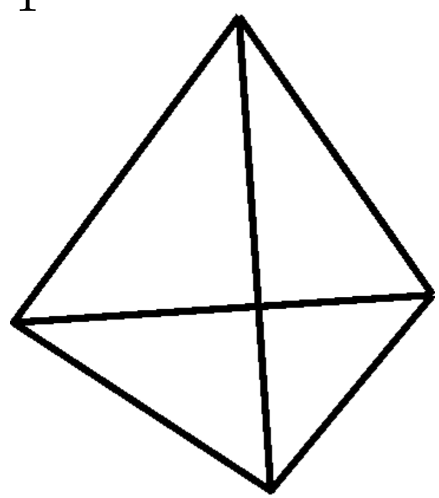
Week 9: Integrals for Volume and Work; Partial Fractions

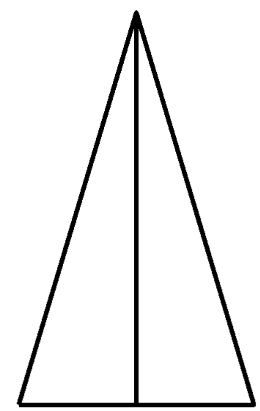
Goals:

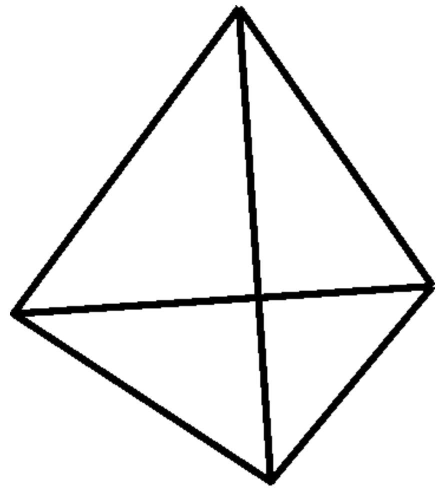
- Applying slicing approaches to construct integrals for volumes and work.
- Evaluating integrals using partial fractions.

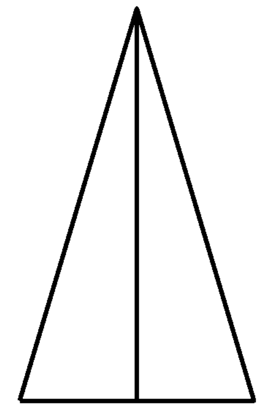
Pyramid Volume

Example: *A pyramid with its base being an equilateral triangle with sides 3 units long, is 5 units high. What is its volume?*
Helpful fact: the area of an equilateral triangle with all sides length a is $\frac{\sqrt{3}}{4}\,a^2$ *square units.*



