Week 2: Vector-Valued Functions

Goals:

- Define vector-valued functions (parametric curves)
- Sketch vector-valued functions
- Determine if vector-valued functions collide or intersect

The Calculus of Moving Objects

Problem. How do we describe a moving object in more than one dimension?

Most students have seen how a single-variable position, e.g. x, can change as a function of time, t. The derivatives we have seen so far apply perfectly to this type of function:

$$\underbrace{x(t)}_{\text{position}} \rightarrow \underbrace{x'(t) \text{ or } v(t)}_{\text{velocity}}$$

However, limiting ourselves single-variable limitation is too constraining for even the first few weeks of studying motion in your physics course: we need a consistent way to describe changing positions (and velocities and accelerations) where the objects move in **2 or more dimensions**.

To start this unit off, we begin with a problem, which we will come back to at the close of the unit. Here is that problem.

Suppose an object moves in three-dimensional space so that at time $t \geq 0$ its coordinates are given as $\langle \cos t, \sin t, e^{-t} \rangle$. What does that tell us about the way this object moves?

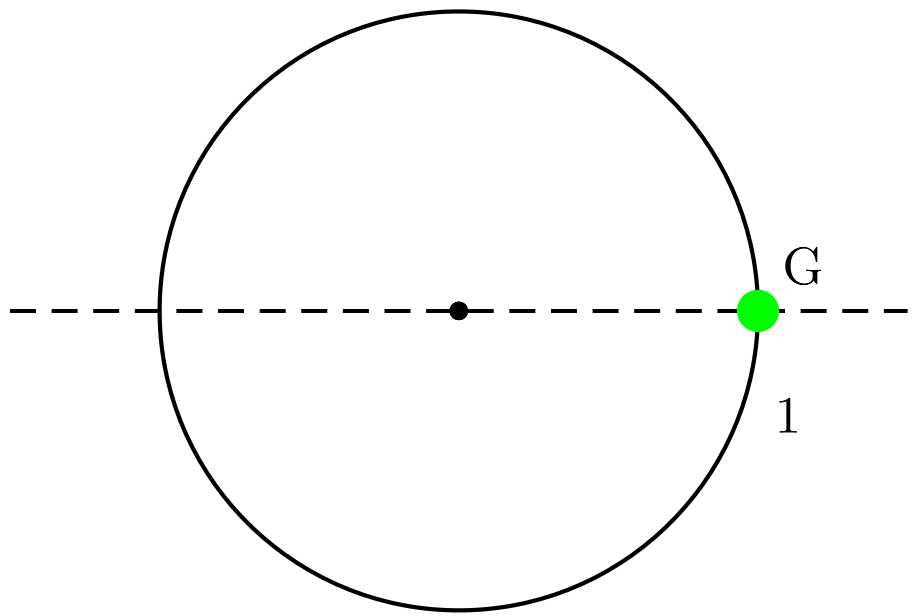
- What would a graph of its trajectory look like?
- What is its speed at, say, time $t = 0$?
- If the object is a vehicle (say a spaceship), following the path indicated by the formula, and if you are a passenger in the vehicle, when will you experience the greatest acceleration?
- If you were to fall out of the vehicle, say at time $t = 1$, what would happen to you, especially if this were to happen in a zero gravity environment?

Four Ways to Represent a Function

A function is a rule or process that assigns
to each *input*
a corresponding *output*.

A function can come in a variety of forms, as the following example will illustrate.

Problem. Suppose we have a wheel of radius 1, centered at the origin, with a green dot G at the extreme right of the wheel. If we turn the wheel counter-clockwise by an angle θ, the height $h(\theta)$ of the dot above (or below) the x-axis will vary with the angle. Discuss the function $h(\theta)$.



