Week 11: Differential Equations and Complex Numbers - Part II

Goals:

- The Complex Exponential Function
- Behaviours of a Spring/Mass System

The Complex Exponential

Previously, we found two interesting relations with exponentials:

- $\dfrac{d^2x}{dt^2} = -x$ has two forms of solution:
  - $x(t) = \cos(t)$, $x(t) = \sin(t)$ are solutions, and so to are
  - $x(t) = e^{it}$, and $x(t) = e^{-it}$.

$i^2 = -1$

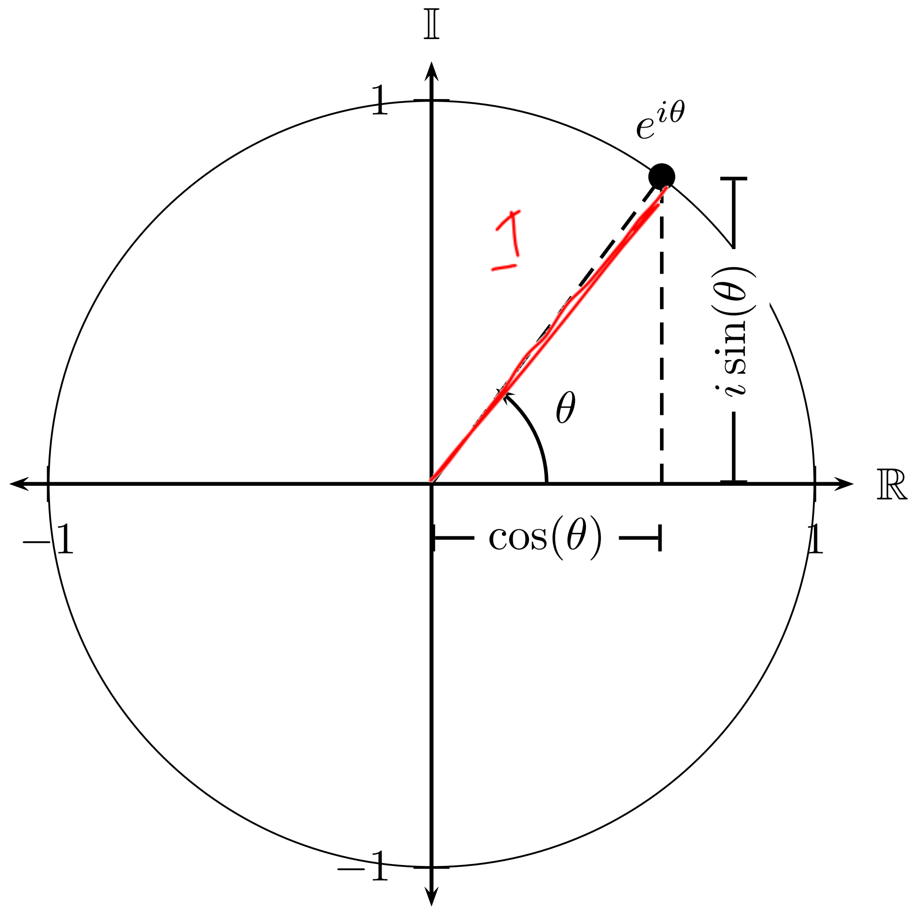
or $i = \sqrt{-1}$

- Complex number multiplication can be framed as **addition** of angles. This is also the way multiplication of exponentials works (multiplying is done by adding the exponents).

These relationships can be extended to arrive at the famous and fundamental expression called **Euler's Formula**. $(\cos\theta, \sin\theta)$

complex exponential →

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$



$\theta = \arg(z)$

Note that means our earlier polar form $r\angle\theta$ notation is equivalent to the exponential form $re^{i\theta}$.

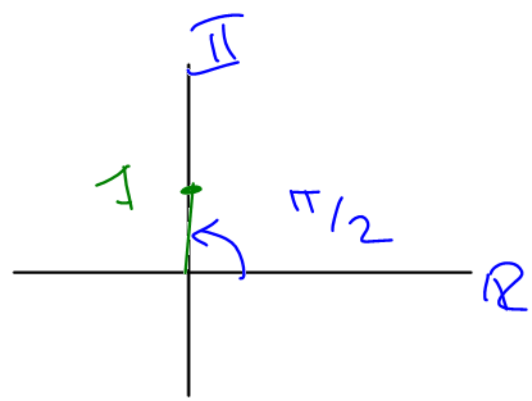
mag angle

$$r\angle\theta = \underbrace{r\ e^{i\theta}}_{\text{mag + angle of } \theta} \qquad\qquad e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

polar form

Problem. Write the following complex numbers in the form $r\,e^{i\theta}$.

(a) $z = i$



$|z| = 1$

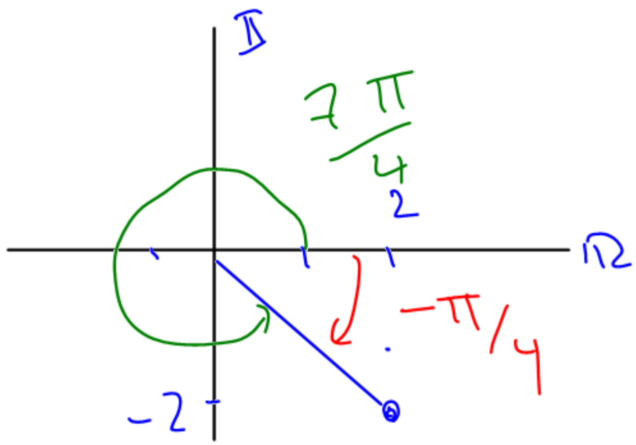
and $\angle$ is $\pi/2$

So

$$z = i = 1\cdot e^{i(\pi/2)}$$

check: $e^{i(\pi/2)} = \cos\underbrace{\cancel{(\pi/2)}}_{0} + i\,\overset{1}{\cancel{\sin(\pi/2)}} = i$

(b) $z = 2 - 2i$



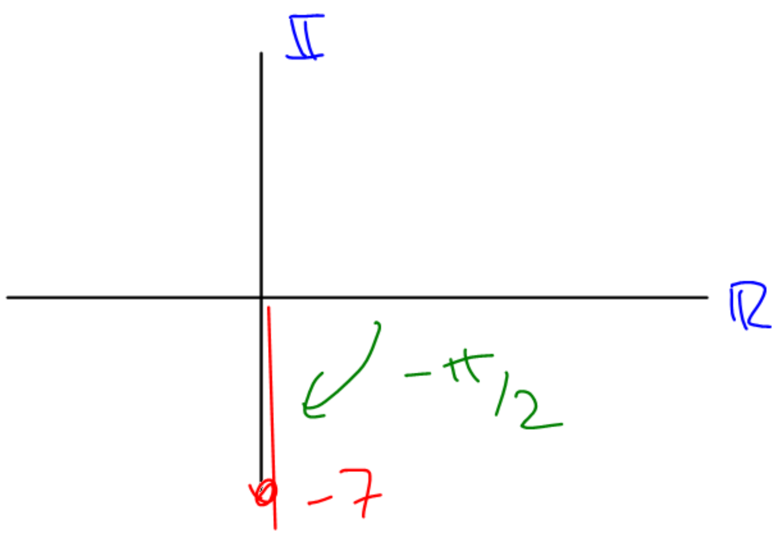
$|z| = \sqrt{(2)^2 + (-2)^2} = \sqrt{8}$

and $\arg(z) = \frac{-\pi}{4}$ or $\frac{7\pi}{4}$

$z = \sqrt{8}\, e^{-i\pi/4}$

(or $z = \sqrt{8}\, e^{i\frac{7\pi}{4}}$)