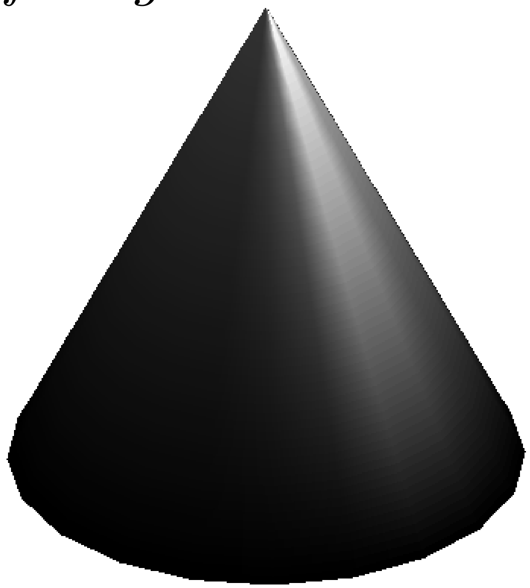


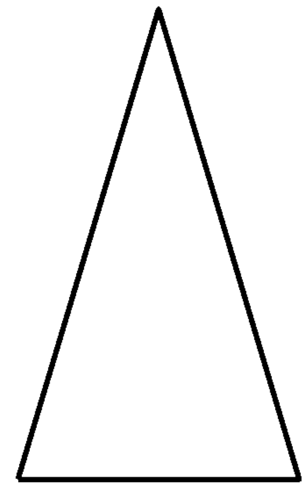


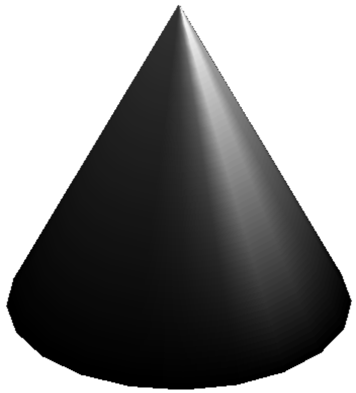


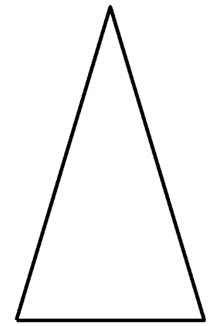
Cone Volume

Example: *Use a similar slicing strategy to find the volume of a cone of height h and bottom radius r.*





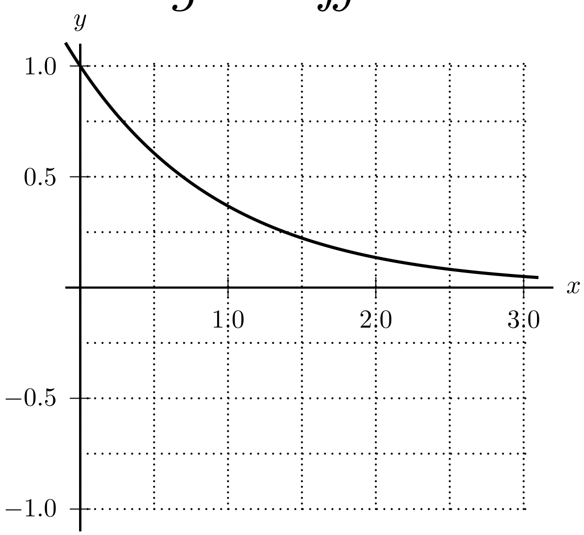


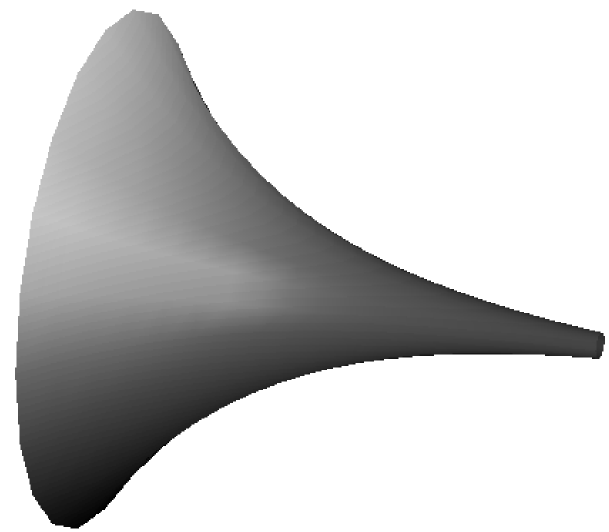


Volumes of Revolution - Introduction

We can extend the cone example to find the volume of more complex "spun" shapes. These are often called **volumes of revolution**.

Example: *Consider the graph of* $y = e^{-x}$ *shown below, and the solid we would build if we "spun" this shape around the* x *axis, cutting it off at* $x = 0$ *and* $x = 3$.





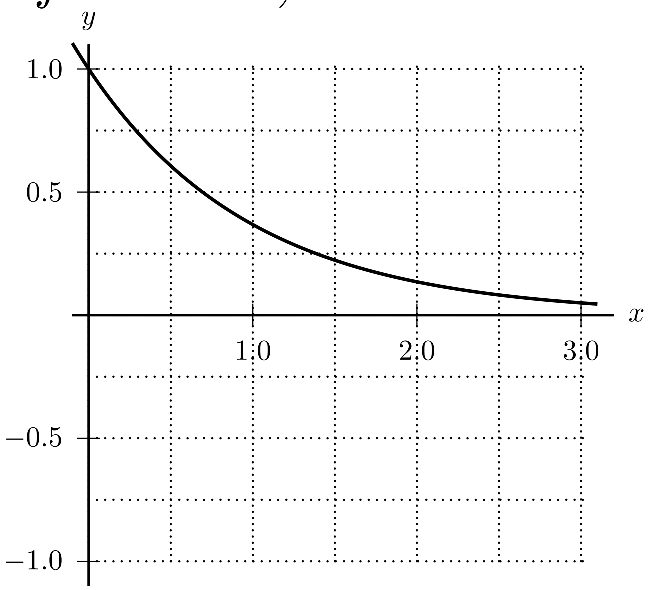
What is the shape of any cut made perpendicular to the x *axis?*

(a) Circle.