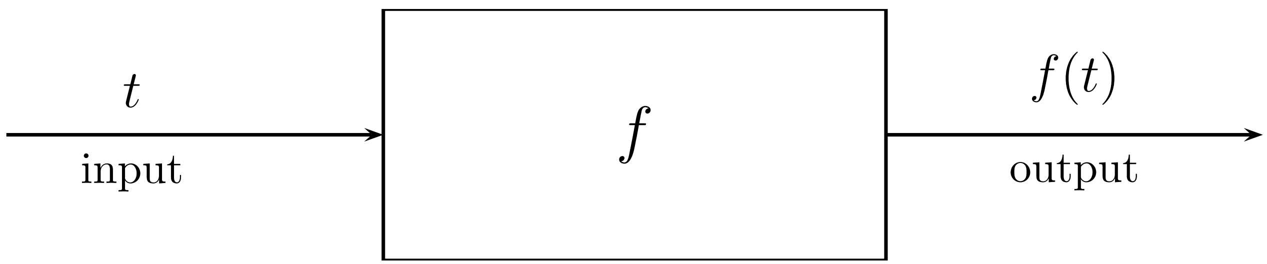
This example shows that there are (at least) **four ways to represent a function**:

- Verbal description (as in the statement of the problem)
- Table
- Graph
- Formula

A function is not necessarily given by a formula.

In every one of the descriptions provided, the function involves two varying quantities, with one of them, the **input**, giving rise to, or "causing", the other one, which is then called the **output**.

The Machine Model of a Function



Review: which variable is the **dependent variable**, and which the **independent** one?

The functions familiar to us from our high school courses are all functions for which both the **input** and the **output** are **single numbers**.

Problem. Give examples of mathematical functions you have seen earlier.

However, we do **not** have to restrict ourselves to functions of that type!

Functions of Several Variables

Problem. If you wanted to find out the average winter temperature at a point on the map below, what information you would need to specify?

In this setting, the temperature is the output of a function of two variables:

$$T = f(x, y)\,.$$

Problem. Sketch the "black box" version of this function:

What would the **domain** of the temperature be in this scenario?

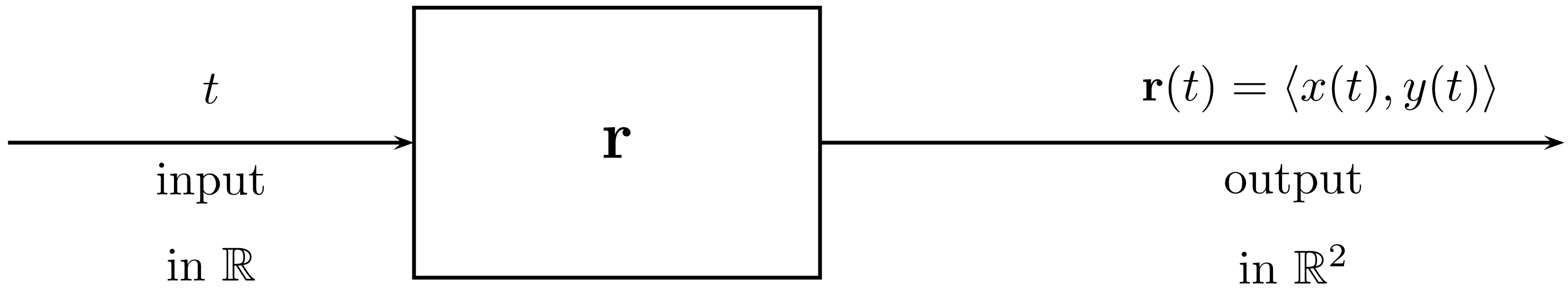
What would the **range** be?

Problem. Give other examples of **functions of several variables**, meaning multiple input variables, but just one output value per input.

