Week 3: Differentiation of Vector-Valued Functions

Goals:

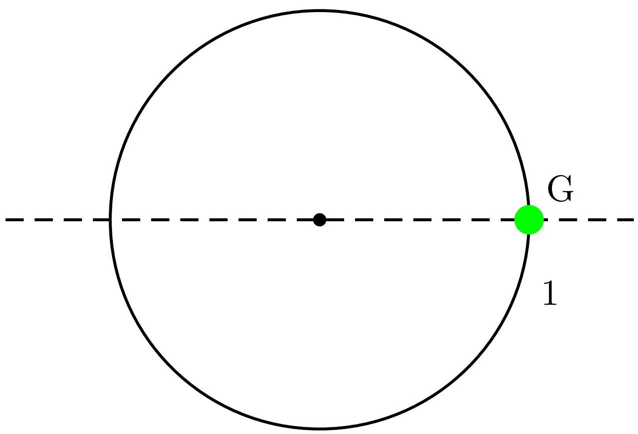
- Compute and use velocity in parametric curves
- Compute and use acceleration in parametric curves

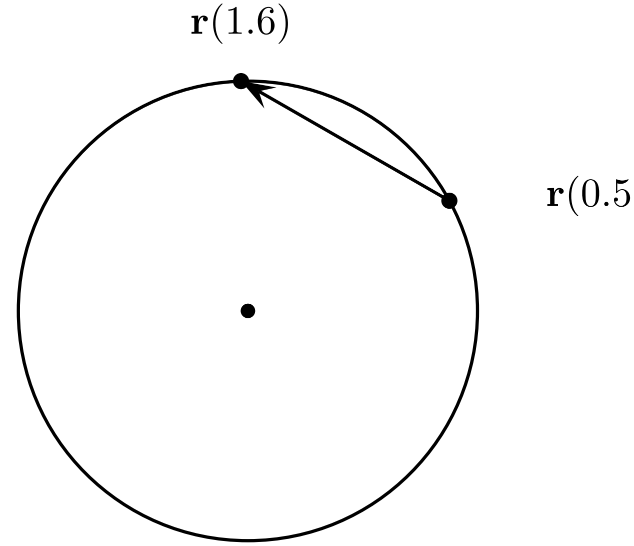
Displacement in Parametric Curves

Using the model of a particle moving along on a circle from earlier,

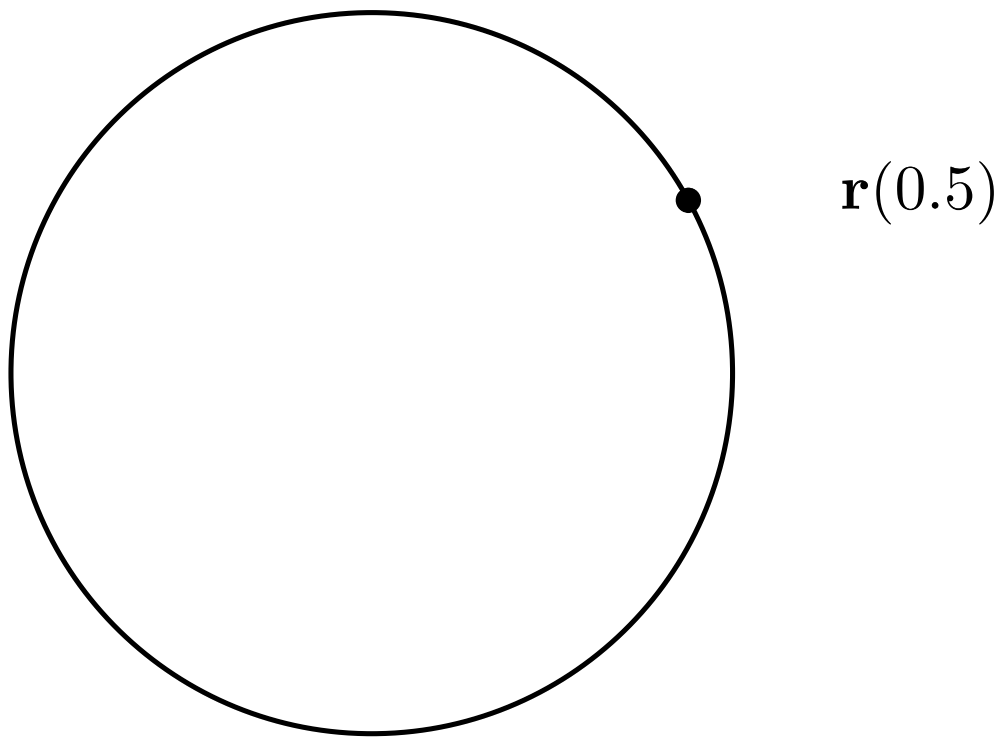
$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle \,,$$

