Week 8: Integration by Parts and Substitution

Goals:

- The guess-and-check method for anti-differentiation.
- The substitution method for anti-differentiation.
- Learning integration by parts.
- Applying a slicing approach to construct integrals for areas.

We now return to the challenge of finding a *formula* for an anti-derivative function. We saw simple cases last week, and now we will extend our methods to handle more complex integrals.

Anti-differentiation by Inspection: The Guess-and-Check Method

Often, even if we do not see an anti-derivative immediately, we can make an educated guess and eventually arrive at the correct answer.

Example: *Based on your knowledge of derivatives, what should the anti-derivative of* $\cos(3x)$, $\int \cos(3x)\ dx$, *look most like?*

(a) $\cos(x) + C$

(b) $\sin(x) + C$

(c) $\cos(3x) + C$

(d) $\sin(3x) + C$

Evaluate $\int \cos(3x)\ dx$ *by guessing and checking your antiderivative.*

Example: *Find* $\int e^{7x-2}\,dx$.

Example: *Both of our previous examples had* linear *'inside' functions. Here is an integral with a* quadratic *'inside' function:*

$$\int xe^{-x^2}\,dx$$

Evaluate the integral.

Why was it important that there be a factor x *in front of* e^{-x^2} *in this integral?*

Integration by Substitution

We can formalize the guess-and-check method by defining an *intermediate variable* the represents the "inside" function.

Example: *Show that* $\int x^3\sqrt{x^4+5}\,dx = \frac{1}{6}(x^4+5)^{3/2} + C$.

$$\int x^3\sqrt{x^4+5}\,dx = \frac{1}{6}(x^4+5)^{3/2} + C$$

Relate this result to the **chain rule**.

Now use the **method of substitution** *to evaluate* $\int x^3\sqrt{x^4+5}\,dx$

Steps in the Method Of Substitution

1. Select a simple function $w(x)$ that appears in the integral.
   - Typically, you will also see w' as a **factor** in the integrand as well.
2. Find $\dfrac{dw}{dx}$ by differentiating. Write it in the form $\ldots dw = \ dx$
3. Rewrite the integral using only w and dw (no x nor dx).
   - If you can now evaluate the integral, the substitution was effective.
   - If you cannot remove all the x's, or the integral became harder instead of easier, then either try a different substitution, or a different integration method.

Example: *Find* $\int \tan(x)\ dx$.

Though it is not required unless specifically requested, it can be reassuring to check the answer.

Verify that the anti-derivative you found is correct.

Example: *Find* $\int x^3 e^{x^4-3} dx$.

Example: *For the integral,*

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2} dx$$

both $w = e^x - e^{-x}$ *and* $w = e^x + e^{-x}$ *are seemingly reasonable substitutions.*

Question: Which substitution will change the integral into the simpler form?

(a) $w = e^x - e^{-x}$

(b) $w = e^x + e^{-x}$

Compare both substitutions in practice.

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\, dx$$

with $w = e^x - e^{-x}$	with $w = e^x + e^{-x}$

Example: *What substitution is most likely to be helpful for evaluating the integral*

$$\int \frac{\sin(x)}{1+\cos^2(x)}dx \ ?$$

(a) $w = \sin(x)$

(b) $w = \cos(x)$

(c) $w = 1 + \cos^2(x)$

$$Evaluate \int \frac{\sin(x)}{1+\cos^2(x)}dx.$$

Using the Method of Substitution for Definite Integrals

If we are asked to evaluate a **definite** integral such as

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx \quad ,$$

where a substitution will ease the integration, we have two methods for handling the limits of integration ($x = 0$ and $x = \pi/2$).

a) When we make our substitution, convert both the *variables* x and the *limits* (in x) to the new variable; or

b) do the integration while keeping the limits explicitly in terms of x, writing the final integral back in terms of the original x variable as well, and *then* evaluating.

Example: *Use method a), converting the bounds to the new variable, to evaluate the integral*

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$$

Example: *Use method b) method, keeping the bounds in terms of* x*, and converting back to* x*'s, to evaluate*

$$\int_9^{64} \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx \quad .$$

Integration by Parts

So far in studying integrals we have used

- direct anti-differentiation, for relatively simple functions, and
- integration by substitution, for some more complex integrals.

However, there are many integrals that can't be evaluated with these techniques.

Try to find $\int xe^{4x}\,dx$.

This particular integral can be evaluated with a different integration technique, **integration by parts.** This rule is related to the **product rule** for derivatives.

Expand

$$\frac{d}{dx}(uv) =$$

Integrate both sides with respect to x and simplify.