Vector-Valued Functions

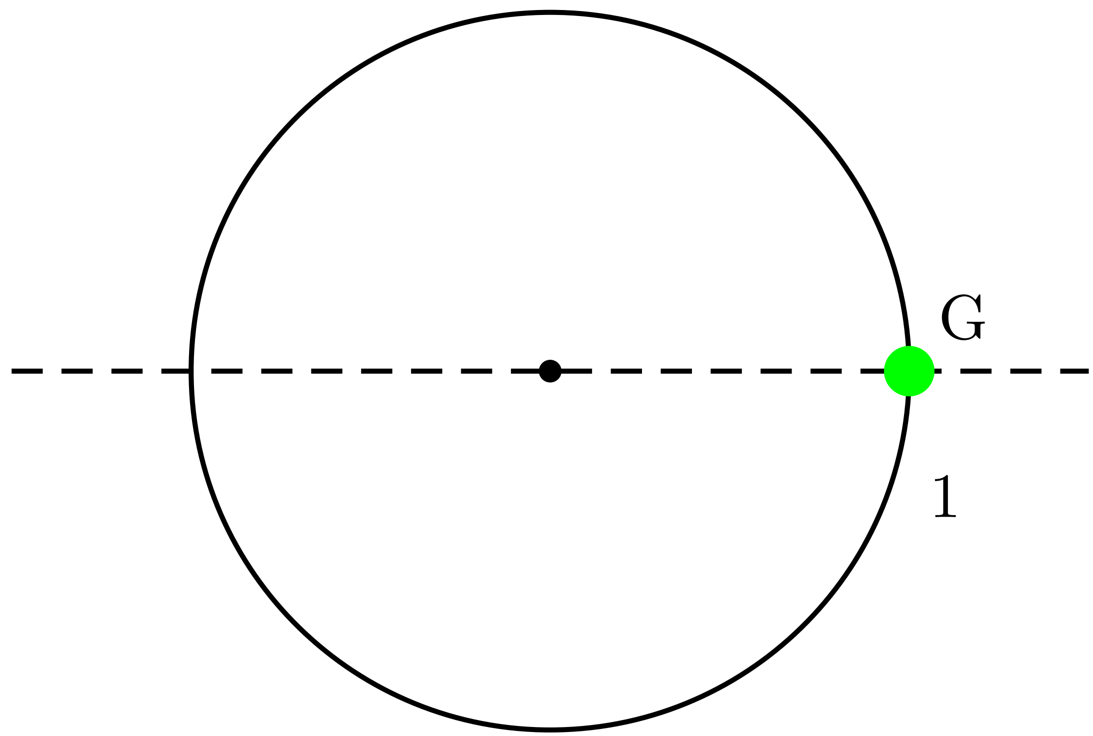
We can also vary the type of function under discussion by taking:

- a **single number** as an **input**, but
- computing a **pair** of numbers as the **output**.



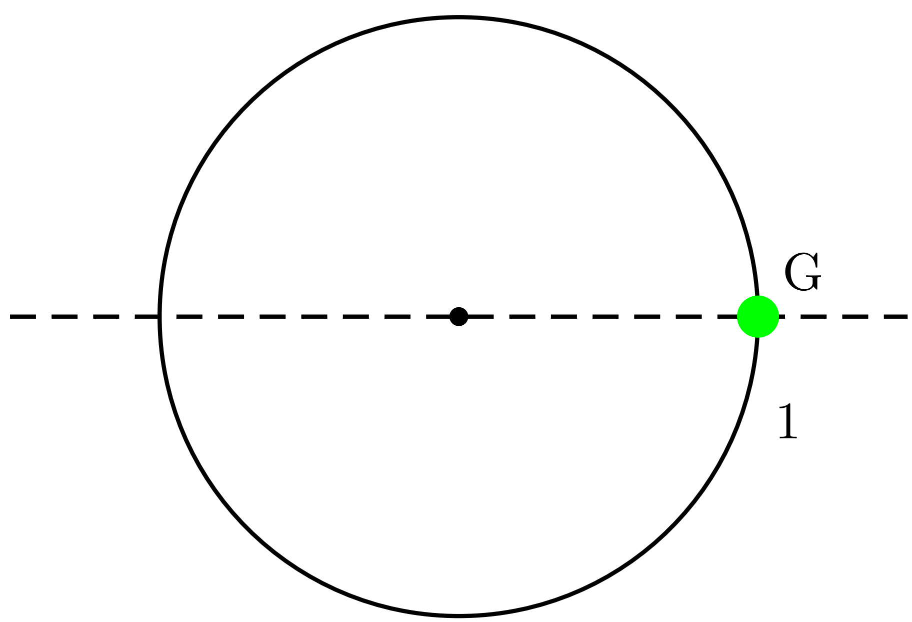
This is precisely what we get if we think of the moving green dot in the problem we discussed earlier.