(c) $z = -7i$



$|z| = 7$

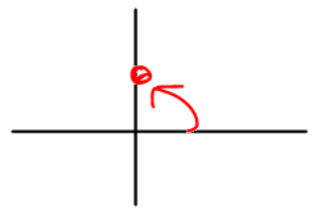
$\arg(z) = -\pi/2$

so $z = -7i = 7e^{-i\pi/2}$

$1 \angle \pi/2$

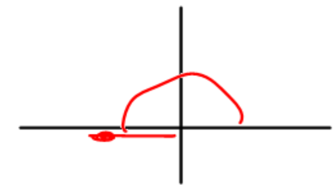
$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Problem. Consider the complex numbers $z_1 = i$ and $z_2 = -1$. $= 1 \angle \pi$



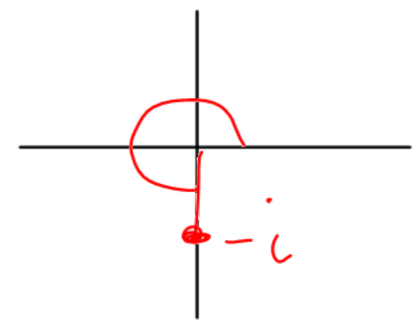
$z_1 = 1 \cdot e^{i\pi/2}$

$z_2 = e^{i\pi}$



(a) Compute $z_1 \cdot z_2$ using direct calculation.

$z_1 \cdot z_2 = (i)(-1) = \underline{-i}$



(b) Compute $z_1 \cdot z_2$ using polar form.

$(1 \angle \pi/2)(1 \angle \pi) = \underline{1 \angle \frac{3\pi}{2}}$

$+$

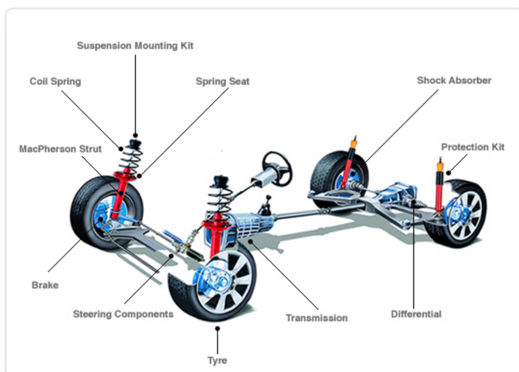
$\times$

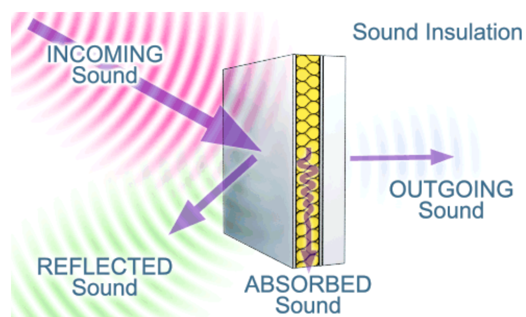
(c) Compute $z_1 \cdot z_2$ using complex exponentials.

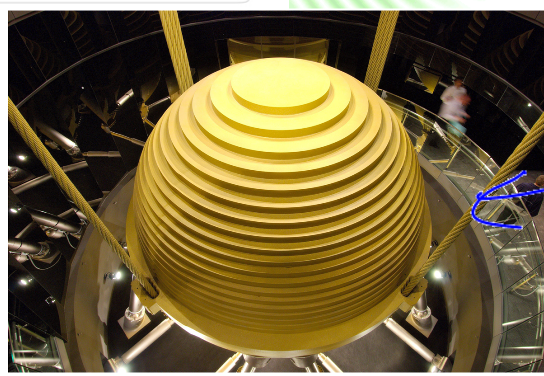
$e^{i\pi/2} \cdot e^{i\pi} = e^{i\frac{\pi}{2} + i\pi} = \underline{e^{i\,3\pi/2}}$

Complex Exponentials and the Spring/Mass System

With our complex exponential in our toolbox now, we return to the spring/mass system. While the mathematics behind this simple system will be very interesting in their own right, we note at the outset that the simple spring/mass model can be applied to a wide variety of not-so-obviously related real-world problems.







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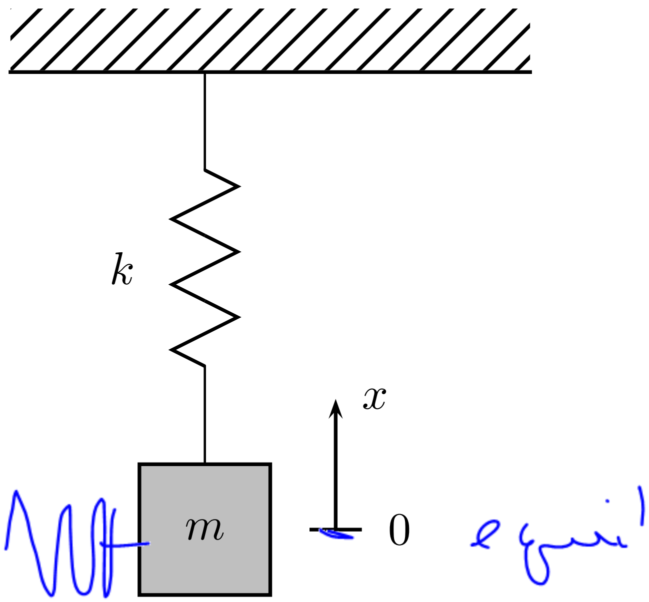


$$\underbrace{mx''}_{ma} = \underbrace{-kx}_{F_{\text{spring}}}$$

Problem. Assume that the motion of the mass is described by $x(t) = e^{\lambda t}$, where λ is an unknown constant.

Sub this proposed $x(t)$ form into the $F = ma$ equation above, and solve for λ.

Need x''

If $x = e^{\lambda t}$

then $x' = \lambda e^{\lambda t}$

and $x'' = \lambda^2 e^{\lambda t}$

Sub into DE.

$m x'' = -Kx$

$m\lambda^2 \cancel{e^{\lambda t}} = -K \cancel{e^{\lambda t}}$

$e^{\lambda t} \neq 0$

$\lambda^2 = \frac{-K}{m}$

$$mx'' = -kx \qquad\qquad x(t) = e^{\lambda t}$$

Continued.

