Week 7: The Fundamental Theorem of Calculus

Goals:

- Identify properties of definite integrals
- Introduce the Fundamental Theorem of Calculus
- Compute simple anti-derivatives and definite integrals

Definite Integrals in Modeling

One of the primary applications of integration is to use a known **rate of change**, and compute the **net change** over some time interval.

Example: *Suppose water is flowing into/out of a tank at a rate given by $r(t) = 200 - 10t$ L/min, where positive values indicate the flow is into the tank.*

Write an integral that expresses the **change** *in the volume of water in the tank during the first 30 minutes of filling.*

(a) $\int_0^{30} (0 - 10)\, dt$

(b) $\int_0^{30} (200 - 10t)\, dt$

(c) $\int_0^{30} (200 - 10t)\Delta t\, dt$

(d) $\int_0^{30} \left(200t - 10\frac{t^2}{2}\right)\, dt$

Estimate the integral using a left-hand rule with three intervals.

Does this information tell you the actual volume in the tank after 30 minutes? Why or why not?

Question: If $h(t)$ represents the height of a child (in cm) at time t (in years), and the child is 120 cm tall at age 10, how would you represent the amount the child grew between $t = 10$ and $t = 18$ years?

A. $\int_{10}^{18} h(t)\, dt$

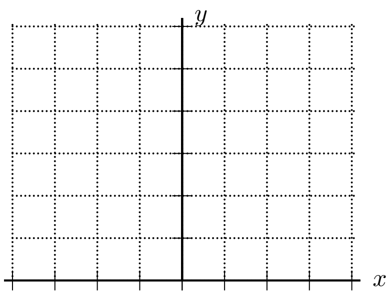
B. $\int_{10}^{18} h(t)\, dt + 120$

C. $\int_{10}^{18} h'(t)\, dt$

D. $\int_{10}^{18} h'(t)\, dt + 120$

Properties of Definite Integrals

Example: *Sketch the area implicit in the integral* $\int_{-\pi/3}^{\pi/3} \cos(x)\, dx$



If you were told that $\int_0^{\pi/3} \cos(x)\ dx = \frac{\sqrt{3}}{2}$, *find the size of the area you sketched.*

(a) $\int_{-\pi/3}^{\pi/3} \cos(x)\ dx = 0$

(b) $\int_{-\pi/3}^{\pi/3} \cos(x)\ dx = 4\frac{\sqrt{3}}{2}$

(c) $\int_{-\pi/3}^{\pi/3} \cos(x)\ dx = 2\frac{\sqrt{3}}{2}$

(d) $\int_{-\pi/3}^{\pi/3} \cos(x)\ dx = \frac{\sqrt{6}}{2}$

This example highlights an important and intuitive general property of definite integrals.

Additive Interval Property of Definite Integrals

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

Explain this general property in words and with a diagram.

A less commonly used, but equally true, corollary of this property is a second property:

Reversed Interval Property of Definite Integrals

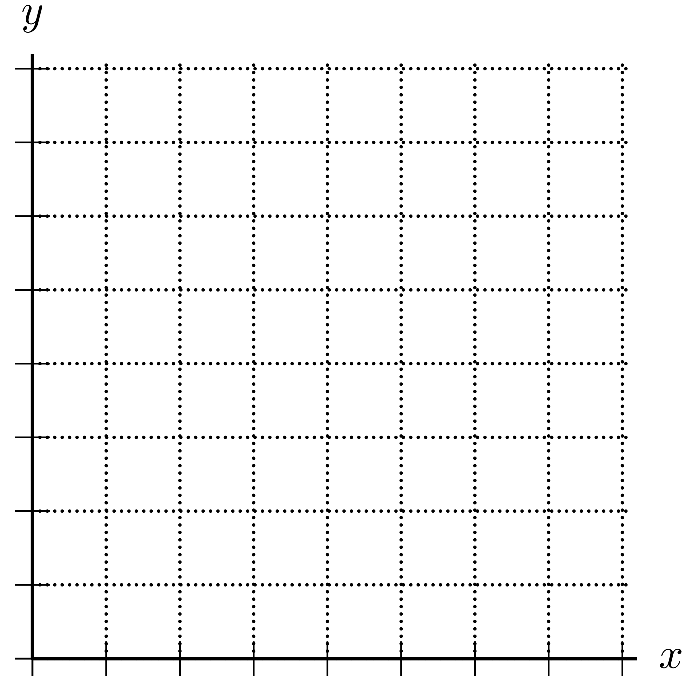
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Use the integral $\displaystyle\int_0^{\pi/3} \cos(x)\,dx + \int_{\pi/3}^{0} \cos(x)\,dx$, *and the earlier interval property, to illustrate the reversed interval property.*