Problem. How do the x and y coordinates of the green dot vary as the angle θ varies?



Suppose the dot is rotating at a constant rate *over time*, say at one revolution per 2π seconds.

Problem. Give a formula for the position of the point at time t (t in seconds).



This is a **vector-valued function**, and it can also be called a **parametric curve**.

Problem. How do those mathematical terms relate to the function?

How could you express the position function as two separate “regular” functions (single input, single output)?

Graphing Parametric Curves with MATLAB

Graphing parametric functions can really help understand what is going on.

Problem. [MATLAB Introduction] To get MATLAB to plot the path of this moving particle, we can use following commands:

```
t = 0:0.01:(2*pi)
```

```
plot( cos(t), sin(t))
```

What do these lines of MATLAB code do?

You can build an animation of a parametric curve in MATLAB with a *for loop*, and *array indexing*. Here is the core of the code (though it will need some tweaking before it works correctly):

```
t = 0:0.01:2*pi

for (i = 1:length(t))
    plot(  cos( t(i)), sin(t(i)), '.g')
end
```

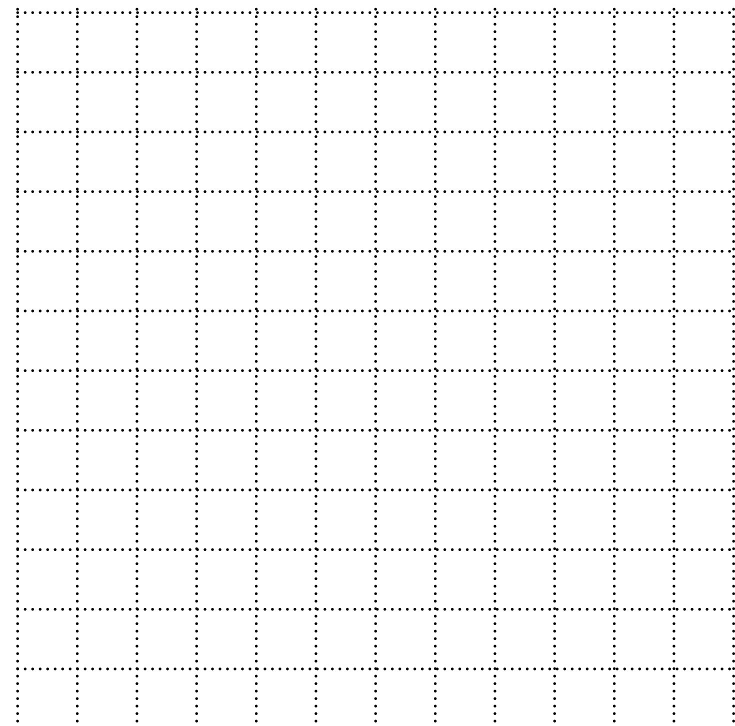
Problem. Update this code so it shows a useful animation of the green point going around in a circle.

