Week 1: Derivatives and Vectors

Goals:

- Evaluate derivatives of all common functions.
- Compute derivatives of more complicated functions using the product, quotient and chain rules.
- Compute and interpret sums of vectors, and scalar and vector products of vectors.

Computing Derivatives

Note: The standard formulas for derivatives are covered in the Grade 12 Ontario curriculum. While they will be reviewed here, **students who are not familiar with them should begin either textbook reading and the QEng Prep as soon as possible.**

In addition the graphical interpretation of derivatives-as-slopes, there are useful algebraic rules. **All of these rules** are based on the **definition** of the derivative,

$$f'(x) = \frac{d}{dx}f = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

By finding common patterns in the derivatives of certain families of functions, we can compute derivatives much more quickly than by using the definition.

Sums, Powers, and Differences

Constant Functions: $$\frac{d}{dx}\,k = 0$$

Power rule: $$\frac{d}{dx}\,x^p = px^{p-1}$$

Sums : $$\frac{d}{dx}\,f(x) + g(x) = \left(\frac{d}{dx}\,f(x)\right) + \left(\frac{d}{dx}\,g(x)\right)$$

Differences: $$\frac{d}{dx}\,f(x) - g(x) = \left(\frac{d}{dx}\,f(x)\right) - \left(\frac{d}{dx}\,g(x)\right)$$

Constant Multiplier: $\frac{d}{dx}\,[kf(x)] = k\left(\frac{d}{dx}\,f(x)\right)$, so long as k is a constant

Example: *Evaluate the following derivatives:*

$$\frac{d}{dx}\left(x^4 + 3x^2\right)$$

$$\frac{d}{dx}\left(2.6\sqrt{x} - \pi x^3 + 4\right)$$

Question: The derivative of $-3x^2 - \frac{1}{x^2}$ is

A. $-6x^3 + 2\frac{1}{x^3}$

B. $-6x + 2\frac{1}{x^3}$

C. $-6x - 2\frac{1}{x^3}$

D. $-x^3 + 2\frac{1}{x}$

Exponentials and Logs

e as a base: $$\frac{d}{dx}\,e^x = e^x$$

Other bases: $$\frac{d}{dx}\,a^x = a^x(\ln(a))$$

Natural Log: $$\frac{d}{dx}\,\ln(x) = \frac{1}{x}$$

Other Logs: $$\frac{d}{dx}\,\log_a(x) = \frac{1}{x}\frac{1}{\ln(a)}$$

Example: *Evaluate the following derivatives:*

$$\frac{d}{dx}\left(4 \cdot 10^x + 10 \cdot x^4\right)$$

$$\frac{d}{dx}\left(e^x + \log_{10}(x)\right)$$

(Exponential and log derivatives are relatively straightforward, until we mix in the product, quotient, and chain rules.)

Product and Quotient Rules

Products: $$\frac{d}{dx} f(x) \cdot g(x) = f'(x)g(x) + f(x)g'(x)$$

Quotients: $$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

Example: *Evaluate the following derivatives:*

$$\frac{d}{dx}\left(4x^2e^x\right)$$

$$\frac{d}{dx}\left(x\ln(x)\right)$$

$$\frac{d}{dx}\left(5\frac{x^2}{\ln(x)}\right)$$

Question: The derivative of $\dfrac{10^x}{x^3}$ is:

