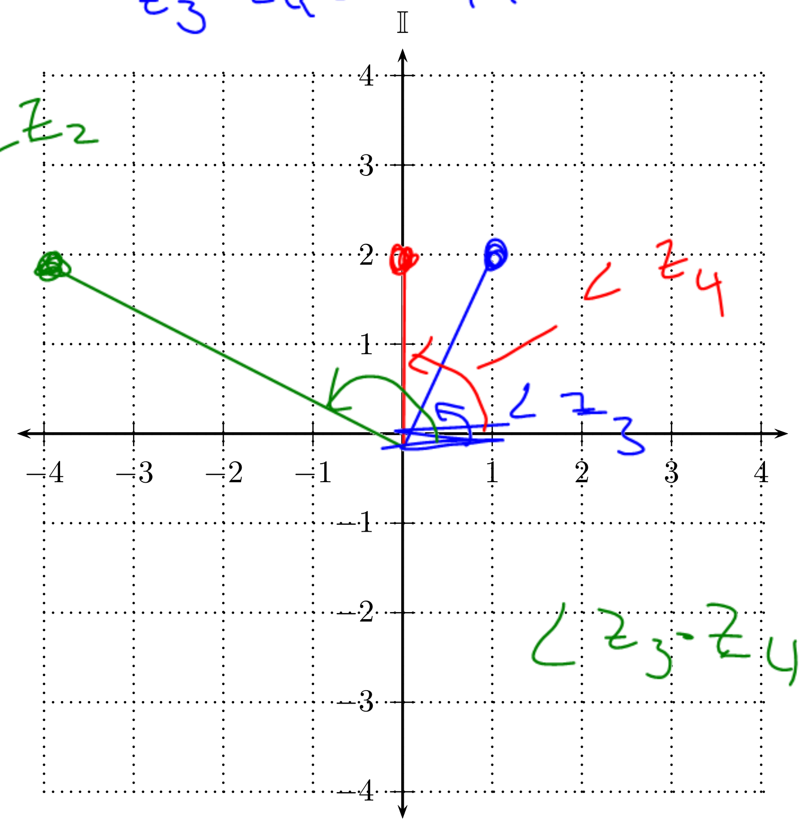
At first glance, there may appear to be little structure to the multiplication of complex numbers. However, a graphical perspective can help us see patterns in what is happening.

Problem. Draw the following complex numbers on the plane, then compute and draw their **product**.

$z_1 = (1 + 2i),\ z_2 = 2$ $\qquad$ $z_3 = 1 + 2i,\ z_4 = \underline{\underline{2i}}$ $\qquad$ $\angle z_4$

$z_1 \cdot z_2 = 2+4i$ $\qquad$ $z_3 \cdot z_4 = -4+2i$





$= \angle z_3 + \angle z_4$

$|z_3| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$ $\quad |z_4| = \sqrt{8}$

$z_1 = 1 + i,\ z_2 = 1 - 1i$ $\qquad z_3 = -2 + 2i,\ z_4 = -2 + 2i$

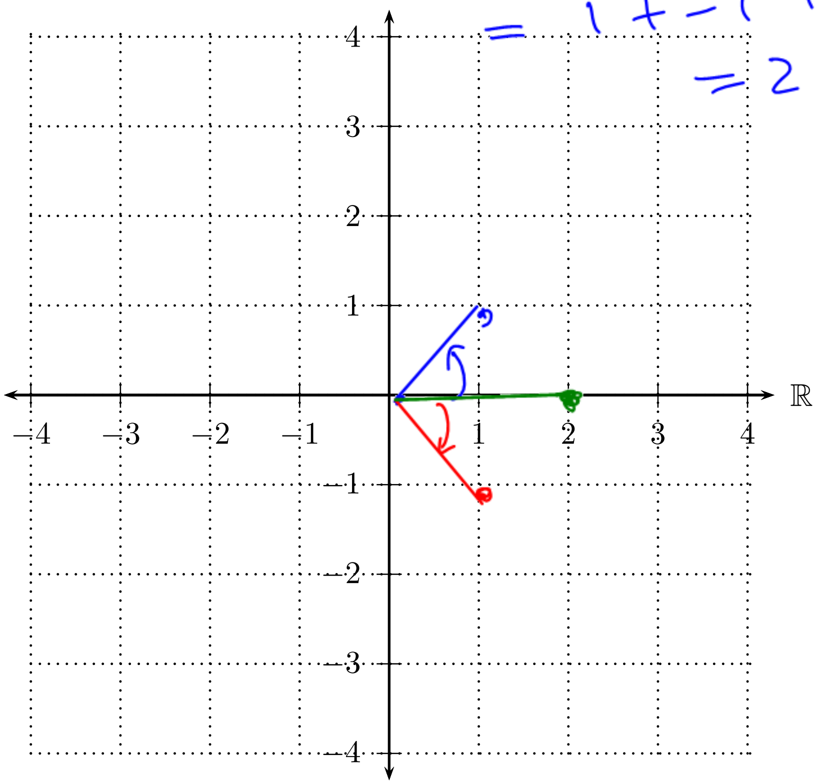
$z_1 \cdot z_2 = 1 + \cancel{i - i} - i^2$