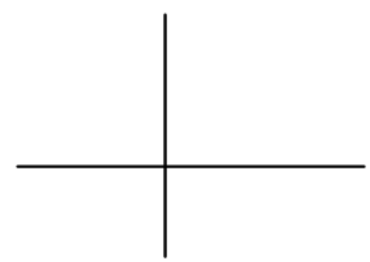
$$\lambda^2 = -\frac{k}{m}$$

or $\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{(-1)\left(\frac{k}{m}\right)}$

$$\lambda = \pm i\sqrt{\frac{k}{m}}$$

$e^{\lambda t}$ is a solution if

$$x_1(t) = e^{\left(i\sqrt{k/m}\right)t}$$



and $x_2(t) = e^{-i\sqrt{\frac{k}{m}}\,t}$

$$mx'' = -kx \qquad\qquad x(t) = e^{\lambda t} \qquad\qquad \lambda = \pm i\sqrt{\frac{k}{m}}$$

Use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to rewrite both of the $x(t)$ solutions.

$$x_1(t) = e^{i\sqrt{\frac{k}{m}}\cdot t} = \cos\left(\sqrt{\frac{k}{m}}\,t\right) + i\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$x_2(t) = e^{i\left(-\sqrt{\frac{k}{m}}\,t\right)} = \cos\left(-\sqrt{\frac{k}{m}}\,t\right) + i\sin\left(-\sqrt{\frac{k}{m}}\,t\right)$$

$$x_2(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right) - i\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$mx'' = -kx$$

$$x_1(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right) + i\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$x_2(t) = \cos\left(-\sqrt{\frac{k}{m}}\,t\right) + i\sin\left(-\sqrt{\frac{k}{m}}\,t\right)$$

Tidy the $x_2(t)$ solution using the fact that

- $\cos(t)$ is symmetric across $t = 0$ (even function), and
- $\sin(t)$ is symmetric through (0,0) (odd function).

$$x_2(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right) - i\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

Based on k and m, find the period of the oscillations for a spring/mass system.

$$\text{period} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}$$

- larger m -> longer per.

recall

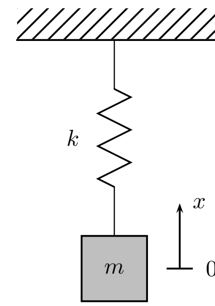
- larger k -> smaller pe

Real Solutions from Complex Forms

$m x'' = -kx$

$$x_1(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right) + i\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$x_2(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right) - i\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$



We have a real problem here though. $x(t)$ is supposed to be the position of our mass, and so should only have real values: no i's allowed in our final answer.

Fortunately, a property you will cover in Linear Algebra will come to our rescue.

If both $x_1(t)$ and $x_2(t)$ are solutions to the equation

$$x'' + cx' + kx = 0,$$

then any linear combination

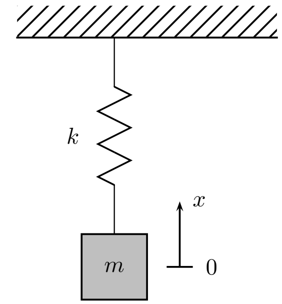
found 2 solutions

$$x(t) = z_1\,x_1(t) + z_2\,x_2(t)$$

is also a solution.

const

$$x_1(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right) + i\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$
$$x_2(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right) - i\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$



$$x(t) = z_1\,x_1(t) + z_2\,x_2(t)$$

Problem. Find at least two choices of z_1 and z_2 that will make $x(t)$ a purely real solution.

can be complex

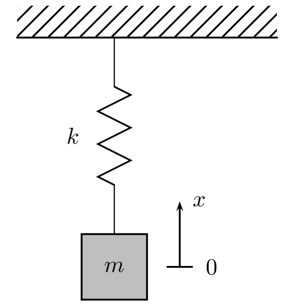
pick $z_1 = 1$, $z_2 = 1$

$$\Rightarrow x(t) = (1)\Big[\underbrace{\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + i\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)}_{x_1}\Big] + (1)\Big[\underbrace{\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) - i\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)}_{x_2}\Big]$$

$$\boxed{x(t) = 2\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)}$$

pure real!

$$x_1(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right) + i\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$
$$x_2(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right) - i\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$



$$x(t) = z_1\, x_1(t) + z_2\, x_2(t)$$

Problem. Find at least two choices of z_1 and z_2 that will make $x(t)$ a purely real solution.

Now pick $z_1 = i$, $z_2 = -i$

Then $x(t) = (i)\left[\cos\left(\sqrt{\frac{k}{m}}\,t\right) + i\sin\left(\sqrt{\frac{k}{m}}\,t\right)\right]$

$+ (-i)\left[\cos\left(\sqrt{\frac{k}{m}}\,t\right) - i\sin\left(\sqrt{\frac{k}{m}}\,t\right)\right]$ $i^2 = -1$

$x(t) = \cancel{0\cos(\quad)} + i^2\sin(\quad) + i^2\sin(\quad)$

$x(t) = -2\sin\left(\sqrt{\frac{k}{m}}\,t\right)$ all real.

Conclusion: if we find two solutions

$$x_1(t) = e^{ibt} \text{ and } x_2(t) = e^{-ibt} \text{ (complex-valued)},$$

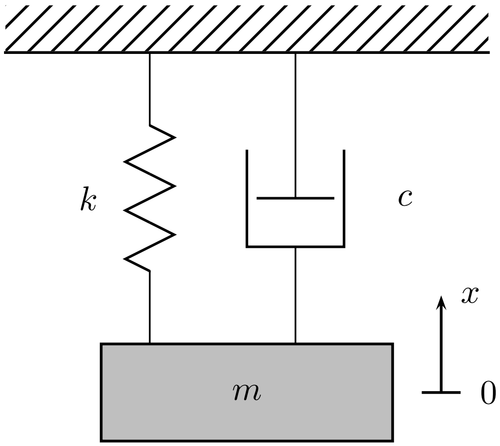
then we can conclude that

linear combinations

$$x_1(t) = \cos(bt) \text{ and } x_2 = \sin(bt) \text{ (purely real)}$$

are also solutions.