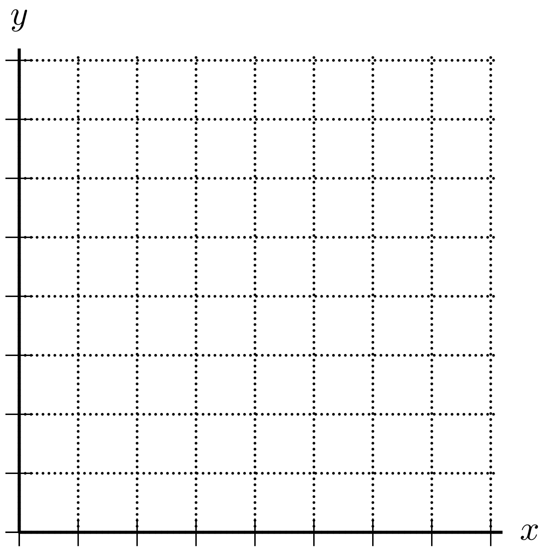
Give a rationale related to Riemann sums for the Reversed Interval property.

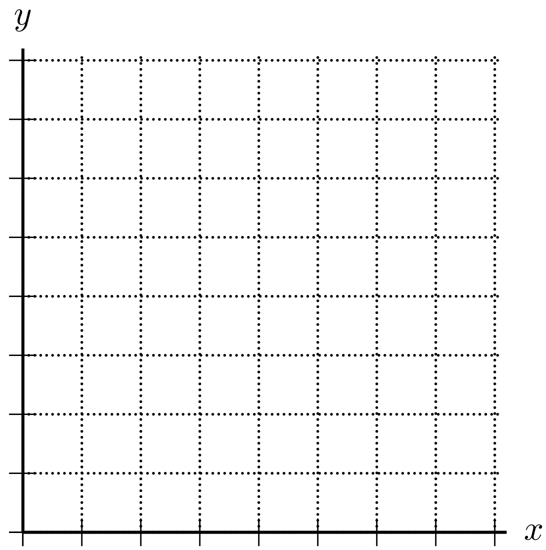
Linearity of Definite Integrals

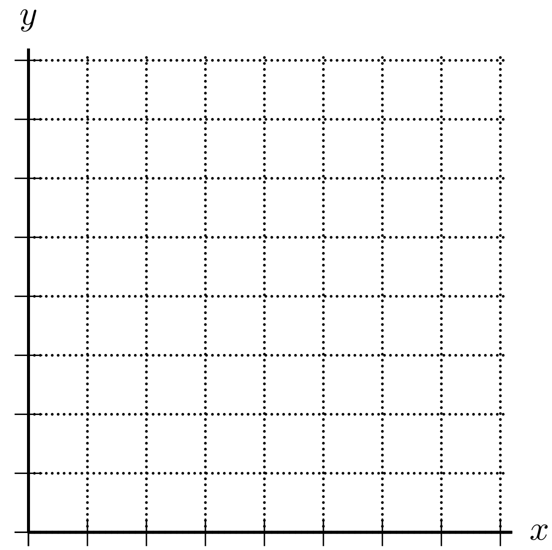
Example: *If* $\int_a^b f(x)\ dx = 10$, *then what is the value of* $\int_a^b 5f(x)\ dx$? *Sketch an area rationale for this relation.*