find the **displacement** of the particle between the times $t = 0.5$ and $t = 1.6$.





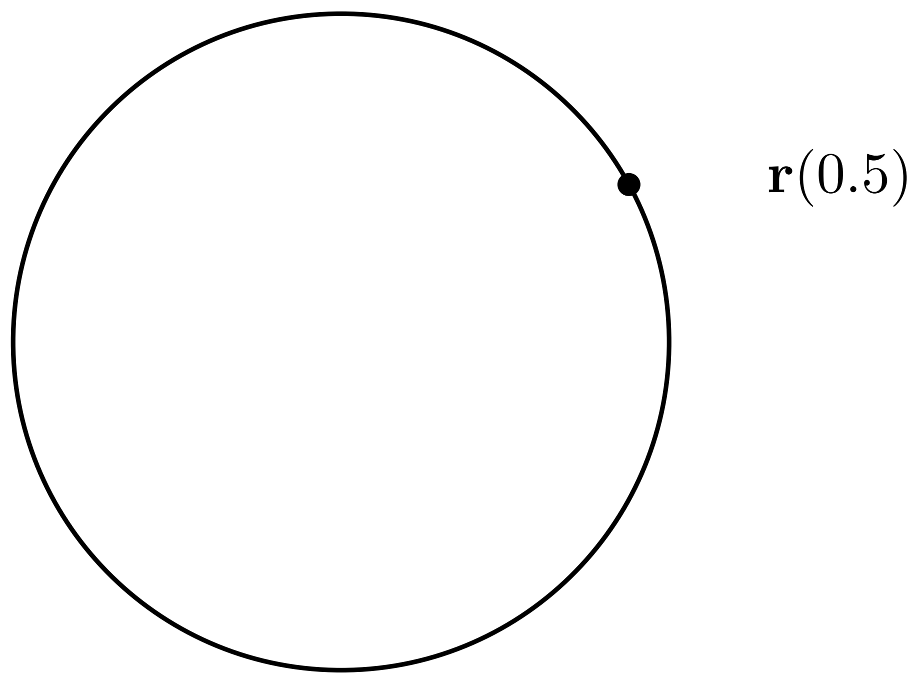
Problem. Suppose we replaced the upper t value ($t = 1.6$), by $0.5 + \Delta t$. Find a formula for this new displacement.



$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle \,,$$
$$\mathbf{d} = \langle \cos(0.5 + \Delta t) - \cos(0.5),\ \sin(0.5 + \Delta t) - \sin(0.5) \rangle$$

Problem. If we let Δt get smaller and smaller ($\Delta t \to 0$) then what will displacement vector do?

A. The displacement vector will keep shrinking as Δt gets smaller.
B. The displacement vector will stay more or less the same length because the speed is the same.