Damped Spring/Mass System - Example 1



$$mx'' = -kx - cx'$$

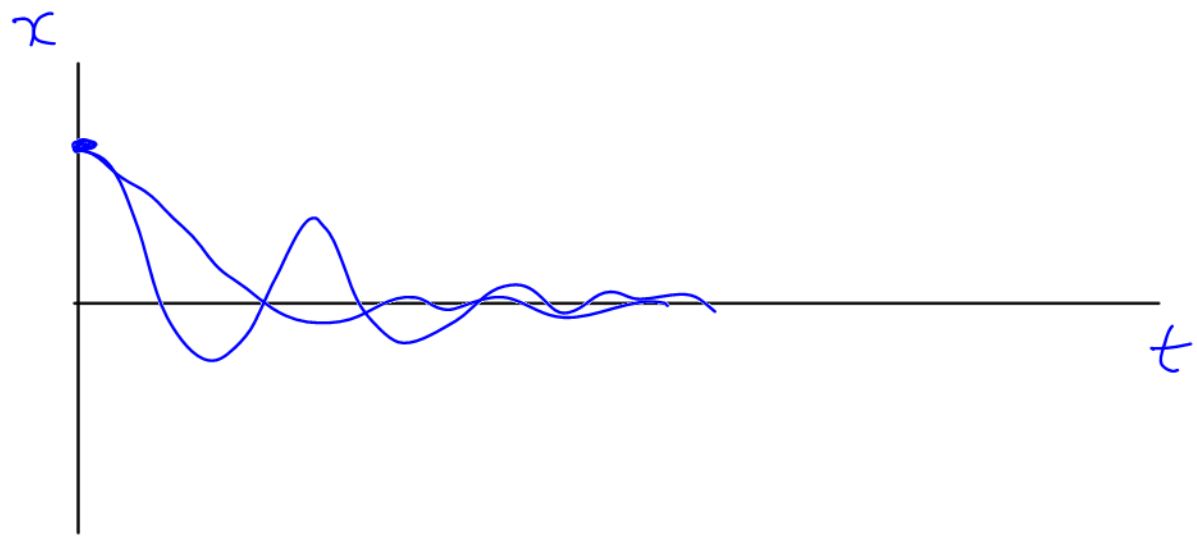
$ma = \Sigma F$ or

$$mx'' + cx' + kx = 0$$

Problem. Find two solutions, $x_1(t)$ and $x_2(t)$ to this system if

- the mass is $m = 1$ kg, and
- the spring constant is $k = 25$ N/m, and
- the damping coefficient is $c = 6$ N/(m/s).

predict motion of mass



unknown (yet)

$$\underbrace{1x'' + 6x' + 25x = 0}_{\text{DE}}$$

Assume $x = e^{\lambda t}$

To sub into DE, need

$x' = \lambda e^{\lambda t}$

and $x'' = \lambda^2 e^{\lambda t}$

Sub into DE:

$$(\lambda^2 e^{\lambda t}) + 6(\lambda e^{\lambda t}) + 25(e^{\lambda t}) = 0$$

Factor

$$\frac{\cancel{e^{\lambda t}}}{\cancel{e^{\lambda t}}}\left[\lambda^2 + 6\lambda + 25\right] = \frac{0}{e^{\lambda t}}$$

$$\lambda^2 + 6\lambda + 25 = 0$$

Quad form $\lambda = \dfrac{-6 \pm \sqrt{6^2 - 4(25)}}{2} = -3 \pm \dfrac{\sqrt{-64}}{2}$

$$1x'' + 6x' + 25x = 0$$

$\lambda = -3 \pm \frac{1}{2}\sqrt{64}\sqrt{-1}$

$= -3 \pm \frac{1}{2} 8 \quad i$

$\boxed{\lambda = -3 \pm 4i}$

Assumed $x = e^{\lambda t}$

true but complex valued

so $x_1(t) = e^{(-3+4i)t}$ and $x_2(t) = e^{(-3-4i)t}$

$x_1(t) = \underbrace{e^{-3t}}_{\text{real}} \cdot e^{4t \cdot i}$ and $x_2(t) = \underbrace{e^{-3t}}_{\text{real}} \cdot e^{-4ti}$

combine w/ Euler form, appropriate multipliers

$x_A(t) = e^{-3t} \cdot \cos(4t)$ and $x_B = e^{-3t}\sin(4t)$

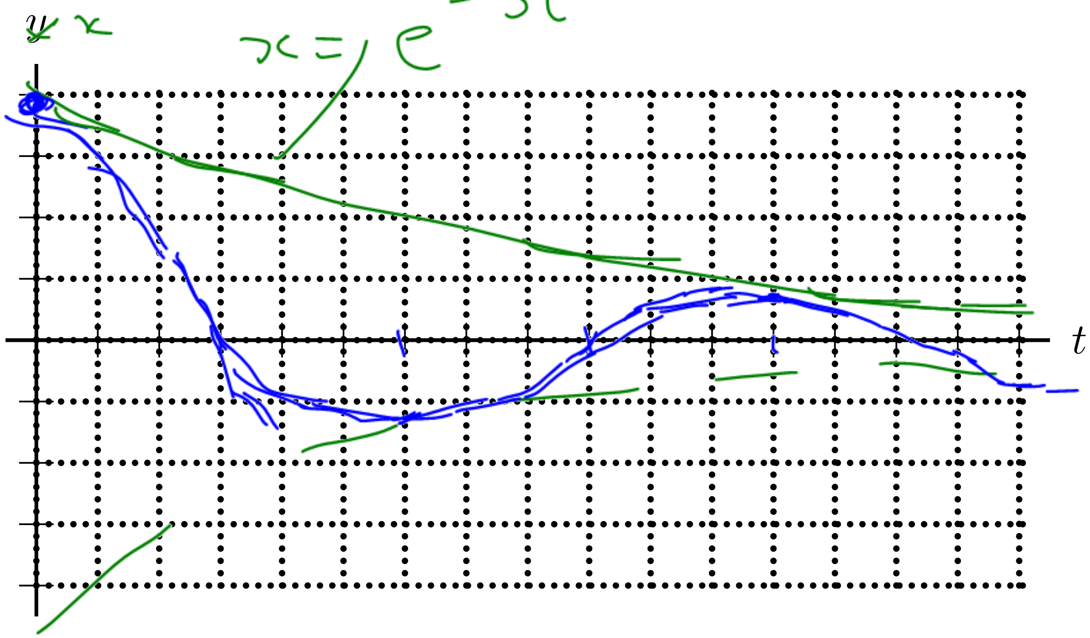
- both all real
- can describe motion of spring/mass

Problem. Sketch your two solutions on the axes below.