Example: *If* $\int_a^b f(x)\ dx = 2$*, and* $\int_a^b g(x)\ dx = 4$ *then what is the value of*
$\int_a^b [f(x) + g(x)]\ \ dx$*? Again, sketch an area rationale for this relation.*







Linearity of Definite Integrals

$$\int_a^b kf(x)\;dx = k\int_a^b f(x)\;dx$$

$$\int_a^b \left(f(x)+g(x)\right)\;dx = \int_a^b f(x)\;dx + \int_a^b g(x)\;dx$$

$$\int_a^b \left(f(x)-g(x)\right)\;dx = \int_a^b f(x)\;dx - \int_a^b g(x)\;dx$$

The Fundamental Theorem of Calculus

We have now drawn a firm relationship between area calculations (and physical properties that can be tied to an area calculation on a graph). The time has now come to build a method to compute these areas in a systematic way.

The Fundamental Theorem of Calculus

If f is continuous on the interval $[a, b]$, and we define a related function $F(x)$ such that $F'(x) = f(x)$, then

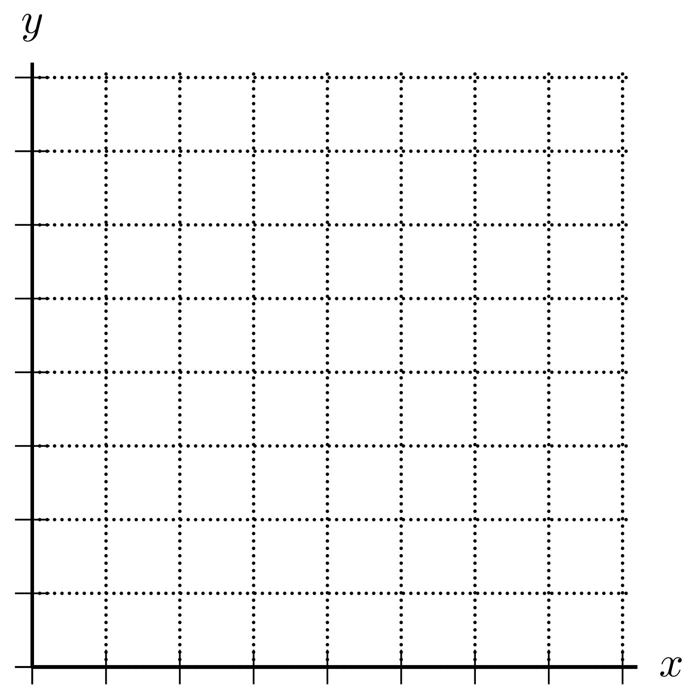
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The fundamental theorem ties the *area* calculation of a definite integral back to our earlier *slope* calculations from derivatives. However, it changes the direction in which we take the derivative:

- Given $f(x)$, we find the *slope* by finding the *derivative* of $f(x)$, or $f'(x)$.
- Given $f(x)$, we find the *area* $\int_a^b f(x)\,dx$ by finding $F(x)$ which is the *anti-derivative* of $f(x)$; i.e. a function $F(x)$ for which $F'(x) = f(x)$.

In other words, if we can find an anti-derivative $F(x)$, then calculating the value of the definite integral requires a simple evaluation of $F(x)$ at two points $(F(b) - F(a))$. This last step is *much* easier than computing an area using finite Riemann sums, and also provides an exact value of the integral instead of an estimate.

Example: *Use the Fundamental Theorem of Calculus to find the area bounded by the x-axis, the line $x = 2$, and the graph $y = x^2$. Use the fact that* $\dfrac{d}{dx}\left(\dfrac{1}{3}x^3\right) = x^2$.



We used the fact that $F(x) = \frac{1}{3}x^3$ *is an anti-derivative of* x^2, *so we were able use the Fundamental Theorem.*

Give another function $F(x)$ *which would also satisfy* $\frac{d}{dx}F(x) = x^2$.

Use the Fundamental Theorem again with this new function to find the area implied by $\int_0^2 x^2 \, dx$.

Did the area/definite integral value change? Why or why not?

Based on that result, give the most general version of $F(x)$ you can think of.

Confirm that you still satify $\dfrac{d}{dx}F(x) = x^2$.