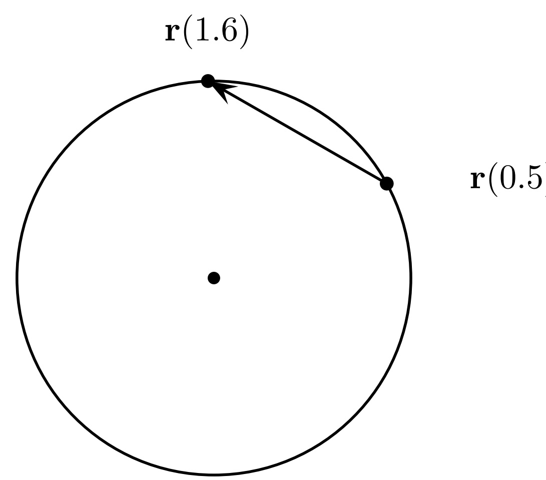
Velocity in Parametric Curves

We continue with our circular motion curve,

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle ,$$

As we shorten a time interval, the **displacement** over that interval will also get smaller: fair enough.
How does this relate to *velocity*?

Problem. Find the **average velocity** of the particle over the period $t = 0.5 \ldots 1.6$.



Problem. (Aside: what is the distinction between **velocity** and **speed**?)

Problem. For our circular motion example, the average velocity formula over the interval $t = 0.5$ to $t = 0.5 + \Delta t$ is:

$$\text{avg. vel.} = \frac{1}{\Delta t}\left\langle \cos(0.5+\Delta t) - \cos(0.5),\ \sin(0.5+\Delta t) - \sin(0.5)\right\rangle.$$

If we let Δt get smaller and smaller ($\Delta t \to 0$), then what will happen to the **velocity vector**?

A. The velocity vector will shrink to a point, because the displacement vector gets smaller and smaller;

B. The velocity vector will get longer and longer because we are dividing the displacement by a smaller and smaller number, Δt;

C. The velocity vector will stay more or less the same length.

We would like to study the effect of shrinking time intervals on average velocity more closely, because it **defines the derivative** for parametric functions.

Defining the Derivative of a Parametric Curve

Let us generalize: using $t = a$ as our first time point, and a time interval of Δt, write the formula of the **average velocity** over this time interval.

Indicate which parts of this calculation are vectors and which are scalars.

Note: since the independent variable isn't always time (t), mathematicians (and textbooks) often use the single-letter-variable h instead of Δt.

Problem. Rewrite the average velocity formula using this notation.

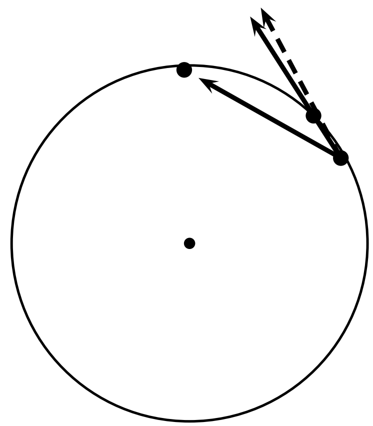
Finally, we capture the idea of **instantaneous** velocity (as opposed to **average** velocity).

Problem. Based on the earlier formula, how would we obtain the instantaneous velocity of $\mathbf{r}(t)$ at the time $t = a$?

Limit Definition of the Vector-Valued Derivative

Graphical Interpretation

If you imagine successive instances of average velocity as h gets smaller and smaller, you might get something like this diagram:



In the diagram you see vectors representing

- average velocity between $t = 0.5$ and 1.6 ($\Delta t = 1.1$) and
- average velocity between $t = 0.5$ and 0.8 ($\Delta t = 0.3$), and
- the *limit* (the dashed arrow) to which these average velocity vectors converge as $\Delta t \to 0$.

The limit vector is called the **derivative** of the vector-valued function $\mathbf{r}(t)$ at $t = a$. This derivative represents the **instantaneous velocity** of $\mathbf{r}(t)$ at $t = a$, and we write it as $\mathbf{r}'(a)$:

$$\mathbf{r}'(a) = \lim_{h \to 0} \frac{1}{h} \left[\mathbf{r}(a+h) - \mathbf{r}(a) \right]$$

or

$$\mathbf{r}'(a) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\mathbf{r}(a+\Delta t) - \mathbf{r}(a) \right]$$

Review: what is the definition of the derivative of a **single-variable** function, $f(t)$ at $t = a$?