(b) Oval.

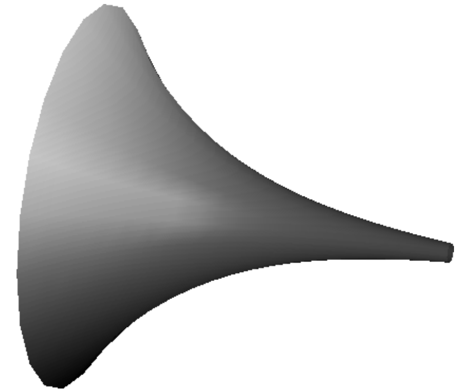
(c) No consistent shape.

Express the **volume** *of a cut, Δx thick, in terms of the location of the cut, x.*



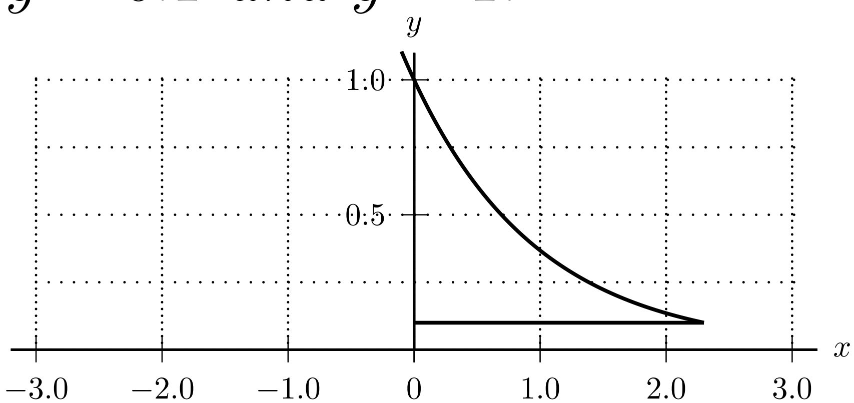
Write down an integral that represents the total volume of the shape.

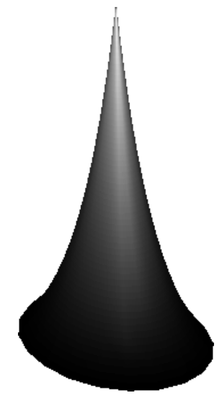
Evaluate the integral for the volume of the shape.



Volumes of Revolution - Changing Rotation Axis

Example: *Now consider the shape we would get if we spun* $y = e^{-x}$ *around the* y *axis. Suppose we cut the region off between* $y = 0.1$ *and* $y = 1$.

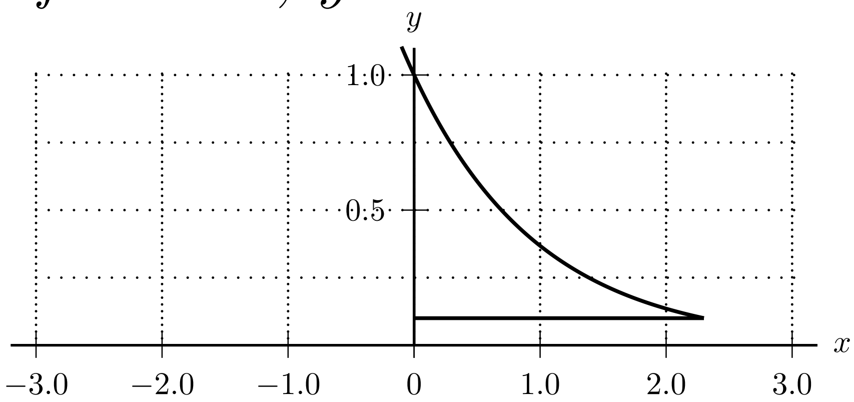




What is the shape of any cut made perpendicular to the x *axis? Would this make a good way to cut up the shape?*

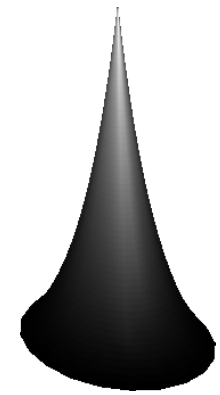
What is the shape of any cut made perpendicular to the y *axis? Would this make a good way to cut up the shape?*

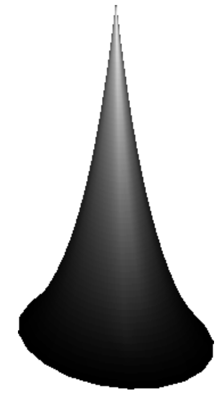
Express the volume of a cut, Δy thick, in terms of the location of the cut, y.