Sketching Parametric Curves

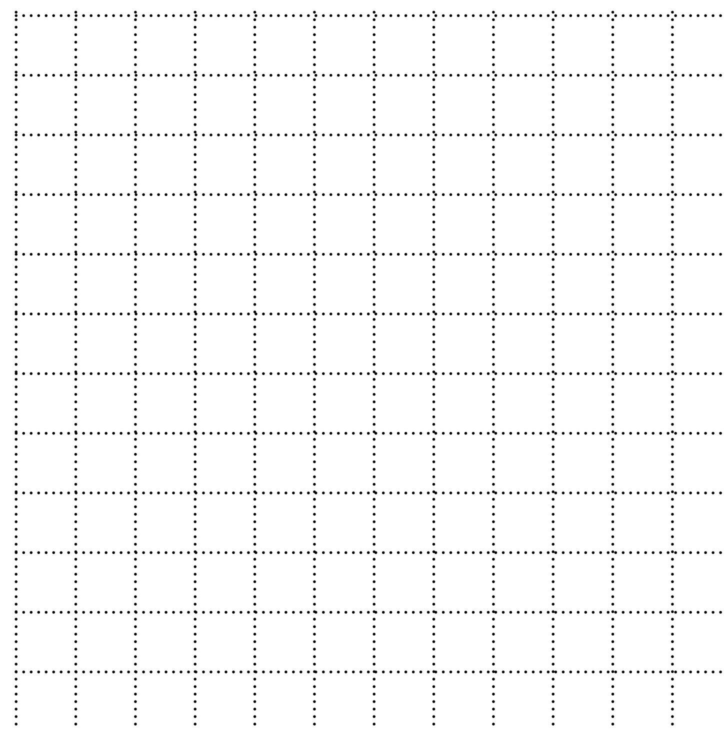
You can sketch a parametric curve given as a formula by creating a table of values and plotting the resulting points.

Problem. Use a table of values to sketch the parametric curve defined by $\mathbf{r} = \langle t^2, t \rangle$.

Problem. Sketch the graph defined by $\mathbf{v} = \langle 4\cos(t), -2\sin(t) \rangle$.



Problem. Sketch the vector-valued function given by $\mathbf{s}(t) = \langle \sqrt{t}, 1/t \rangle$.



Eliminating the Parameter

As an alternative to computing a table of $x(t), y(t)$ values and then plotting, another sketching method that is often useful is the method of **eliminating the parameter**.

Problem. Starting with the function $\mathbf{r}(t) = \langle t^2 - 2t, t + 1 \rangle$, create separate functions for $x(t)$ and $y(t)$ and then, **eliminate the variable** t from the equations.

Problem. On your own, use a *table of values* to obtain the rough shape of the graph of

$$\mathbf{r}(t) = \langle -t^2, t^4 \rangle$$

Eliminate the parameter to build an x/y formula (with no t) for the curve.

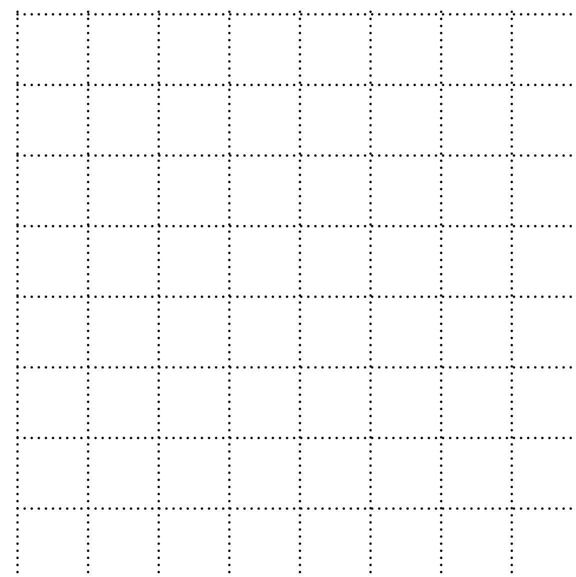
$$\mathbf{r}(t) = \langle -t^2, t^4 \rangle$$

Sketch the resulting graph.

Compare the two approaches and the information that they provide.

Eliminating the Parameter - Examples

Problem. Eliminate the parameter from $\mathbf{r}(t) = \langle 3t - 2, 4t - 3 \rangle$. Sketch the resulting graph.



Problem. One form of ellipse centered at the origin is the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are the lengths of the 'radii' along the x and y axes respectively.

First: note how the ellipse equation is realted to the equation of a circle centered at the origin: $x^2 + y^2 = r^2$.