Problem. Suppose a vector-valued function is given by the formula $\mathbf{r}(t) = \langle t, t^2 \rangle$. What is its derivative at time $t = a$?

When the position of an object is given by a vector-valued function

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle$$

then its velocity at a can be calculated by taking derivatives of the components and evaluating them at $t = a$:

$$\mathbf{r}'(a) = \langle x'(a), y'(a) \rangle$$

Velocity - Further Examples

Problem. Find the velocity of the particle whose position is given by

$$\mathbf{r}(t) = \langle \cos(t), e^t \rangle.$$

Problem. Find the velocity of the particle whose position is given by

$$\mathbf{q}(t) = \langle t^2 - \sqrt{t}, e^{4t} - t \rangle.$$

Problem. Find the velocity of the particle whose position is given by

$$\mathbf{s}(t) = \left\langle \ln(t), \frac{1}{t} \right\rangle.$$

Plotting Vector-Valued Derivatives With MATLAB

The following generates a plot of the trajectory defined by $\mathbf{r} = \langle t, t^2 \rangle$, over the time interval $-3 \leq t \leq 3$.

```
% Trajectory
t = -3:0.01:3;
plot(t, t.^2);
```

To include a velocity vector, say at t=0.5, you can use the `quiver` command.

```
% Velocity at t = 0.5
hold on;
quiver(0.5, 0.5^2, 1, 2*(0.5));
```

What do the elements in the `quiver` command represent?

Modifying our earlier animation, the following shows the velocity vector as the particle moves along the trajectory.

```
t = -3:0.05:3;
for (i = 1:length(t))
    clf;
    hold on;
    ti = t(i);  % current time in the loop
    x = ti;  % compute the current x and y coords
    y = ti^2;
    vx = 1; % compute the current velocity components
    vy = 2 * ti;
    plot(t, t.^2, '-b',  x, y, '.r', 'markersize', 25);
    quiver(x, y, vx, vy, 'r');
    ylim([-2,9]);
    drawnow;
end
```

Velocity - Components

Problem. A particle moves so that its location at time t is given by

$$\mathbf{r}(t) = \left\langle \frac{1}{3}t^3 - t,\ t \right\rangle$$

where the first coordinate measures position along a horizontal axis, and the second coordinate measures vertical position.

What is the formula for the **velocity** of the particle at any time t?

A. $\mathbf{v}(t) = \langle 3,\ 1 \rangle$

B. $\mathbf{v}(t) = \langle 3t^3 - t,\ 1 \rangle$

C. $\mathbf{v}(t) = \langle t^2 - 1,\ 1 \rangle$

D. $\mathbf{v}(t) = \langle t^2 - 1,\ t \rangle$

Problem. The particle will be moving in different directions at different times. What would be special about the velocity at the moments when the particle is moving directly **upwards** or **downwards**?

A. The **position**'s *vertical* component would be zero.

B. The **position**'s *horizontal* component would be zero.

C. The **velocity**'s *vertical* component would be zero.

D. The **velocity**'s *horizontal* component would be zero.

$$\mathbf{r}(t) = \left\langle \frac{1}{3}t^3 - t,\ \ t \right\rangle$$
$$\mathbf{r}'(t) = \mathbf{v}(t) = \langle t^2 - 1, 1 \rangle$$

Problem. At what value(s) of t is the particle moving in a perfectly vertical direction?