$x = e^{-3t}$

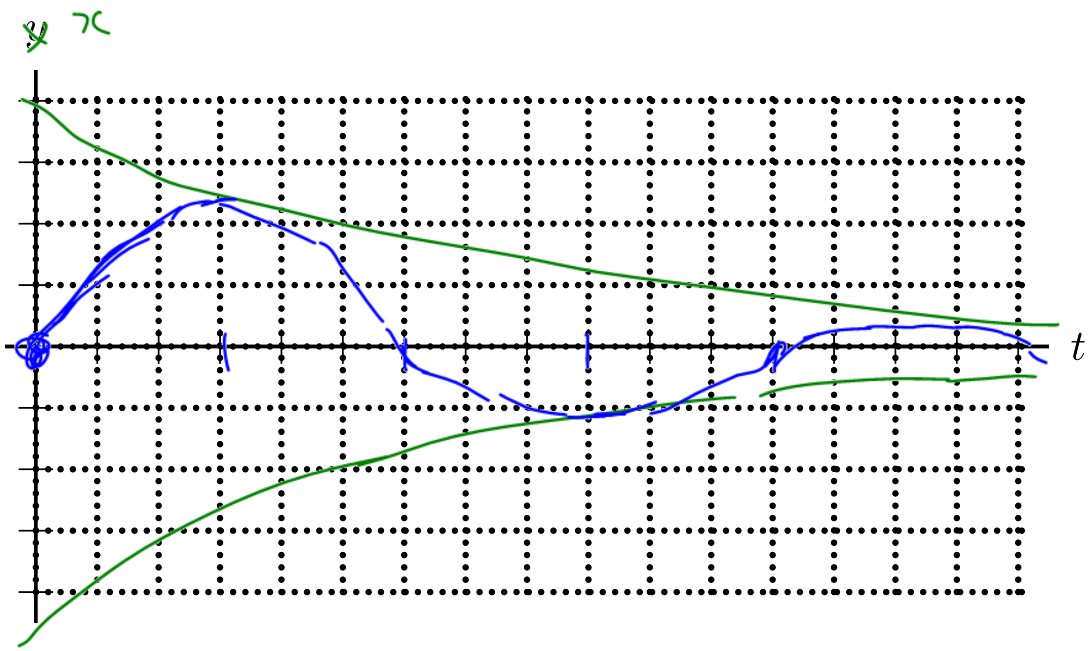
$x_A(t)$

$= \underbrace{e^{-3t}}_{\text{amplitude}} \cos(4t)$



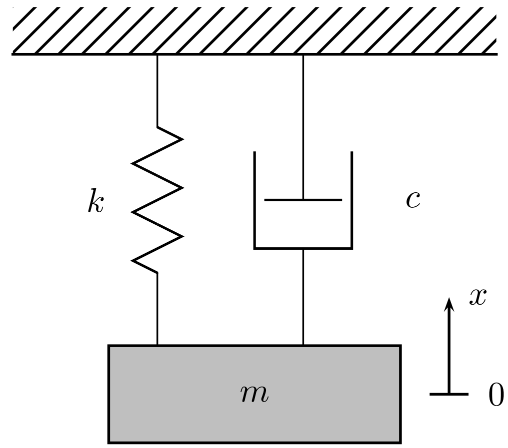
$x_B(t)$

$= e^{-3t} \sin(4t)$



Problem. You are given the additional details about the initial conditions of the system:

- $x(0) = 4$, and
- $x'(0) = 0$.



Describe in words what these values mean about the spring system.

$x(0)=4$ → lifting the mass to $x=4$

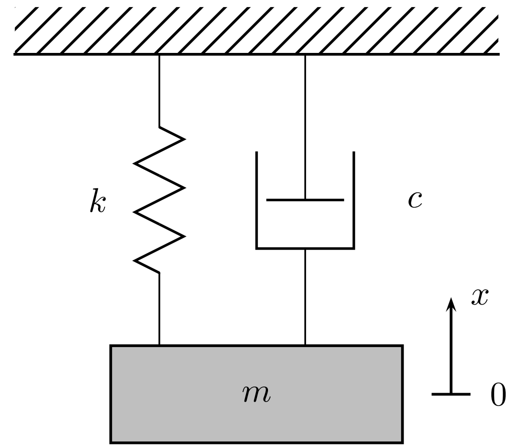
$x'(0)=0$ → mass starts w/o velocity

Lift mass to $x=4$,

hold steady,

Let it go.

Problem.



- $x(0) = 4$, and
- $x'(0) = 0$.

Find the full solution for the position of the mass over time that satisfies these initial conditions.

$$\text{General solution} = \underbrace{A}_{\text{const}}\ \underbrace{e^{-3t}\cos(4t)}_{x_A} + \underbrace{B}_{\text{const}}\ \underbrace{e^{-3t}\sin(4t)}_{x_B}$$

Now, use $x(0) = 4$, $x'(0) = 0$ to find A, B.

$$t = 0,\ x = 4 \rightarrow 4 = A e^0 \underbrace{\cos(0)}_{1} + \underbrace{\cancel{B e^0 \sin(0)}}_{0}$$

$$\boxed{A = 4}$$

$$x(t) = A e^{-3t}\cos(4t) + B e^{-3t}\sin(4t)$$

- $x(0) = 4$, and
- $x'(0) = 0$.

so

$$x'(t) = A\left[-3e^{-3t}\cos(4t) + e^{-3t}(-4)\sin(4t)\right]$$

$$+ B\left[-3e^{-3t}\sin(4t) + e^{-3t}(4)\cos(4t)\right]$$

$x'(0) = 0$ $\quad t = 0$

$$0 = A\left[-3e^{0}\underbrace{\cancel{\cos(0)}}_{1} + \cancel{e^{0}(-4)\sin(0)}\right]$$

$$+ B\left[\cancel{-3e^{0}\sin(0)} + e^{0}(4)\underbrace{\cancel{\cos(0)}}_{1}\right]$$

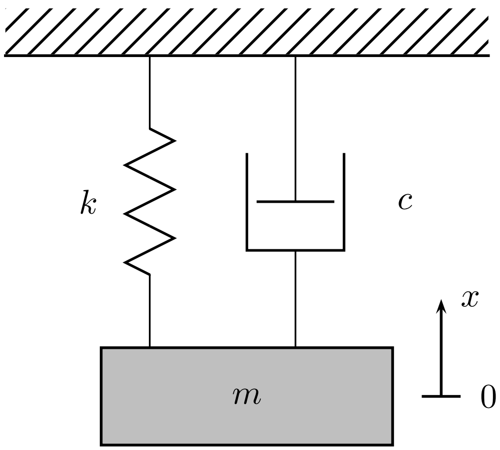
$$0 = -3A + 4B \qquad A = 4$$

$$+3(\cancel{4}) = \cancel{4}B$$

$$B = 3$$

$$x(t) = 4e^{-3t}\cos(4t) + 3e^{-3t}\sin(4t)$$

Damped Spring/Mass System - Example 2



$$m\underline{x''} = -\underline{kx} - \underline{cx'}$$

ma or ΣF

$$mx'' + cx' + kx = 0 \quad (1)$$

Problem. Find two solutions, $x_1(t)$ and $x_2(t)$ to this system if

- the mass is $\underline{m = 1}$ kg, and
- the spring constant is $\underline{k = 25}$ N/m, and
- the damping coefficient is $c = 15$ N/(m/s).

most damping.