A. $\dfrac{10^x}{\ln(10)}x^{-3} + 10^x(-3x^{-4})$

B. $\dfrac{10^x \ln(10)x^3 - 10^x(3x^2)}{x^6}$

C. $\dfrac{10^x \frac{1}{\ln(10)}x^3 - 10^x(3x^2)}{x^6}$

D. $\ln(10)10^x x^{-3} + 10^x(-3x^{-4})$

Chain Rule

Nested Functions: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

Liebnitz form $\frac{d}{dx} f(g(x)) = \frac{df}{dg}\frac{dg}{dx}$

Example: *Evaluate the following derivatives:*

$$\frac{d}{dx}\, e^{x^2}$$

$$\frac{d}{dx} \ln(x^4)$$

$$\frac{d}{dx}\left(\frac{1}{1+x^3}\right)$$

$$\frac{d}{dx}\left(x^4 + 10^{3x}\right)$$

Question: The derivative of $e^{\sqrt{x}}$ is

A. $\frac{1}{2}e^{\frac{1}{\sqrt{x}}}$

B. $e^{\sqrt{x}}\left(\sqrt{x}\right)$

C. $\frac{1}{2}e^{\sqrt{x}}\left(\frac{1}{\sqrt{x}}\right)$

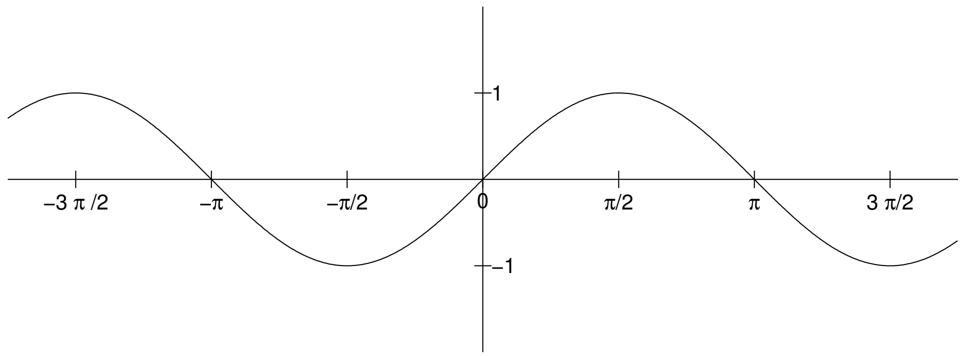
D. $\frac{1}{2}e^{\sqrt{x}}\left(\sqrt{x}\right)$

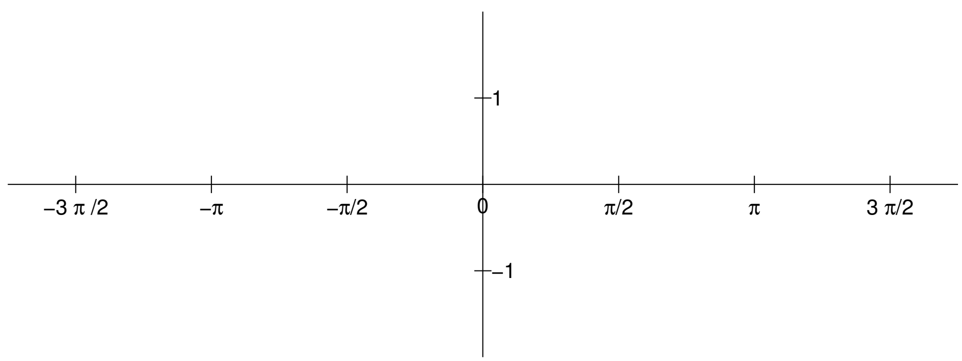
Derivatives of Trigonometric Functions

The derivation of the formulas for the derivatives of sin and cos are an interesting study in both limits and trigonometric identities. For those who are interested, many such derivations can be found on the web[1]. However, it is in some ways more useful to derive the formula in a graphical manner.

[1] For example, http://www.math.com/tables/derivatives/more/trig.htm#sin

Below is a graph of $\sin(x)$. Use the graph to sketch the graph of its derivative.





From this sketch, we have evidence (though not a proof) that

Theorem

$$\frac{d}{dx}\sin x =$$

Most students will also be familiar with the other derivative rules for trig functions:

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

Prove the secant derivative rule, $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$, *using the definition* $\sec(x) = \frac{1}{\cos(x)}$ *and the other derivative rules.*

Question: Find the derivative of $4 + 6\cos(\pi x^2 + 1)$

A. $4 - 6\sin(\pi x^2 + 1) \cdot (2\pi x)$

B. $-6\cos(\pi x^2 + 1) \cdot (2\pi x)$

C. $-6\sin(\pi x^2 + 1) \cdot (2\pi x)$

D. $-6\sin(\pi x^2 + 1) \cdot (\pi x^2 + 1)$

E. $6\sin(2\pi x)$

Vectors

An essential concept in math and the sciences is the idea of quantities which also have an associated **direction**. We will use **vectors** to represent these directional quantities.

Vectors

A **vector** is a quantity that has both **magnitude and direction**.