Write down an integral that represents the total volume of the shape.

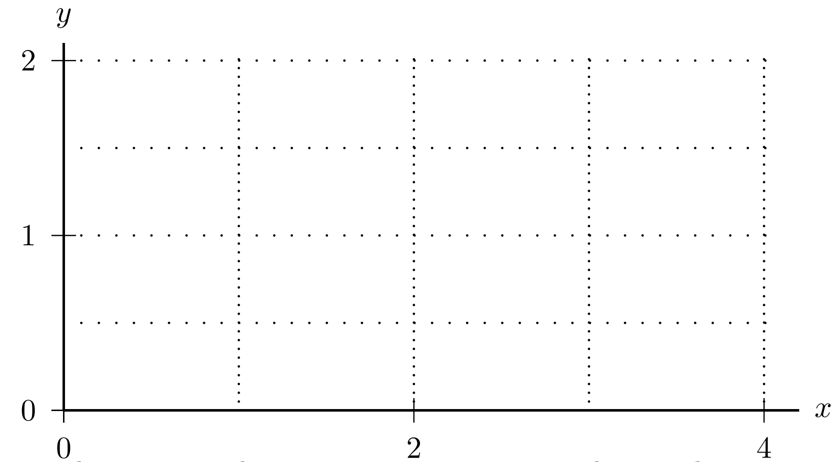
Evaluate the integral from the previous page to find the volume of the shape.





Volumes of Revolution - Slices As Rings

Example: *The region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 4$ is rotated around the y-axis.*



What shape will you get if you slice this volume perpendicular to the y axis?

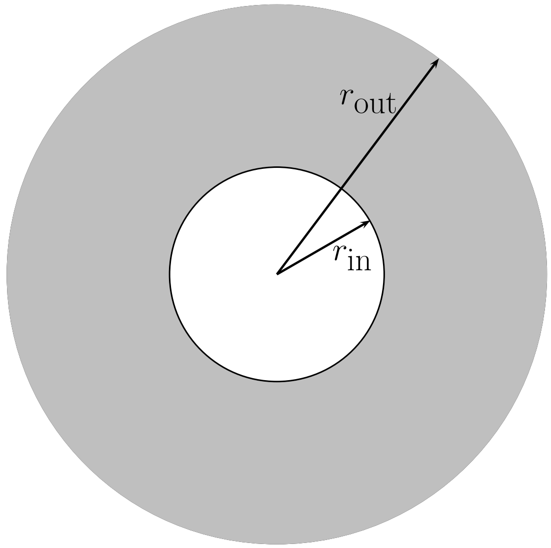
(a) Circles.

(b) Ovals.

(c) Rings.

(d) No consistent shape.

For a ring with the two radii shown, what is the area of the ring shown in grey?