Assume $x = e^{\lambda t}$, sub into DE (1)

need $x' = \lambda e^{\lambda t}$

and $x'' = \lambda^2 e^{\lambda t}$

$$\left.\begin{matrix} x' = \lambda e^{\lambda t} \\ x'' = \lambda^2 e^{\lambda t} \end{matrix}\right\} \to (1) \quad 1 \cdot [\lambda^2 e^{\lambda t}] + 15[\lambda e^{\lambda t}] + 25[e^{\lambda t}] = 0$$

Factor: $\quad \cancel{e^{\lambda t}}[\lambda^2 + 15\lambda + 25] = \cancel{0}$

$$\frac{\cancel{e^{\lambda t}}}{\cancel{e^{\lambda t}}} \qquad \frac{0}{e^{\lambda t}}$$

$$1x'' + 15x' + 25x = 0$$

$$\lambda^2 + 15\lambda + 25 = 0$$

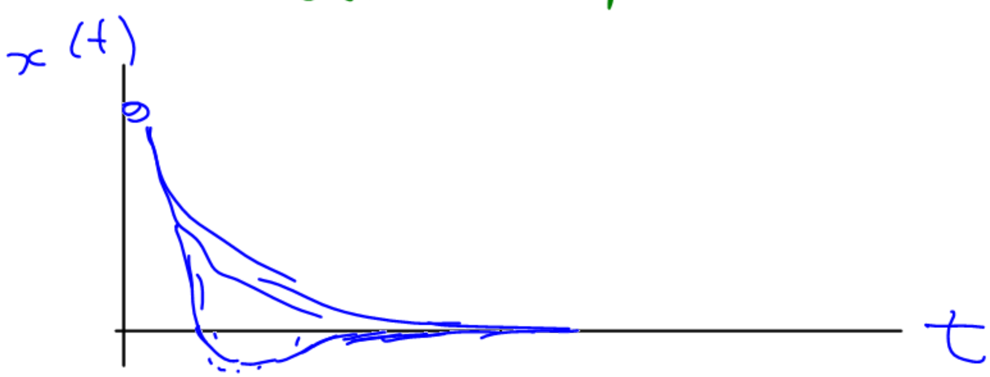
$$\lambda = \frac{-15 \pm \sqrt{15^2 - 4(25)}}{2}$$

$$= -\frac{15}{2} \pm \frac{\sqrt{125}}{2} \doteq -7.5 \pm 5.6$$

$$\doteq -13.1, \ -1.9$$ both real

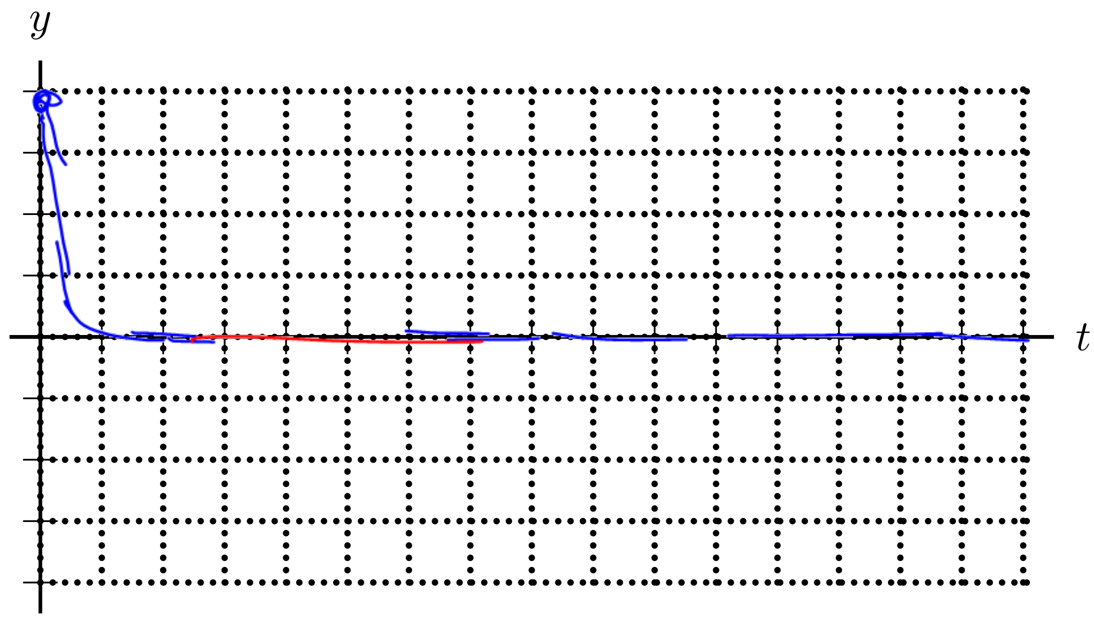
so $x_1 = e^{-13.1t}$, $x_2 = e^{-1.9t}$ all real solutions

No sine/cosine expressions this time.

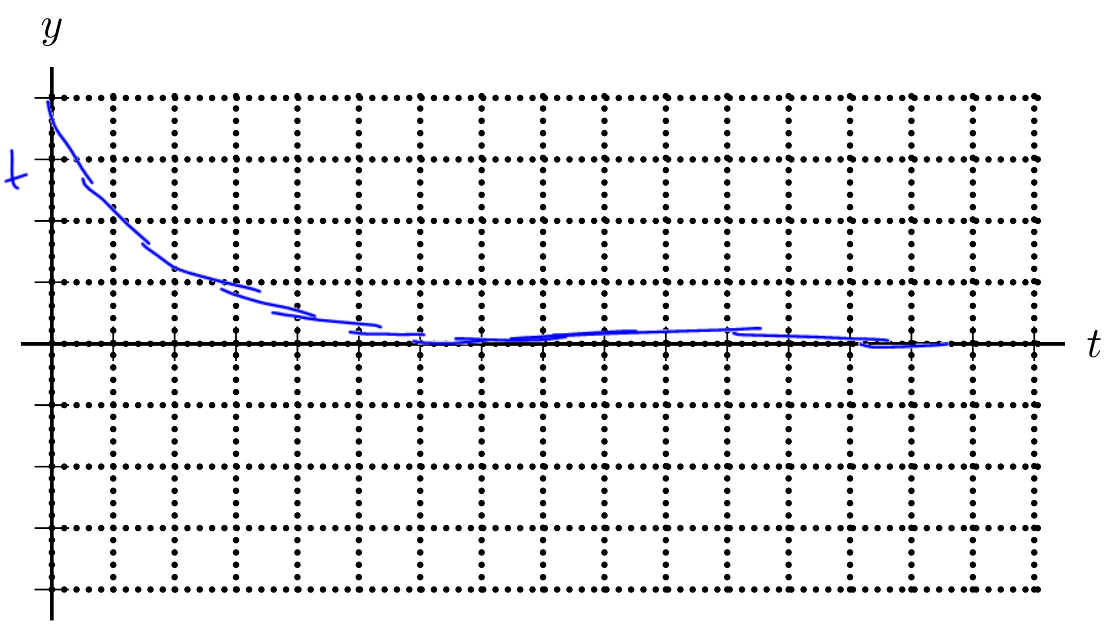


Problem. Sketch your two solutions on the axes below.

$x_1(t) = e^{-13.1t}$



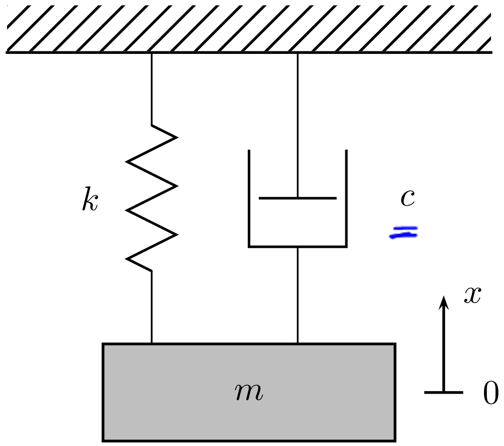
$x_2(t) = e^{-1.4t}$



Sin / Cos

From these examples, we can see that as we move from zero damping to damping so heavy that oscillations can't even happen, the mathematical form of the solutions change. We will investigate this in the general case next.