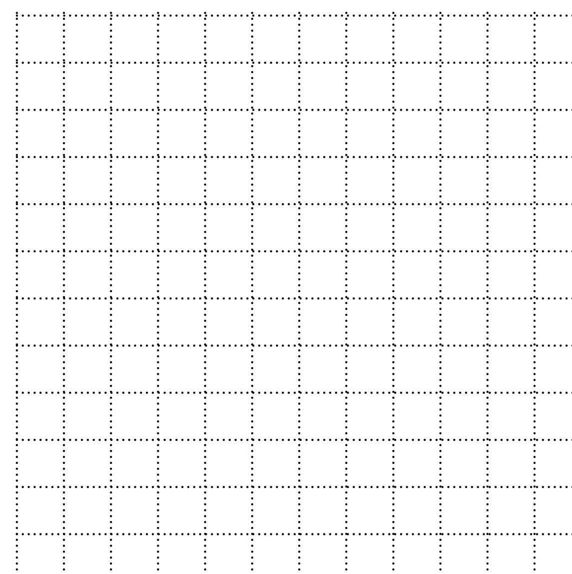
Eliminate the parameter in $\mathbf{u}(t) = \langle 4\cos(t), -2\sin(t)\rangle$ to show that this trajectory follows an ellipse. Sketch the graph of the trajectory.

Collisions and Intersections

If two objects travel through space along two different curves, it's often important to know whether they will collide. The curves might intersect, but we need to know whether the objects are in the same position at the same time.

Example: *Show through a sketch that the following two curves intersect.*

$$\mathbf{r}(t) = \langle t^2, t \rangle$$
$$\mathbf{u}(t) = \langle 3 - 2t, 4t - 3 \rangle$$



Find characteristics of a **collision** between the two trajectories.

$$\mathbf{r}(t) = \langle t^2, t \rangle$$
$$\mathbf{u}(t) = \langle 3 - 2t, 4t - 3 \rangle$$

Find characteristics of an **intersection** between the two trajectories.

Determine whether the two particles **collide**, and if so where and when.

$$\mathbf{r}(t) = \langle t^2, t \rangle$$
$$\mathbf{u}(t) = \langle 3 - 2t, 4t - 3 \rangle$$

Determine whether the two particles' trajectories **intersect**, and if so where.

$$\mathbf{r}(t) = \langle t^2, t \rangle$$
$$\mathbf{u}(t) = \langle 3 - 2t, 4t - 3 \rangle$$

$$\mathbf{r}(t) = \langle t^2, t \rangle$$
$$\mathbf{u}(t) = \langle 3 - 2t, 4t - 3 \rangle$$

MATLAB animation

```
%% Collision and Intersection Example
close all
figure('units','normalized','outerposition',[0 0 1 1]);  % make new figure full-screen size
% Setup the time scale, and the r(t) and u(t) trajectories
t = -3:0.02:2;
rx = t.^2;
ry = t;
ux = 3-2*t;
uy = 4*t-3;

% Animation
for i = 1:length(t)
    clf
    plot(rx, ry, ':b', ux, uy, ':k'); % whole trajectories
    xlim([-10, 6]);  ylim([-15, 5]);
    axis equal
    hold on;
    plot(rx(i), ry(i), '.b', 'markersize', 25) % moving points
    plot(ux(i), uy(i), '.k', 'markersize', 25)
    text(4, 0, sprintf('t = %.2f', t(i)), 'fontsize', 20);
    drawnow
end
```

Collisions and Intersections in 3D

In the previous example, it was relatively straightforward to determine at least that the paths given intersected, because we could draw a 2D sketch. However, if the particles are moving in 3D, we almost have to use an algebraic approach rather than diagrams.

Example: *Two particles are moving along the trajectories given by the parametric curves*

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$
$$\mathbf{u}(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$

Do the trajectories cross? If they do, find the location of the crossing(s).

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$
$$\mathbf{u}(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$
$$\mathbf{u}(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$

Would the particles collide? If they do, at what time and at what (x, y) location would the collision(s) occur?

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$
$$\mathbf{u}(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$