With our extensive practice with derivatives earlier, we should find it straightforward to determine some simple *anti*-derivatives.

Complete the following table of anti-derivatives.

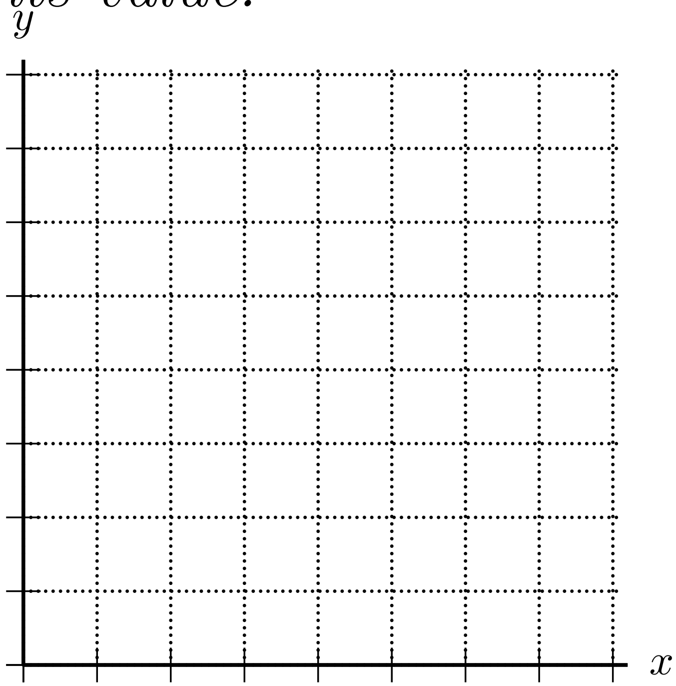
function $f(x)$	anti-derivative F(x)
x^2	$\frac{x^3}{3} + C$
x^n	
$x^2 + 3x - 2$	

function $f(x)$	anti-derivative F(x)
$\cos x$	
$\sin x$	
$x + \sin x$	

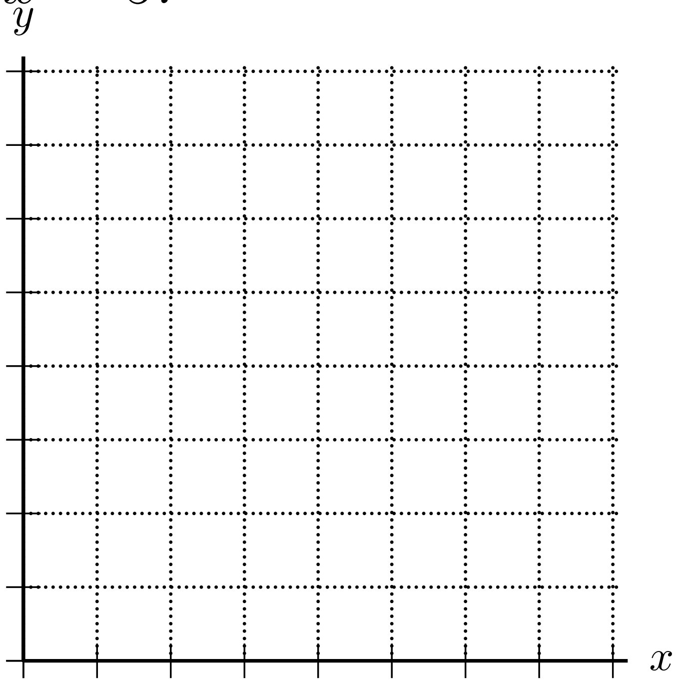
function $f(x)$	anti-derivative F(x)
e^x	
2^x	
$\frac{1}{\sqrt{1-x^2}}$	
$\frac{1}{1+x^2}$	
$\frac{1}{x}$	

The chief importance of the **Fundamental Theorem of Calculus (F.T.C.)** is that it enables us (potentially at least) to find values of definite integrals more accurately and more simply than by the method of calculating Riemann sums. In principle, the F.T.C. gives a precise answer to the integral, while calculating a (finite) Riemann sum gives you no better than an approximation.