Velocity and force are examples of vector quantities.

Scalars

Quantities that have magnitude only are called **scalars**.

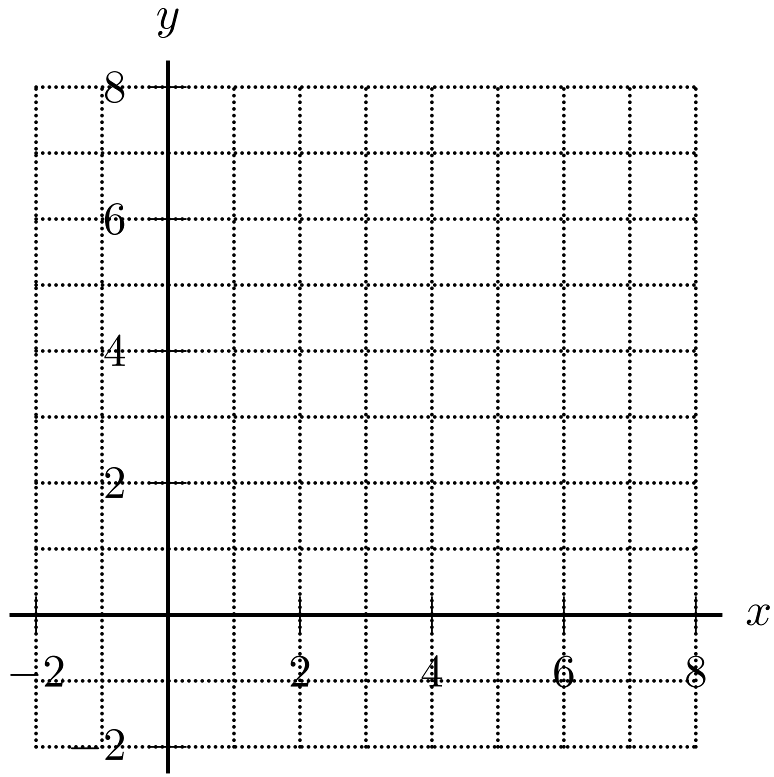
Length, volume and speed are examples of scalar quantities.

It is often convenient to use arrows to represent vector quantities. The length of the arrow corresponds to the magnitude of the vector and the direction of the arrow tells you the direction of the vector. A natural question to ask is "does it matter where the vector is located"? The answer to this question is that it does **not** matter. Two arrows that have the same direction and magnitude are two representations of the same vector.

Which of the following arrows represent the same vector

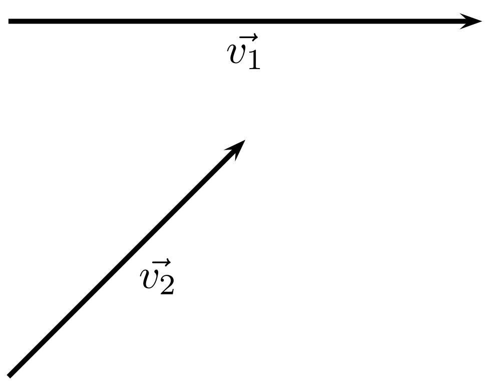
(i) *the arrow from* $(0,0)$ *to* $(3,4)$*;* *(ii) the arrow from* $(2,2)$ *to* $(5,-2)$*;*

(iii) *the arrow from* $(-1,3)$ *to* $(2,7)$*;* *(iv) the arrow from* $(1,2)$ *to* $(-2,-2)$*?*



It is possible to combine vector quantities in much the same way as we combine numbers (by addition and subtraction).

For the vectors $\vec{v_1}$ *and* $\vec{v_2}$ *below,*



sketch:

- $\vec{v_1} + \vec{v_2}$
- $\vec{v_1} - \vec{v_2}$
- $\vec{v_2} - \vec{v_1}$

Sketch the vectors $(\vec{v_1} - \vec{v_1})$, *and* $(\vec{v_2} - \vec{v_2})$.

The result of these last two differences is called the **zero vector**. It is a vector with zero length, and is the *only vector for which we can assign no direction.*

Vector Components and Magnitudes

To manipulate vectors without needing to draw arrows, we need a symbolic representation for them. One of the most useful representations is the **component** form of a vector.