A. At $t = 1$

B. At $t = 0$

C. At $t = 0$ and $t = \pm\sqrt{3}$

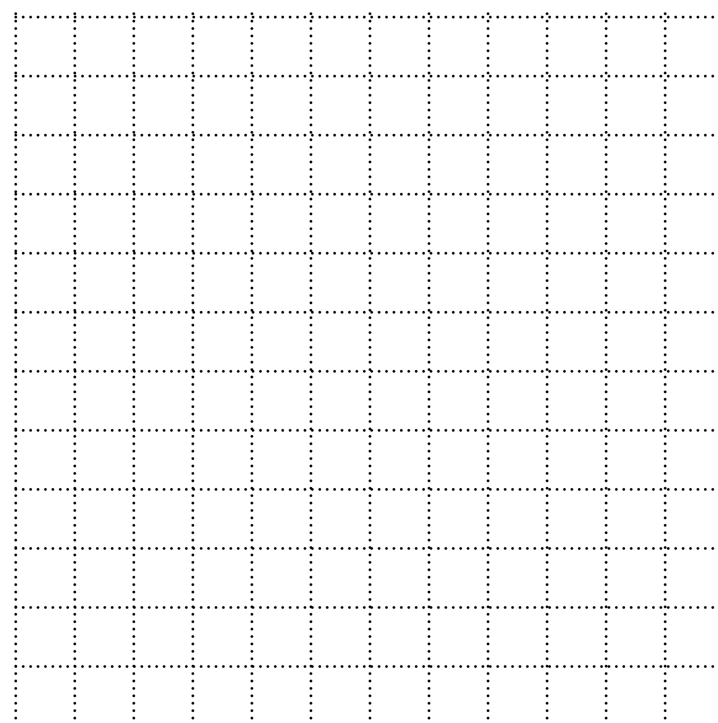
D. At $t = \pm 1$

Analyzing and Designing Circular Motion

Now that we have a method for computing velocities and speeds for vector-valued functions, we can use this to better understand and design trajectories with desired behaviours.

Problem. Sketch the trajectory defined by

$$\mathbf{r}(t) = \langle 5\cos(2t) + 3, -5\sin(2t) - 2\rangle$$



$$\mathbf{r}(t) = \langle 5\cos(2t) + 3, -5\sin(2t) - 2 \rangle$$

Analyze the relationship between the various elements in the formula and the resulting graph.

Problem. Design a circular trajectory that satisfies the following criteria (all units in meters and seconds):

- Radius of 8 m.
- Center at (4, 7).
- Constant speed of 15 m/s.
- Starts at the bottom of the circle (lowest y at time $t = 0$), and rotates clockwise.

- Radius of 8 m.
- Center at (4, 7).
- Constant speed of 15 m/s.
- Starts at the bottom, rotates clockwise.

- Radius of 8 m.
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Straight Line Trajectories

Problem. Indiana Jones is hanging on to a cart which is following a twisty track defined by its (x, y) position,

$$\mathbf{r}(t) = \left\langle 2 + t,\ t^3 - t \right\rangle$$

Unaware, the villain is counting his ill-gotten gold at coordinates $(5, -3)$. When should Indy jump so that he is launched on the perfect trajectory to tackle the evil-doer?

Hint: Newton's First Law is handy here...

$$\mathbf{r}(t) = \left\langle 2 + t,\ t^3 - t \right\rangle$$

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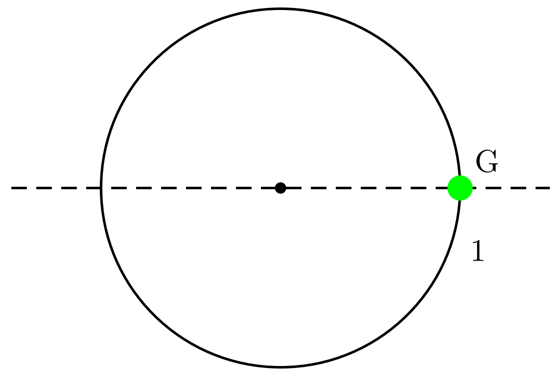
Acceleration in Parametric Curves

We will now return to our continuing (and simpler) example of a green dot moving in a circular path.