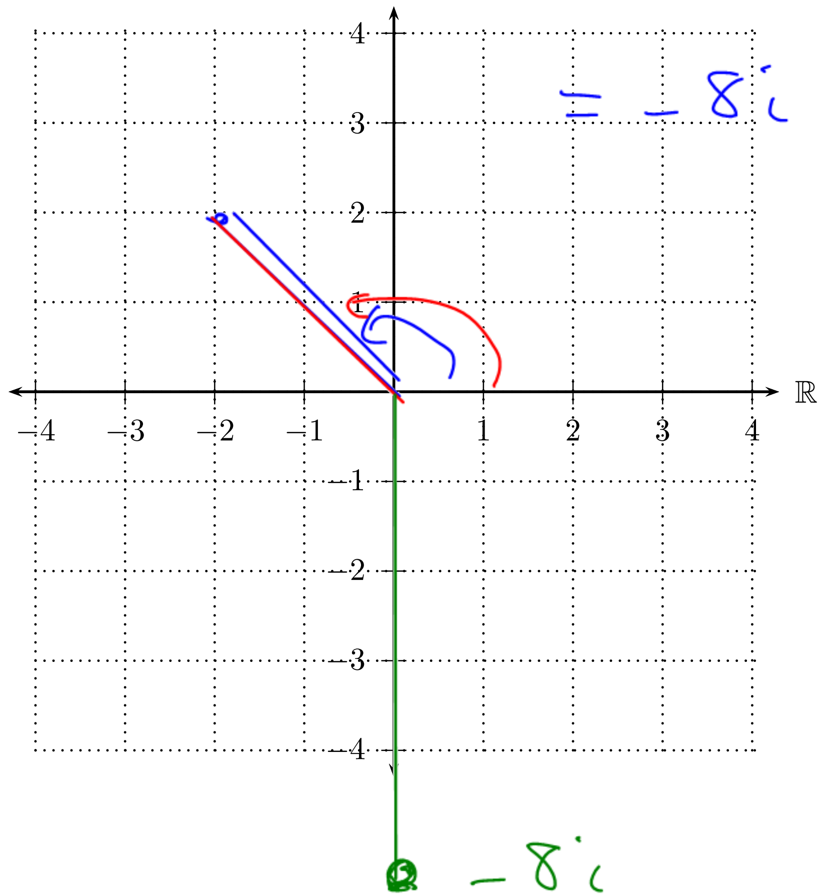
$= 1 + -(-1)$

$= 2 + 0i$

$z_3 \cdot z_4 = \cancel{4} - 4i - 4i + \cancel{4i^2}$ $\quad (-1)$

$= -8i$





$-8i$

Problem. What patterns do you notice in the multiplication of complex numbers?