Damped Spring/Mass System - Patterns of Behaviour



$$mx'' = -kx - cx'$$

ma $=$ or ΣF

$$mx'' + cx' + kx = 0$$

$$\downarrow$$

Assume $x(t) = e^{\lambda t}$

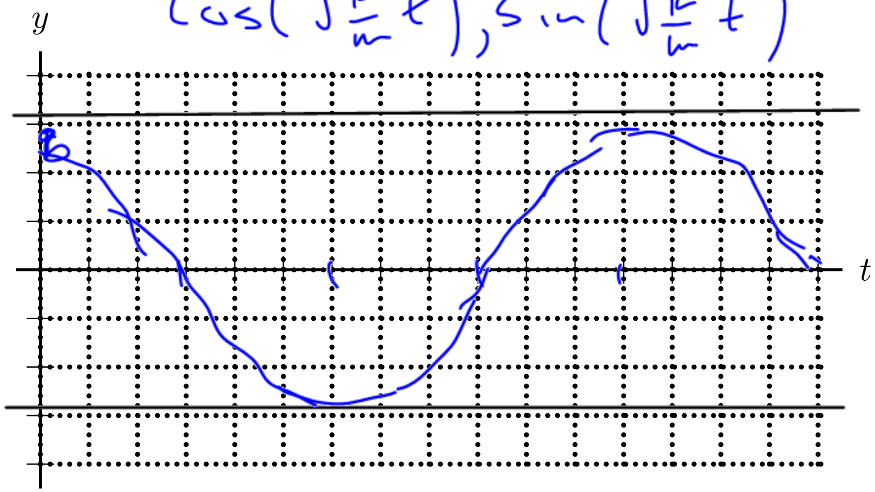
$$\downarrow$$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

Damping	$c^2 - 4km$	λ	Description
None $c = 0$	$\frac{\sqrt{-4km}}{2m}$	$\lambda = \pm\sqrt{\frac{k}{m}}\,i$	oscillations constant amplitude
Light $c^2 < 4km$	$\frac{\sqrt{c^2-4km}}{2m}$	$\lambda = a \pm bi$	e^{-at} $\cos(bt)$ $\sin(bt)$ damped oscillations
$c^2 = 4km$	0	$\lambda = -\lambda_1, -\lambda_1$	critical damping.
Heavy $c^2 > 4km$	pos	$\lambda = -\lambda_1$ $\lambda = -\lambda_2$	No oscillations

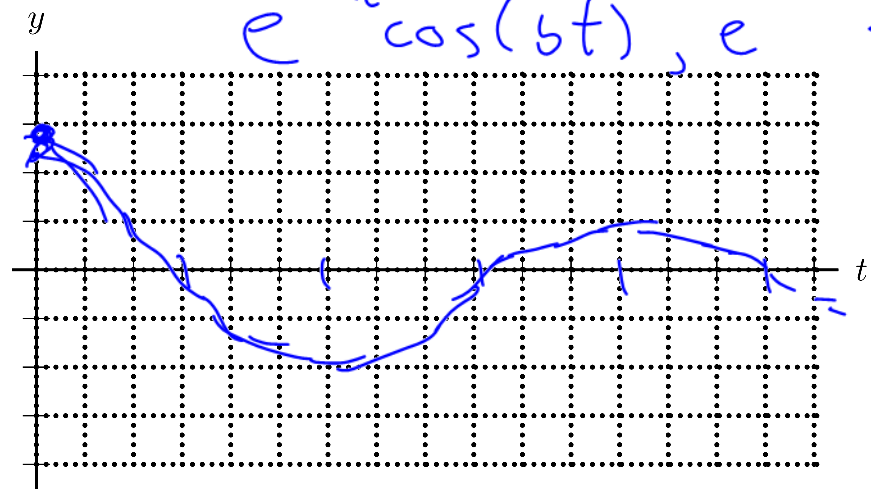
Problem. Sketch possible solutions for all four spring/mass cases.

$\cos\left(\sqrt{\frac{k}{m}}\,t\right), \sin\left(\sqrt{\frac{k}{m}}\,t\right)$

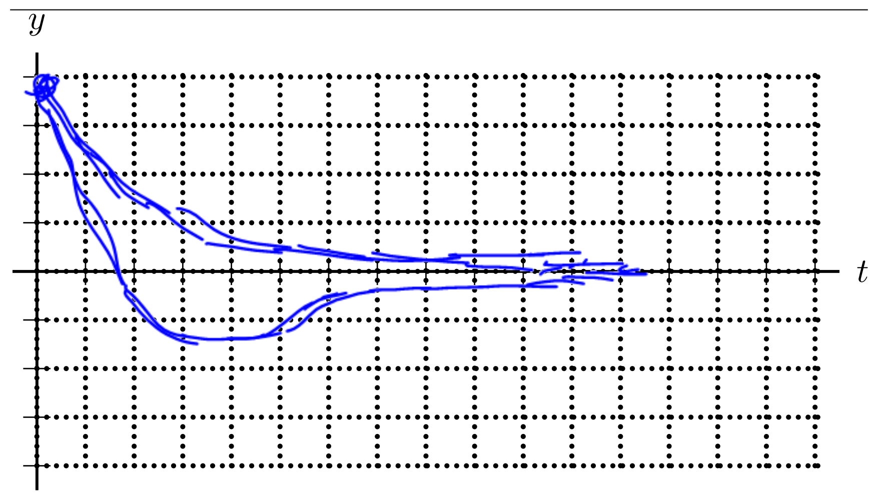


Undamped

$e^{-at}\cos(bt),\ e^{-at}\sin(bt)$

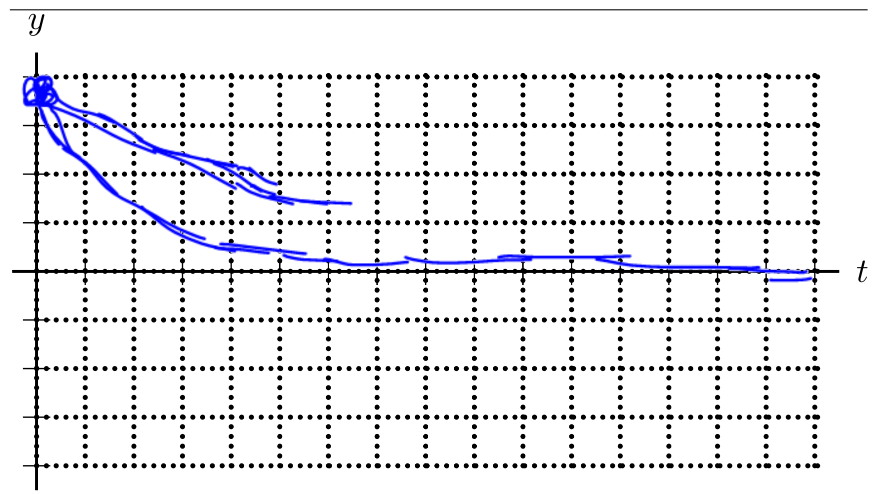


Damped



Critically Damped

$e^{-\lambda_1 t},\ te^{-\lambda_1 t}$



Overdamped

$e^{-\lambda_1 t},\ e^{-\lambda_2 t}$

Demonstration - Spring/Mass

We will demonstrate how the solutions to the damped spring/mass DE change as the damping is gradually increased.

$$my'' + cy' + ky = 0$$

In this demonstration, we will use $m = 1$ kg, and $k = 25$ N/m.

$$y'' + cy' + 25y = 0 \qquad r = \frac{-c \pm \sqrt{c^2 - 4(25)(1)}}{2(1)}$$

Problem. What damping level will produce *critical damping* (transition between oscillations and no oscillations in the solution)?