Example: *Consider the area of the triangle bounded by* $y = 4x$, $x = 0$ *and* $x = 4$. *Compute the area based on a sketch, and then by constructing an integral and using anti-derivatives to compute its value.*



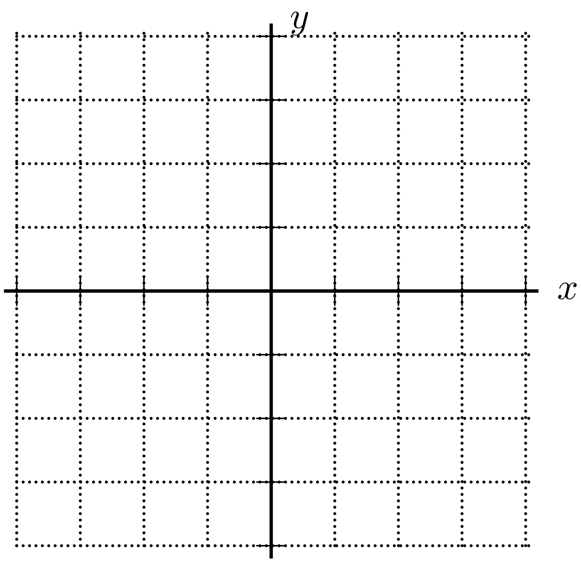
Example: *Use a definite integral and anti-derivatives to compute the area under the parabola* $y = 6x^2$ *between* $x = 0$ *and* $x = 5$.



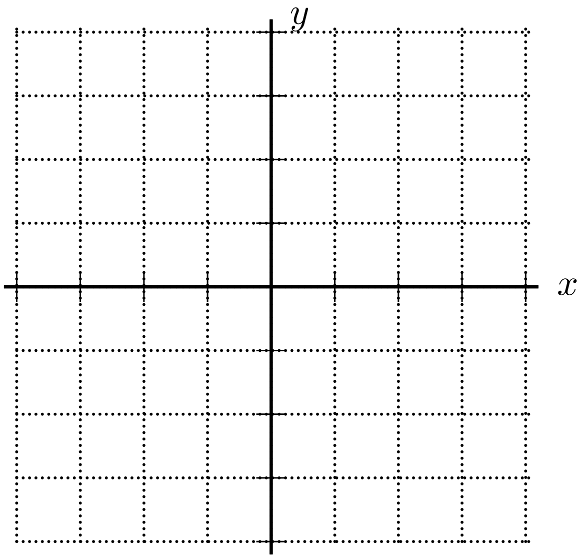
Basic Anti-Derivatives - 1/x

The last entry in our anti-derivative table was $f(x) = \frac{1}{x}$. It is a bit of a special case, as we can see in the following example.

Example: *Sketch the region implied by the integral* $\int_{-3}^{-1} \frac{1}{x}\, dx$.



Example: *Now use the anti-derivative and the Fundamental Theorem of Calculus to obtain the exact area under* $f(x) = \dfrac{1}{x}$ *between* $x = -3$ *and* $x = -1$. *Make any necessary adaptations to our earlier anti-derivative table.*


y
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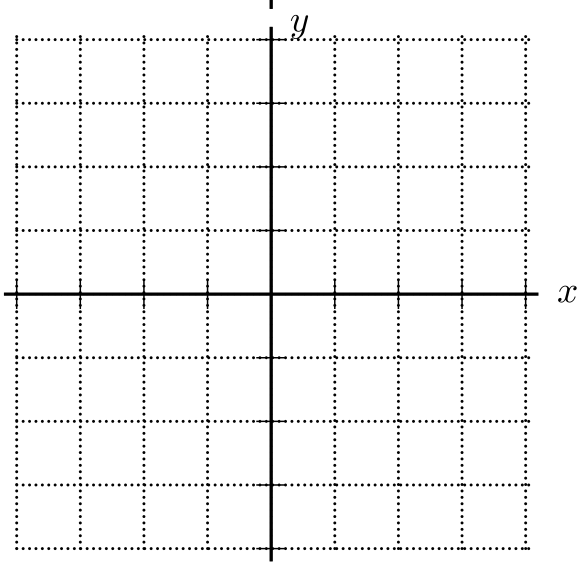
y
x

Anti-derivatives and the Fundamental Theorem of Calculus

The F.T.C. tells us that if we want to evaluate

$$\int_a^b f(x)\,dx$$

all we need to do is find an anti-derivative $F(x)$ of $f(x)$ and then evaluate $F(b) - F(a)$.