Component Unit Vectors

We define

- $\vec{i}$ a vector of length 1 in the direction of the x axis
- $\vec{j}$ a vector of length 1 in the direction of the y axis
- $\vec{k}$ a vector of length 1 in the direction of the z axis

Components of a vector

If we expression a vector in the form

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

we call $v_1\vec{i}$, $v_2\vec{j}$, and $v_3\vec{k}$ the **components** of $\vec{v}$

Alternate Component Form

If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$,

a shorter form for the component representation is

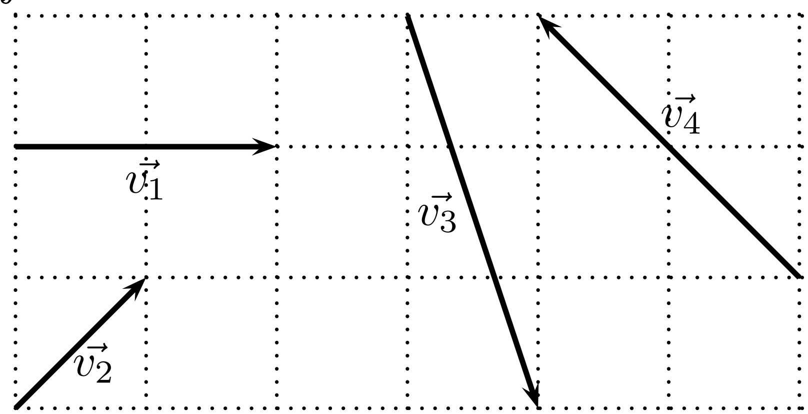
$$\vec{v} = \langle \underbrace{v_1}_{\vec{i}}, \underbrace{v_2}_{\vec{j}}, \underbrace{v_3}_{\vec{k}} \rangle$$

Note: sometimes mathematicians and other scientists use different bracket shapes

$$\vec{v} = [v_1, v_2, v_3] \text{ or } \vec{v} = (v_1, v_2, v_3)$$

to indicate that the set of values represents a vector, rather than a point. We will continue to use either the vector components, $\vec{i}$, $\vec{j}$ and $\vec{k}$, or angled parentheses,
'$\langle$' and '$\rangle$'.

Example: *Express the following vectors in both component forms:*



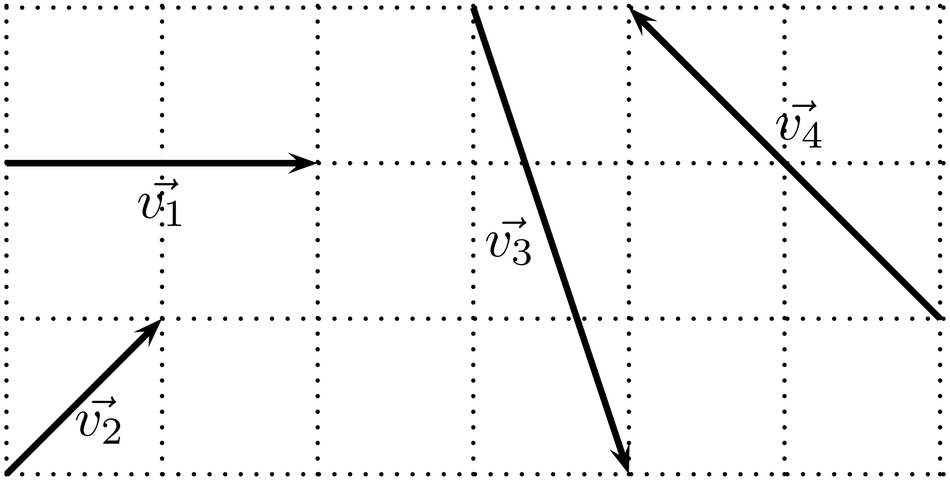
Question: Which of the following vectors represents $\vec{v_1} + \vec{v_3}$?

(a) $\langle 2, -3 \rangle$

(b) $\langle 3, -3 \rangle$

(c) $\langle 3, 2 \rangle$

(d) $\langle 3, 3 \rangle$



Question: Which of the following vectors represents $\vec{v_4} - \vec{v_3}$?