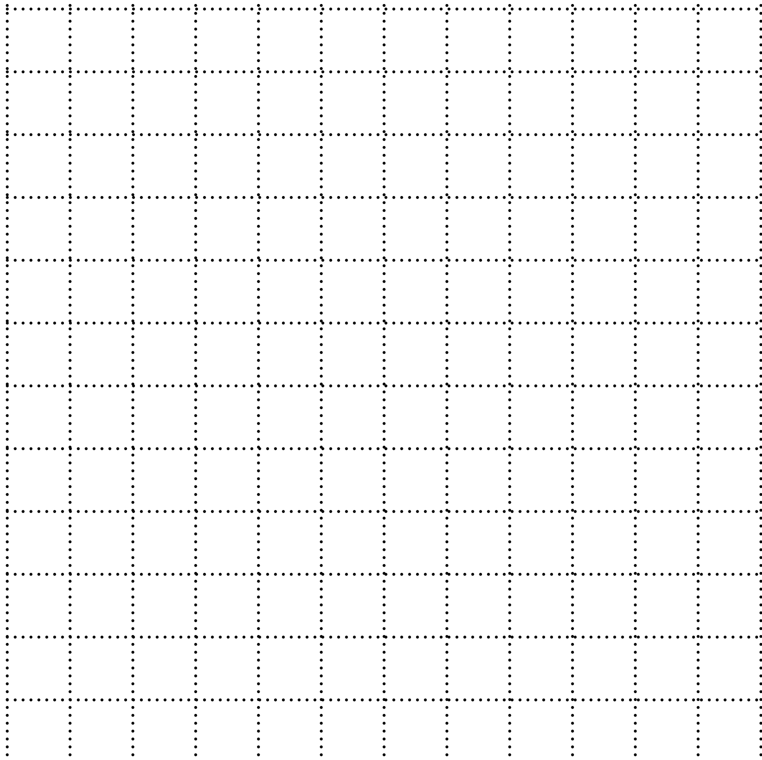


The path of the green dot was given, as a function of time, by the formula

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle \ .$$

Problem. Find the formula for the instantaneous **velocity** of the particle.

Problem. Sketch the path of the particle, and the velocity vector at several points.



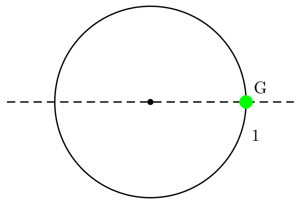
Should the velocity vectors all have the same length, or should they grow or shrink over time?

Now imagine what would it mean to take a **second** derivative of a vector-valued function. In physics we associate the **second derivative of position** with **acceleration**.

In calculus, in high school, you will have learned to relate the second derivative to the **concavity** of the graph of a function. We will see that both points of view continue to make sense when we deal with vector-valued functions.

The **second derivative of the position** of a moving particle measures how fast and in which direction a particle is **accelerating**. Also, when the acceleration vector points in a direction different from that of the velocity, it also indicates the "amount" by which the path **curves or bends**.

Think of the green dot as a vehicle with you as a passenger. As you zip around the circle, you will find yourself pushed against the side of the vehicle. This indicates that you are accelerating in the direction in which the vehicle pushes against your body. A greater speed or a tighter circle will produce a greater acceleration.

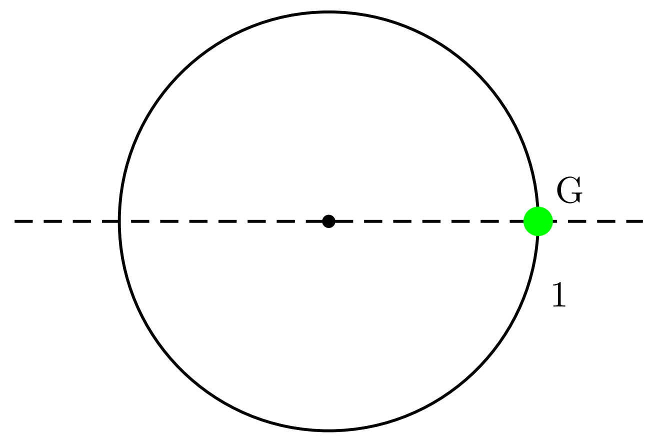


$$\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle,$$
$$\mathbf{r}'(t) = \mathbf{v}(t) = \langle -\sin(t), \cos(t) \rangle$$

Problem. For the circular motion defined by $\mathbf{r}(t)$, use derivative rules to compute the **instantaneous** acceleration function for the particle's motion.