THERE IS A CATCH. While in many cases this really is very clever and straightforward, in other cases finding the anti-derivative can be surprisingly difficult. This week, we will stick with simple anti-derivatives; in later weeks we will develop techniques to find more complicated anti-derivatives.

Some general remarks at this point will be helpful.

Remark 1

Because of the importance of finding an anti-derivative of $f(x)$ when you want to calculate $\int_a^b f(x)\,dx$, it has become customary to denote the anti-derivative itself by the symbol

$$\int f(x)\,dx$$

The symbol $\int f(x)\,dx$ (with no limits on the integral) refers to the anti-derivative(s) of $f(x)$, and is called the **indefinite integral** of $f(x)$

Note that the definite integral is a number, but the indefinite integral is a function (really a family of functions).

Remark 2

Since there are always infinitely many anti-derivatives, all differing from each other by a constant, we customarily write the anti-derivative as a family of functions, in the form $F(x) + C$. For example,

$$\int x^2\,dx = \frac{x^3}{3} + C$$

Note that *an anti-derivative* is a single function, while the *indefinite integral* is a family of functions.

Remark 3

Since the last step in the evaluation of the integral $\int_a^b f(x)\,dx$, once the anti-derivative $F(x)$ is found, is the evaluation $F(b) - F(a)$, it is customary to write $F(x)\Big|_a^b$ in place of $F(b) - F(a)$, as in

$$\int_0^4 x^2\,dx = \frac{x^3}{3}\Bigg|_0^4 = \frac{4^3}{3} - \frac{0^3}{3}$$

Remark 4

The variable x in the definite integral $\int_a^b f(x)\,dx$ is called the *variable of integration*. It can be replaced by another variable name without altering the value of the integral.

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(\theta)\,d\theta$$

Net Change Theorem

Note that we create an anti-derivative $F(x)$, we are building it such that $f(x) = F'(x)$. This means that f gives the rate of change of F. Notice that this observation was made much earlier, when we started our discussion of integration: when an integral is associated with a process of **accumulation** then the **rate of accumulation** is always precisely the integrand.

Consider $F(x)$ as the quantity we are tracking, so F' is its rate of change. Another statement of the Fundamental Theorem of Calculus Part 2 would then be

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

"The integral of a rate of change is the total change". The textbook calls this the **Net Change Theorem**.

Problem. If a car is moving at $v(t) = 2t$ m/s from $t = 1$ to $t = 5$, what does the quantity $\int_1^5 v(t)\, dt$ represent?

A. The position of the car at $t = 5$, starting at $t = 1$.

B. The net change in position of the car between $t = 1$ and $t = 5$.

C. The velocity of the car at $t = 5$, starting at $t = 1$.

D. The net change in velocity of the car between $t = 1$ and $t = 5$.

Problem. If $h(t)$ represents the height of a child (in cm) at time t (in years), and the child is 120 cm tall at age 10, which of the following would represent the amount the child grew between $t = 10$ and $t = 18$ years?

A. $\displaystyle\int_{10}^{18} h(t)\ dt$

B. $\displaystyle\int_{10}^{18} h(t)\ dt + 120$

C. $\displaystyle\int_{10}^{18} h'(t)\ dt$

D. $\displaystyle\int_{10}^{18} h'(t)\ dt + 120$

Problem. Suppose water is flowing into/out of a tank at a rate given by $r(t) = 200 - 10t$ L/min, where *positive* rates indicate flow *in*. By how much does the water level in the tank change during the first 45 minutes after $t = 0$?

What is an assumption you would have to make about the initial amount of water in the tank for this to make sense?