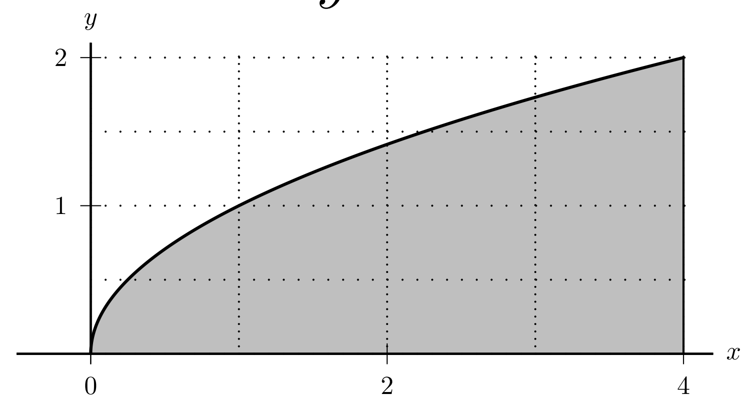


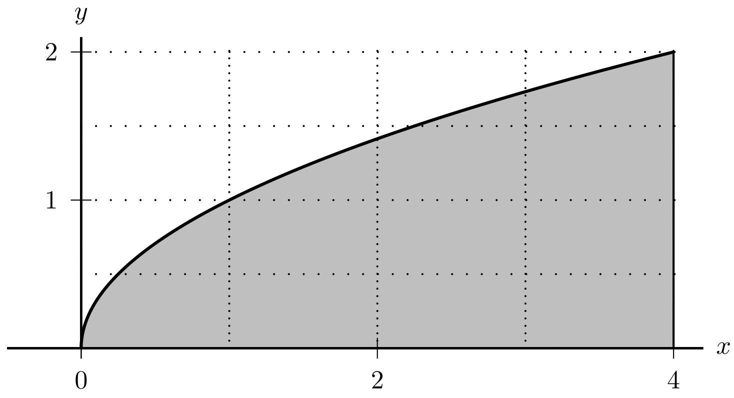
(a) Grey area $= r_{\text{out}}^2 - r_{\text{in}}^2$

(b) Grey area $= \pi r_{\text{out}}^2 - \pi r_{\text{in}}^2$

(c) Grey area $= \pi(r_{\text{out}} - r_{\text{in}})^2$

Compute the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 4$ around the y-axis.




y
2
1
0
2
4
x

Partial Fractions

This is the last of the techniques of integration (that is, techniques for anti-differentiation) covered in this course. The method of *Partial Fractions* is purely algebraic. It consists of a series of algorithmic steps that simplify the integrand so that an anti-derivative can be easily found.

For example, consider the integral

$$\int \frac{x+5}{x^2+x-2}\,dx.$$

List the methods of integration you might try to evaluate this integral, and why they might or might not work.

The method of *partial fractions* is used only for expressions of the form

$$\frac{P(x)}{Q(x)} \quad \text{(rational function)},$$

where P and Q are polynomials.
A *proper* rational function is one for which the degree of P is strictly less than the degree of Q. To be able to use the method of Partial Fractions, we *must first* make sure that the integrand is a proper rational function; this is our first step.

Step 1.

Turn $\frac{P(x)}{Q(x)}$ into an expression involving a proper rational function and a polynomial:

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}.$$

Separate the rational function $\frac{x^2 - 4x + 3}{x - 5}$ into a polynomial and a proper rational function.

Problem. Find $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx.$

The next step (after we have turned the integral into one involving a proper rational) consists of four cases, distinguished by what happens when you factor the denominator.

<u>Step 2. CASE I</u>:

The denominator is the product of *distinct linear* factors. Say

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k).$$

The goal is to look for numbers $A_1, \cdots, A_k$ so that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_k}{a_kx + b_k}.$$

“Partial Fractions”

Partial Fractions - Examples

Problem. Find $\displaystyle\int \frac{x}{x^2+3x+2}\,dx.$

$$\int \frac{x}{x^2 + 3x + 2} \, dx.$$

Problem. Find $\displaystyle\int \frac{3x^2 - 2}{(x-1)(x-2)(x+1)}\, dx.$

$$\int \frac{3x^2 - 2}{(x-1)(x-2)(x+1)}\,dx.$$

$$\int \frac{3x^2 - 2}{(x-1)(x-2)(x+1)}\, dx.$$

Step 2. CASE II:

The polynomial $Q(x)$ is a product of linear factors, some of which are repeated. **Rule**: If $(ax+b)$ occurs to the power r, then instead of one term, put down the following r terms:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_r}{(ax+b)^r}.$$

Problem. Find $\displaystyle\int \frac{dx}{x^2(x-1)^2}$.

$$\int \frac{dx}{x^2(x-1)^2}.$$

$$\int \frac{dx}{x^2(x-1)^2}.$$

Why do we have to learn these rules for setting these problems up?

It is natural at this point to wonder why we have to set up the question precisely as we did. Why could we not reduce the number of variables by letting

$$\frac{1}{x^2(x-1)^2} = \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} \quad ?$$

If we did this we would get the equations

$$\begin{aligned} C &= 0 \\ B - C + D &= 0 \\ 2B &= 0 \\ B &= 1 \end{aligned}$$

You can see immediately that this system of equations does not have a solution. That is, there do not exist numbers B, C, and D that satisfy all four equations at once.