$z_1 \cdot z_2 \rightarrow$ magnitude of $z_1 \cdot z_2 = |z_1||z_2|$

and $\angle$ of $z_1 \cdot z_2$

$= (\angle z_1) + (\angle z_2)$

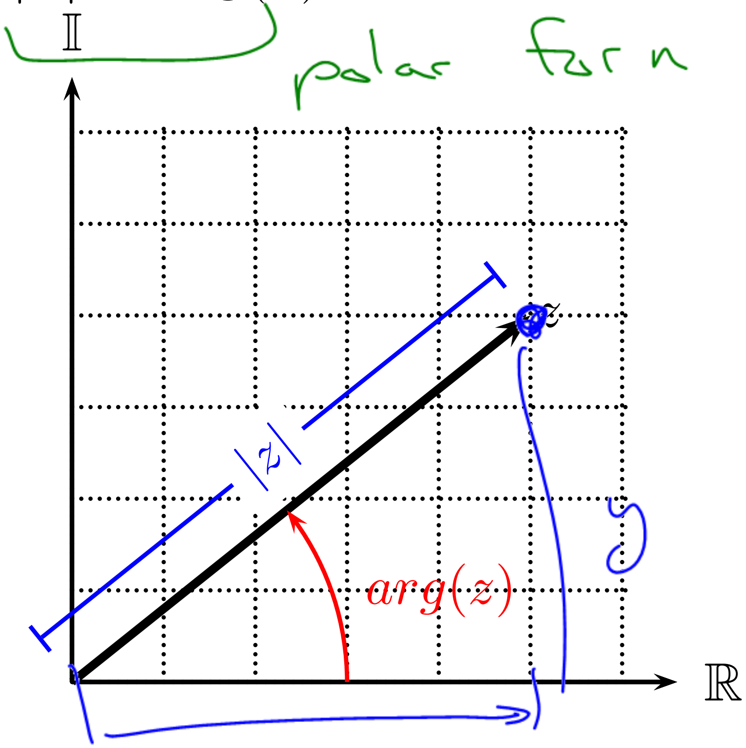
Polar form of Complex Numbers

Computing products of complex numbers can be made easier if we write our complex numbers z in **polar form**:

- A **length** or **magnitude** of the number, $|z|$, and
- An **angle** in radians, measured counter-clockwise from the positive $\mathbb{R}$ axis, $\arg(z)$.

We can then write complex numbers as either

$z = \underbrace{a + bi}_{\text{cartesian form}}$ **or** $z = \underbrace{|z| \angle \arg(z)}_{\text{polar form}}$.

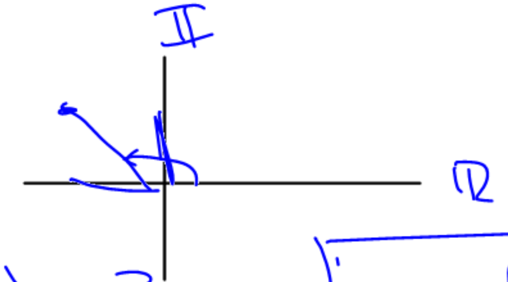


Problem. Find the polar form of the following complex numbers. Write the result using the notation $(|z| \angle \phi)$.

(a) $z_1 = -2 + 2i$

$|z_1| = \sqrt{(-2)^2 + (2)^2}$

$= \sqrt{8}$



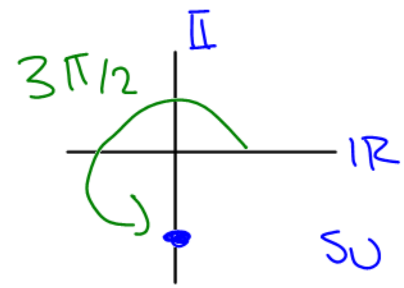
$\arg(z) = \frac{3\pi}{4}$

$\boxed{z_1 = \sqrt{8} \angle \frac{3\pi}{4}}$

(b) $z_2 = -i = 0 + (-1)i$

$|z_2| = \sqrt{0^2 + (-1)^2}$

$= \sqrt{1} = 1$

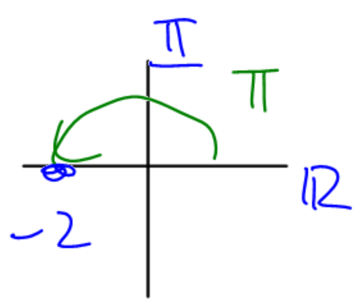


so $\boxed{z_2 = 1 \angle \frac{3\pi}{2}}$

(c) $z_3 = -2 + 0i$

$|z_3| = \sqrt{(-2)^2 + 0^2}$

$= \sqrt{4} = 2$



$\boxed{z_3 = 2 \angle \pi}$

Problem. Use the polar forms to compute the products of the following complex numbers. You can leave the results in polar form.

(a) $z_1 = (-2+2i) = (\sqrt{8}\angle 3\pi/4)$,

$z_2 = -i = (1\angle -\pi/2)$

- multiply magnitudes
- add angles/arguments

$$z_1 \cdot z_2 = \sqrt{8} \angle \left(\frac{3\pi}{4} + \left(-\frac{\pi}{2}\right)\right)$$

$$= \sqrt{8} \angle \pi/4$$

(b) $z_1 = (-2+2i) = (\sqrt{8}\angle 3\pi/4)$,

$z_3 = -2 = (2\angle \pi)$

$$z_1 \cdot z_3 = 2\sqrt{8} \angle \left(\frac{3\pi}{4} + \pi\right)$$

$$= 2\sqrt{8} \angle \frac{7\pi}{4}$$

Foreshadowing

When we **multiply** complex numbers, the actual process involves the **addition** of the angular components of the numbers.

In high school, you saw a similar situation: a computation where multiplication is actually calculated by using an $\boxed{\text{addition}}$ operation.

Can you think of the situation where you used that approach?

$$e^{2} \cdot e^{3} = e^{(2+3)}$$

$$e^{\pi i} \cdot e^{-\pi/2\, i} = e^{\pi/2\, i}$$