(a) $c = 0$

(b) $c = 5$

(c) $c = 10$

(d) $c = 25$

$$y'' + cy' + 25y = 0$$

What will the form of the solutions be when damping is **below** critical?

What will the form of the solutions be when damping is **above** critical?

During demonstration, try to ask yourself the following questions:

- As damping increases in general, does the **graph** of the solution change gradually or dramatically?

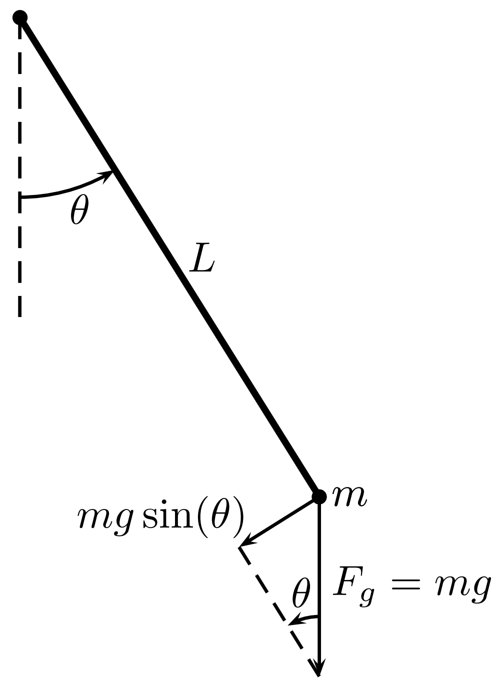
  What about the **mathematical form** of the solution?

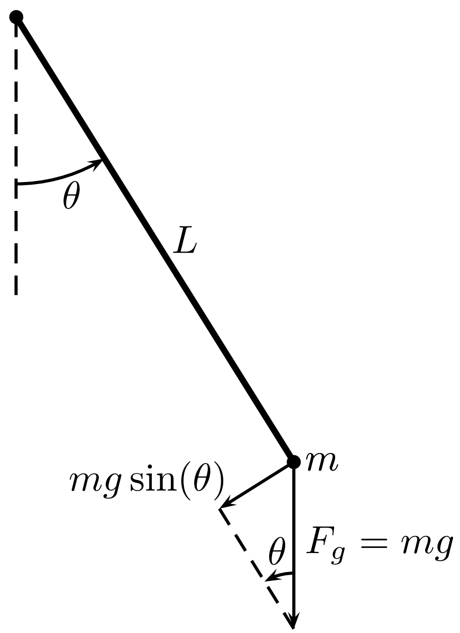
- Near critical damping specifically, does the **graph** of the solution change gradually or dramatically?

  What about the **mathematical form** of the solution?

Applications - Pendulum

Problem. Consider the simple pendulum (mass at the end of a rod) shown below.





If we start from the rotational (torque) version of Newton's Second Law,

$$(\text{moment of inertia}) \cdot (\text{angular accel}) = \sum \text{torques}$$

we obtain

a

$$(mL^{\cancel{2}}) \cdot (\theta'') = -mg\cancel{L}\sin(\theta)$$

or, in (almost) standard form:

mL mL

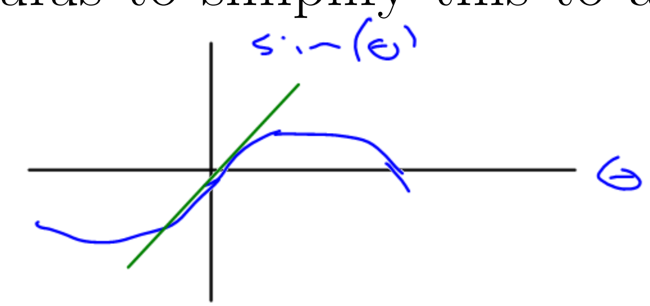
$$\theta'' + \frac{g}{L}\sin(\theta) = 0$$

accel

nonlinear eq'n

Use a well-known approximation from calculus to simplify this to a linear DE:

$\sin(\theta) \approx \theta$



What limitations does this put on our interpretation of the solution?

θ is small, or ~0 (< 10-15 degrees)

Find the general solution to the linearized differential equation

$$(1)\ \theta'' + \frac{g}{L}\theta = 0 \qquad \text{like} \quad x'' + \frac{k}{m}x = 0$$

for the angular position of the pendulum over time.

Assume $\theta = e^{\lambda t}$

need $\theta' = \lambda e^{\lambda t}$

so $\theta'' = \lambda^2 e^{\lambda t}$

Sub into DE (1): $(\lambda^2 e^{\lambda t}) + \frac{g}{L}(e^{\lambda t}) = 0$

Factor $\quad \frac{\cancel{e^{\lambda t}}\left(\lambda^2 + \frac{g}{L}\right)}{\cancel{e^{\lambda t}}} = \frac{0}{e^{\lambda t}}$

$\lambda^2 = -g/L$ or $\boxed{\lambda = \pm\sqrt{\frac{g}{L}} \cdot i}$

$$\theta'' + \frac{g}{L}\theta = 0$$

so $\theta_1(t) = e^{+\sqrt{g/L}\, i\, t}$ and $\theta_2(t) = e^{-\sqrt{g/L}\, i\, t}$

Euler's formula

or

$\theta_1(t) = \cos\left(\sqrt{\frac{g}{L}}\, t\right)$ and $\theta_2(t) = \sin\left(\sqrt{\frac{g}{L}}\, t\right)$

purely real pair of solutions

$$\theta = c_1 \cos\left(\sqrt{\frac{g}{L}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{L}}t\right)$$

Use your solution to predict the period of the oscillations of a pendulum, given g and the length of the rod, L.

$$\text{period} = \frac{2\pi}{\sqrt{g/L}} = \sqrt{\frac{L}{g}} \cdot 2\pi \quad \text{seconds}$$

Up Next

- APSC 172 Calculus II - Like APSC 171, but with multiple variables.

- APSC 174 Linear Algebra - Vectors, but deeper.

- APSC 112 Physics II - Starts with Harmonic Motion.

- MTHE 224, 225, 231, 235, 237 - Differential Equations.