Express $\int u\frac{dv}{dx}\,dx$ *relative to the other terms.*

Integration by Parts

For short, we can remember this formula as

$$\int u dv = uv - \int v du$$

Integration by parts:

- Choose a part of the integral to be u, and the remaining part to be dv.
- **Differentiate** u to get du.
- **Integrate** dv to get v.
- Replace $\int u\ dv$ with $uv - \int v du$.
- Hope/check that the new integral is easier to evaluate.

$$\int u\,dv = uv - \int v\,du$$

Example: *Use integration by parts to evaluate* $\int xe^{4x}\,dx$.

Verify that your anti-derivative is correct.

$$\int u dv = uv - \int v du$$

Guidelines for selecting u and dv

- Try to select u and dv so that either
  - u' is simpler than u or
  - $\int dv$ is simpler than dv
- Ensure you can actually integrate the dv part by itself

Example: *Consider the integral* $\int x \cos x \; dx$.

Based on the guidelines, what choice of u and dv should you try first?

(a) $u = x\cos(x)$, $dv = dx$.

(b) $u = 1$, $dv = x\cos(x)\; dx$.

(c) $u = x$, $dv = \cos(x)\; dx$.

(d) $u = \cos(x)$, $dv = x\; dx$.

Evaluate the integral $\int x \cos x \; dx$.

Verify that your anti-derivative is correct.

Example: *Evaluate the slightly more challenging integral*

$$\int x^2 \cos x \; dx$$

What choice of u and dv would be most likely to be helpful?

(a) $u = 1$, $dv = x^2 \cos(x) \; dx$.

(b) $u = x$, $dv = x \cos(x) \; dx$.

(c) $u = x^2$, $dv = \cos(x) dx$.

(d) $u = x^2 \cos(x)$, $dv = dx$.

$$\int x^2 \cos x \; dx$$

$$\int x^2 \cos x \, dx$$

Integration By Parts - Definite Integrals

When using integration by parts to evaluate *definite* integrals, you need to apply the limits of integration to the **entire** anti-derivative that you find.

Example: *Evaluate* $\int_0^{\pi} x \sin(4x)\ dx$

Don't forget that dv does not require any other factors besides dx. That can help when there is only a single factor in the integrand.

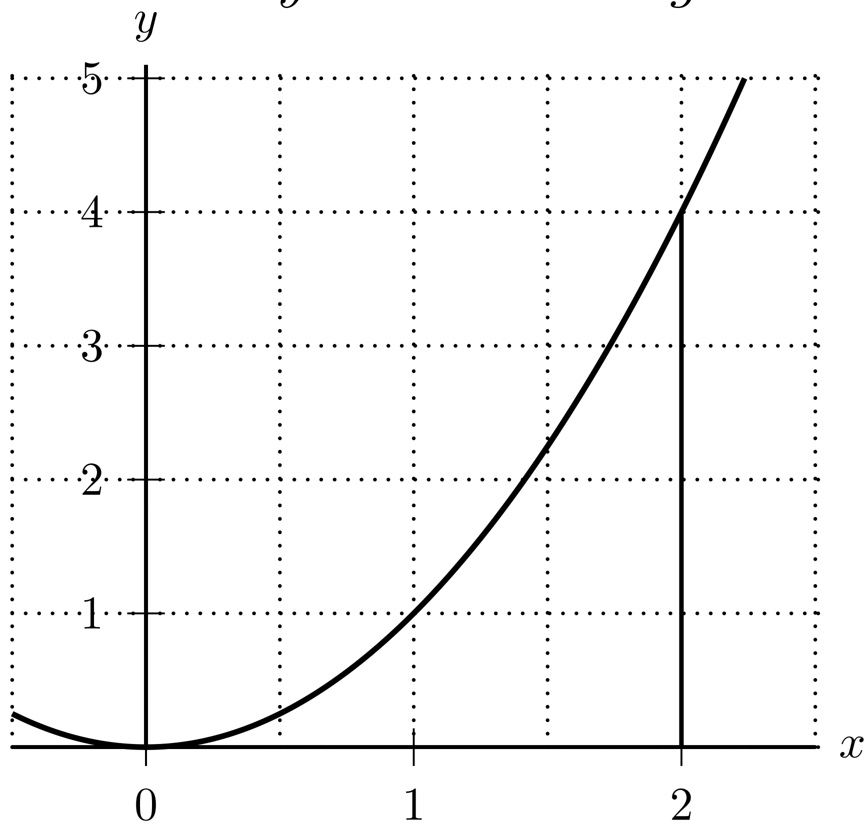
Example: *Find* $\displaystyle\int_1^2 \ln x \; dx$