Compute the **instantaneous** acceleration at $t = 0$.

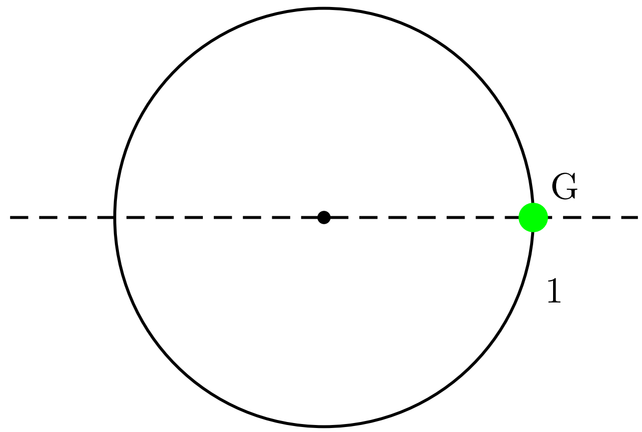
Problem. What statement is true about the **acceleration** of our orbiting dot?



A. The acceleration vector is constant.

B. The magnitude of the acceleration is constant, but the direction changes.

C. The acceleration magnitude alternates between increasing and decreasing.



Problem. Can you find a simple relationship between the position $\mathbf{r}$ and $\mathbf{r}''$?

Express this relationship in words.

Acceleration - General Circular Motion

Our simple circular motion is as simple as it can be (radius 1, and 'default' speed for sine and cosine).

Problem. Show that, for **any** uniform-speed circular motion, the acceleration of a particle must **always** be directly towards the center of the circle.
(Note that we'll first need a general formula for any uniform-speed circular motion!)

(continued)

Motion Over Time - Taxi Problem

Problem. Suppose you are a passenger in the back seat of a taxi that is speeding along so that its location at time t (seconds) is $\mathbf{r}(t) = \langle 100t^2, 10t \rangle$, measured in meters.

What is the shape of the path of the taxi?

$$\mathbf{r}(t) = \langle 100t^2, 10t \rangle$$

Problem. What is your acceleration at time $t = 0$?

$$\mathbf{r}(t) = \langle 100t^2, 10t \rangle$$

Problem. Given that your mass is 70 kilograms, and assuming that you forgot to put on your seat belt, with what force is the car door pressing against you at the instant $t = 0$?

Problem. In the same taxi scenario, as t increases from 0 to 10, which of the following alternatives best describes what happens?

A. The force of the door against your body stays the same.

B. The force of the door against your body decreases, but the force of the back of your seat against your body increases.

C. The force of the door against your side remains the same, but you find yourself sliding forward towards the back of the seat in front of you.

Spaceship Problem

We now complete this first unit by revisiting the problem posed at its beginning.

Suppose a spaceship moves in three-dimensional space so that at time $t \geq 0$ its coordinates are given as

$$\mathbf{r}(t) = \langle \cos(t),\ \sin(t),\ e^{-t} \rangle.$$

in meters, with time measured in seconds.

Problem. What does that tell us about the way this object moves? Try focusing on the xy coordinates and the z coordinates separately to simplify the problem.

$$\mathbf{r}(t) = \langle \cos(t),\ \sin(t),\ e^{-t} \rangle.$$

.

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), e^{-t} \rangle, \ t \geq 0$$

Problem. What is the spaceship's speed at time $t = 0$?

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), e^{-t} \rangle, \ t \geq 0$$

Problem. When will the spaceship experience the greatest acceleration?

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), e^{-t} \rangle, \;\; t \geq 0$$

Problem. If you were to fall off at time $t = 1$, what trajectory would you follow? (Assume we are in space, with effectively zero gravity.)

Problem. Something to challenge you: How far from the origin will you "land" on the (x, y)-plane if you fall off the spacecraft at time t?