(a) $\langle 3, -3 \rangle$

(b) $\langle -3, 5 \rangle$

(c) $\langle 1, -1 \rangle$

(d) $\langle 1, 1 \rangle$

Magnitude or Length From Components

$$\text{If } \vec{v} = v_1\vec{i} + v_2\vec{j}$$

$$\text{The length of a vector } ||\vec{v}|| = \sqrt{v_1^2 + v_2^2}$$

$$\text{In 3 dimensions, where } \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k},$$

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

How long is the vector $\langle 2, -3 \rangle$*?*

(a) -1

(b) $\sqrt{5}$

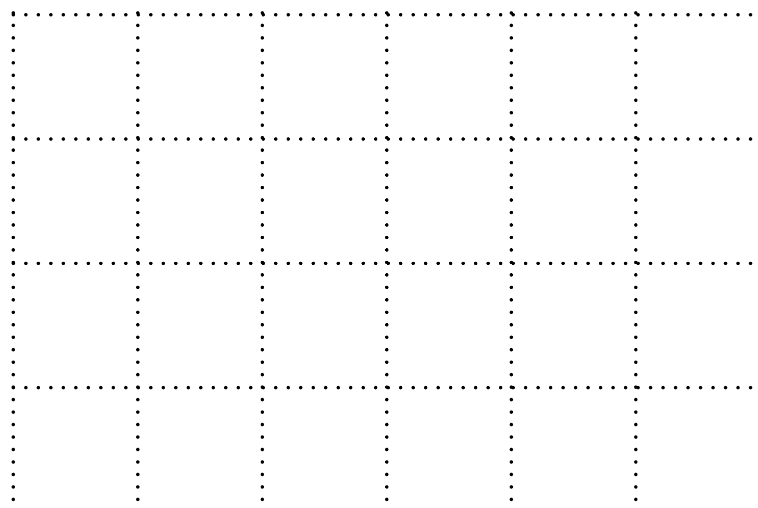
(c) $\sqrt{13}$

(d) 13

Parallel Vectors

It is often important to know whether two vectors are parallel to each other, that is that they point along the same line, but might have different magnitudes.

Example: *Are the vectors* $\vec{v} = \langle 1, -2 \rangle$ *and* $\vec{w} = \langle 2, -4 \rangle$ *parallel?*



Without drawing them, how can we determine that the vectors $\vec{v} = \langle 1, -2 \rangle$ *and* $\vec{w} = \langle 2, -5 \rangle$ *are* **not** *parallel?*

Parallel Vectors

Two vectors, $\vec{v}$ and $\vec{u}$, are parallel to one another if there exists a scalar multiplier c such that

$$\vec{v} = c\,\vec{u}.$$

Example: *Show that the vector* $\langle 3, 4, 5 \rangle$ *is not parallel to* $\langle 1, 2, 3 \rangle$.

Vector Multiplication

Unlike for addition and subtraction, vector quantities differ from scalars in that **vector multiplication can be defined in several ways**. There are two such operations that we will need to use:

- scalar multiplication
- dot product

Scalar multiplication: $\lambda\vec{v}$

- combines a **scalar**, e.g. λ, with a **vector**, e.g. $\vec{v}$ to produce a new **vector**, $\lambda\vec{v}$.
- the magnitude of the new vector is $|\lambda|$ times the original vector length e.g. $2\,\vec{v} = \vec{v} + \vec{v}$ twice as long as the original.
- If $\lambda > 0$, $\lambda\vec{v}$ is a vector in the same direction as $\vec{v}$
- If $\lambda < 0$, $\lambda\vec{v}$ is a vector in the **opposite** direction as $\vec{v}$

Example: *Choose a vector $\vec{v}$ and then draw*

- $2\vec{v}$,

- $0\ \vec{v}$, *and*

- $(-1.5)\vec{v}$.

Example: *For the vector $\vec{v} = \langle 5, -2 \rangle$, express the following in component form:*

- $2\vec{v}$,

- $0\ \vec{v}$, *and*

- $(-1.5)\vec{v}$.

Linearity of Vector Operations

Addition, subtraction, and scalar multiplication all obey consistent rules of operation familiar from your experience with scalar operations. These properties are summarized on page 617 of Hughes-Hallett. For convenience we repeat them here.