Integrals as Areas - Review

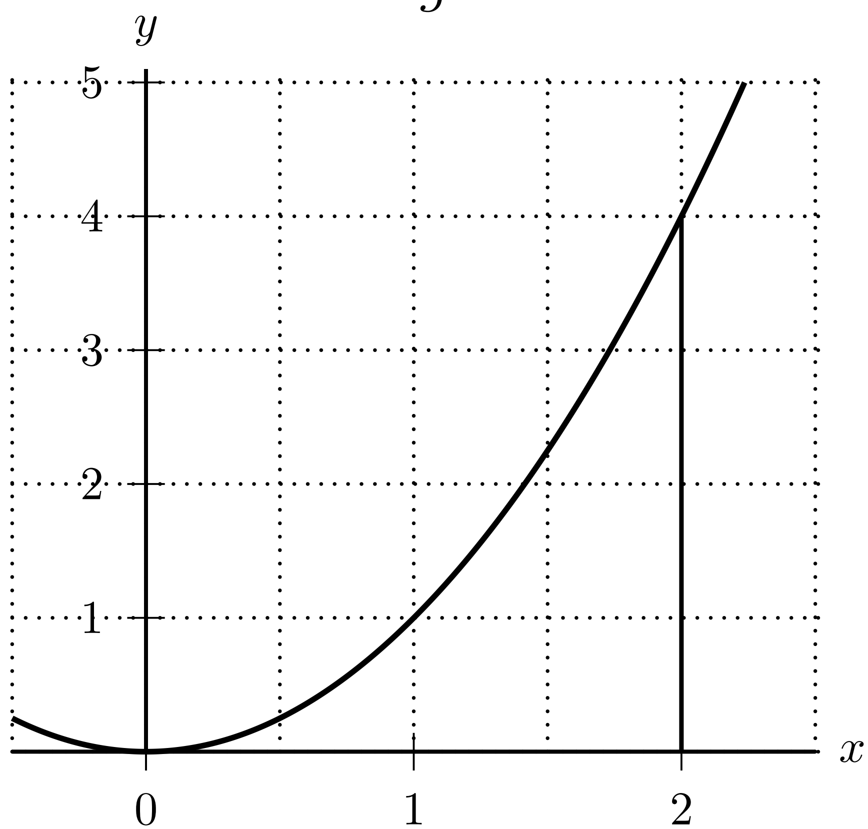
Example: *Write the integral that represents the area underneath the graph $y = x^2$ between $x = 0$ and $x = 2$.*

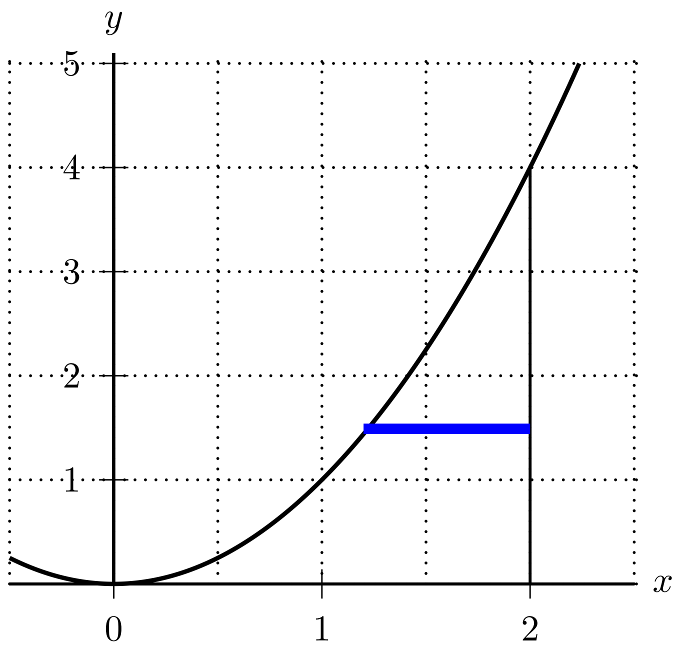
Evaluate the integral to find the area.

Illustrate how this region under the graph of $y = x^2$ can be constructed by accumulating small rectangles of varying heights.



Now show how the exact same area can be constructed by using **horizontal** *rectangles. Write the analogous intervals, widths, etc. on the diagram.*





What is the **width** of each rectangle, given its y location?