Problem. Suppose we want to integrate $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\,dx$. How should we set up the partial fractions?

(A.) $\dfrac{A}{x-1} + \dfrac{B}{x+2}$

(B.) $\dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x+2}$

(C.) $\dfrac{A}{x-1} + \dfrac{B}{(x+2)^2}$

(D.) $\dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$

(E.) $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2}$

Problem. Find $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\,dx$.

$$\int \frac{1}{(x-1)(x+2)^2}\,dx.$$

$$\int \frac{1}{(x-1)(x+2)^2}\,dx.$$

Step 2. CASES III AND IV:

The polynomial $Q(x)$ contains quadratic factors. **These cases will be included only if lecture time remains.**

Academic Note: The method of partial fractions is included here as an integration technique. However, it will also re-appear in a differential course next year for many of you (MTHE 225 being the most common). In that course, partial fractions will be used to simplify an integration-related transform called the *Laplace transform* which is frequently used in analyzing engineering control systems.

Work

The basic formula for work is the product of **force times distance**. If force is measured in Newtons and distance in meters, the answer is in Joules.

Problem. If you pull an object, using a force of 5 N, and you move it from $x = 0$ to $x = 15$ m, how much work have you done? Give units in your answer.

Problem. If the force you applied was changing as you pulled, with $F = (5 - 0.1x)$ N, how would this affect how you can calculate the total work?

Problem continued.

Problem. When a particle is x meters from the origin, a force measuring $\cos\left(\frac{\pi x}{3}\right)$ N acts on it. How much work is done by moving the particle from $x = 1$ to $x = 2$?

Problem. An aquarium 2 m long, 1 m wide and 1 m deep is full of water. Find the **minimum** amount of work needed to pump half of the water out of the aquarium.

Problem continued.

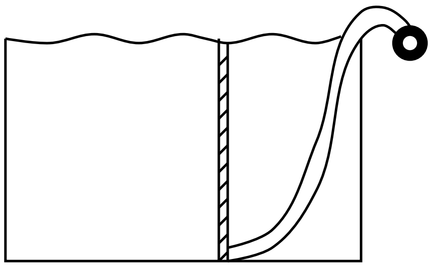
Comments on the aquarium problem

(1) Strictly speaking, the assumptions behind the problem are highly idealized. In any real situation the water would come out of the hose with some amount of kinetic energy, and this extra energy adds to the work done. To calculate the **minimum** amount of work required is to ignore these effects. Even if in real life the amount of work required is always somewhat more than what this calculation tells, it is nevertheless helpful to know that it gives the absolute minimum that could be reached.

(2) The method used really hinges on the conservation of energy: energy gained = work done. We calculated this work by calculating the increase in (potential) energy in the horizontal slabs of water.

(3) To minimize the work needed, we imagine the pumping done "slowly" so no kinetic energy is created.

(4) In principle, if we knew what happened to each particle of water, we could do a more detailed and "realistic" analysis. It would require knowing where the hose is placed (on the bottom of the aquarium or higher) and a calculation of the work done on or by each individual water particle as it is pushed down the tube and then up again, or (in other cases) as it sinks closer to the bottom of the aquarium.



In practice this picture becomes far too complicated to use. The power of the principle of energy conservation is in its ability to simplify the problem.

Problem. A large cylindrical tank is filled with water. There is a drain in the center of the bottom of the tank, two meters above the surface of a lake. A hose is attached to the drain, and the tank is allowed to empty through the hose onto the surface of the lake. We want to calculate the loss of potential energy of the water as it runs from the tank to the surface of the lake.

Problem. How should we choose the "slices" of water for our integral, and why?

A. Horizontal slabs because it worked last time.

B. Horizontal slabs because all the points in a horizontal slab are the same distance above the surface of the lake.

C. Horizontal slabs because when it is at rest, water surface is always horizontal.

D. Cylindrical shells because the tank is cylindrical.

E. Cylindrical shells because each such shell is at a constant radius from the center, where the drain is located.

Problem. The parabola $y = x^2$ is rotated about the y-axis, and filled with water to the level $y = 3$. How much work is required to pump the water out through a hole located at $y = 4$? (Assume all scales are in meters.)

Problem continued.