Commutativity

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

Associativity

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

Distributivity

$$(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$$

$$\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$$

Identity

$$1\vec{v} = \vec{v}, \; 0\vec{v} = \vec{0}$$

$$\vec{v} + \vec{0} = \vec{v}$$

Note that for any vector $\vec{v}$, $(-1)\vec{v}$ is a vector with the same magnitude/length as $\vec{v}$ and **opposite direction**.
Because of this property we write $(-1)\vec{v} = -\vec{v}$.

Dot Product of Vectors: $\vec{v} \cdot \vec{w}$

Remember that the *scalar* product multiplies a scalar times a vector. Another possible multiplication between **two vectors** is called the **dot product**. The dot product

- combines two **vectors**, e.g. $\vec{v}, \vec{w}$ to produce a **scalar**, $\vec{v} \cdot \vec{w}$
- If $\theta \in [0, \pi]$ is the angle between two vectors $\vec{v}$ and $\vec{w}$, then

$$\vec{v} \cdot \vec{w} = ||\vec{v}||\ ||\vec{w}||\ \cos(\theta)$$

Question: Use this definition to find $\vec{i} \cdot \vec{i}$.

(a) -1

(b) 0

(c) 1

(d) 2

$$\vec{v} \cdot \vec{w} = ||\vec{v}|| \; ||\vec{w}|| \; \cos(\theta)$$

Question: Use this definition to find $\vec{i} \cdot \vec{j}$.

(a) -1

(b) 0

(c) 1

(d) 2

$$\vec{v} \cdot \vec{w} = ||\vec{v}||\ ||\vec{w}||\ \cos(\theta)$$

Suppose that $\vec{v}$ and $\vec{w}$ are perpendicular to one another. What can you say about $\vec{v} \cdot \vec{w}$?

What can you conclude if $\vec{v} \cdot \vec{w} = 0$?

The previous definition of dot product involved the **angle** between the two vectors. It is also helpful to compute the dot product purely in terms of the **components** of the vectors.

Component Definition of Dot Product

$$\text{If } \vec{v} = \lambda_1 \vec{i} + \lambda_2 \vec{j} + \lambda_3 \vec{k}$$

$$(\text{or } = \langle \lambda_1, \lambda_2, \lambda_3 \rangle)$$

$$\text{and } \vec{w} = \mu_1 \vec{i} + \mu_2 \vec{j} + \mu_3 \vec{k},$$

$$(\text{or } = \langle \mu_1, \mu_2, \mu_3 \rangle)$$

then

$$\vec{v} \cdot \vec{w} = \lambda_1 \, \mu_1 + \lambda_2 \, \mu_2 + \lambda_3 \, \mu_3.$$

It is not at all obvious that this is the same as the other definition!

The fact that the two definitions always give the same result is proven in your textbook. We will study an example demonstrating this general property to see a specific instance of this general rule.

Example: *Use* both *definitions of the dot product to calculate*

$$\langle 1,1\rangle \cdot \langle 0,3\rangle$$

in two different ways.

Example: *Find a vector* $\vec{u} = \langle a, b \rangle$ *of magnitude/length* 1 *which is perpendicular to the vector* $3\vec{i} + 7\vec{j}$.

$$\vec{u} = \langle a, b \rangle \ \textit{of magnitude/length } 1, \ \textit{perpendicular to } 3\vec{i} + 7\vec{j}.$$

Are there other possibilities than the perpendicular vector you found?

Product Confusion *Is* $(\vec{v_1} \cdot \vec{v_2})\vec{v_3} = \vec{v_1}(\vec{v_2} \cdot \vec{v_3})$*?*

(a) Yes, the results are equal.

(b) No, the results will be different because of the grouping.

(c) No,the results will be different because the product types are different.

Example: *Which pairs (if any) of vectors from the following list*

(a) Are perpendicular?

(b) Have an angle **less than** $\pi/2$ *between them?*

(c) Have an angle of **more than** $\pi/2$ *between them?*

$\vec{a} = \langle 1, 0, -2 \rangle$ $\vec{b} = \langle 1, 3, 0 \rangle$ $\vec{c} = \langle 2, 1, 1 \rangle$