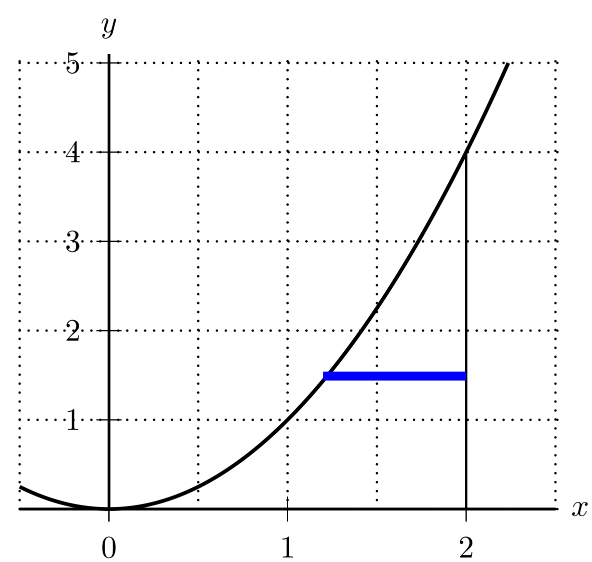
(a) width $= \sqrt{y}$

(b) width $= \sqrt{y} - 2$

(c) width $= 2 - \sqrt{y}$

(d) width $= 4 - \sqrt{y}$

Write out first a sum, then a new integral, that would represent the exact area in the sketch. Evaluate the integral.

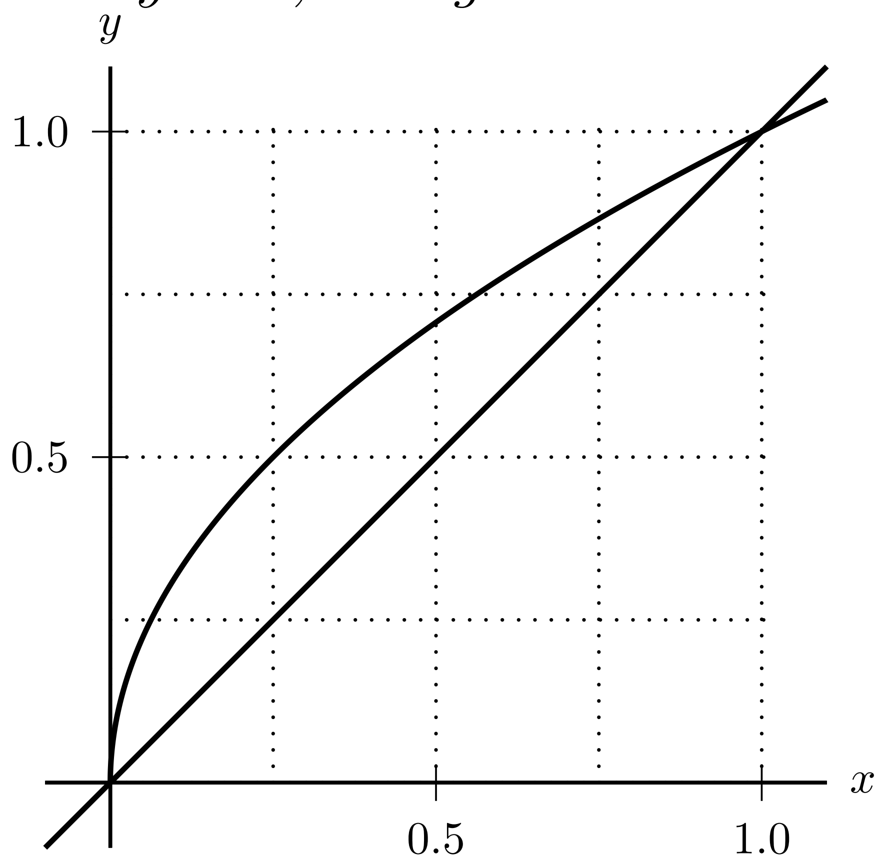


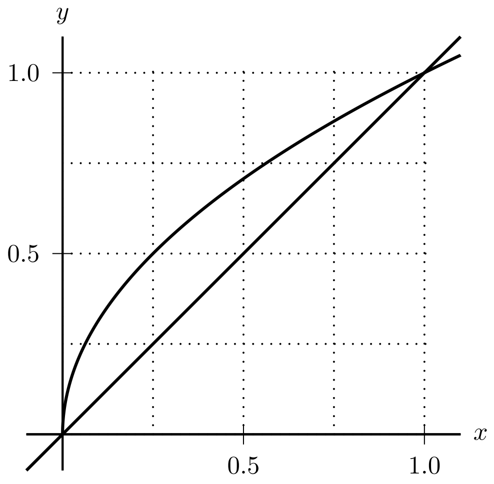
Integrals as Sums of Slices

Most people visualize this approach as *slicing* the shape into thin pieces. To find the total area, the process is:

- decide along which axis you want to slice (sayslices perpendicular to x)
- find the size of a **generic** slice, as a function of the position x
- write out the sum that represents to the total you want
- transform the sum into an integral
- evaluate the integral

Example: *Find the area enclosed between the graphs of* $x = y^2$ *and* $y = x$, *using horizontal slices.*




y
1.0
0.5
x
0.5
1.0