Week 10: Differential Equations and Complex Numbers - Part 1

Goals:

- Modeling with Differential Equations
- Harmonic Motion
- Introduction to Complex Numbers

Differential Equations

As pointed out on several other occasions in the course, most laws of nature and science, once they are translated into mathematics, take the form of a **differential equation**.

A differential equation is quite unlike the other equations we have studied. For one thing, it involves derivatives, often even second or higher derivatives. But this is not the most important difference between a differential equation and the kinds of equations we have been used to and continue to study in our courses.

- Solutions to **algebraic** equations are **real values** (or sets of values)

- Solutions to **differential** equations are **functions** (or sets of functions)

Problem. Which of the following **functions** is a solution to the **differential** equation $x''(t) = -36\, x(t)$?

A. $x(t) = -6t^3$

B. $x(t) = \cos(6t)$

C. $x(t) = e^{-6t}$

D. $x(t) = -e^{-6t}$

Problem. Confirm your answer.

$x''(t) = -36\ x(t)$

A. $x(t) = -6t^3$

B. $x(t) = \cos(6t)$

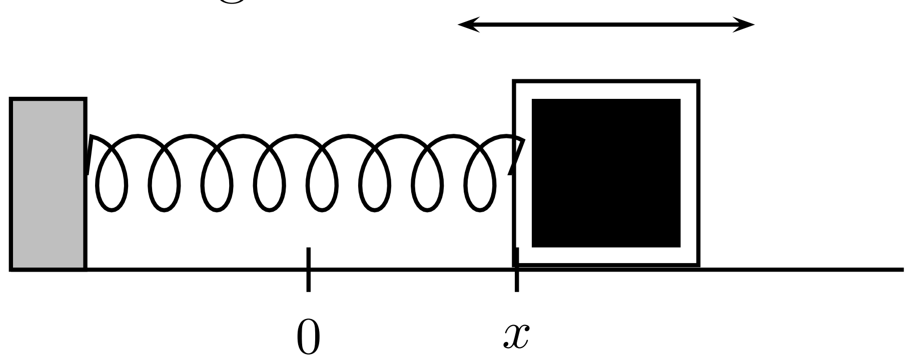
C. $x(t) = e^{-6t}$

D. $x(t) = -e^{-6t}$

Modeling with Differential Equations

Harmonic Motion

We will begin by studying the possible solutions of a particular differential equation that will be very important in your Physics course next term. Thinking about these solutions will help us understand what we mean when we say a function is a solution of a differential equation. The differential equation we saw in the previous concept question arises when we study the motion of a block at the end of a spring, as in the next diagram:



In this system, how would you describe x in words?

Problem. Draw a free-body diagram for the mass. Indicate the magnitude of the forces, assuming

- the mass of the block is m kg, and
- the spring constant (in N/m) is given by the constant k.

Let us work with our intuition about this system before beginning the mathematics.

Problem. If the spring is very stiff, is k large or small?

(a) k will be **large**.

(b) k will be **small**.

Problem. If we exchange a soft spring for a stiff spring, do you expect the *period* of the oscillations to increase or decrease?

(a) The period will be **longer** if the spring is stiffer.

(b) The period will be **shorter** if the spring is stiffer.

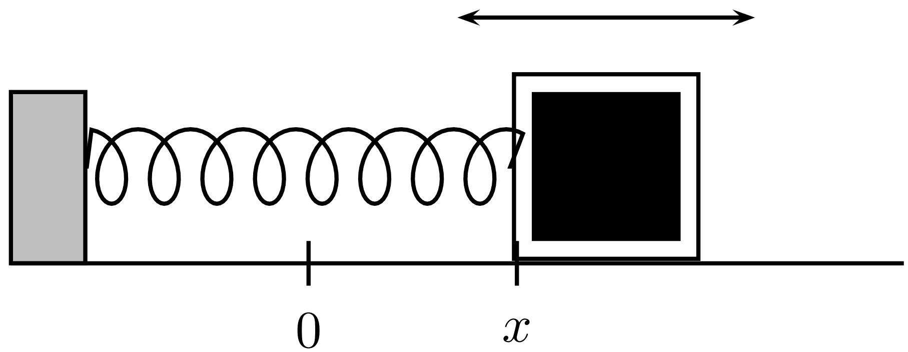
Follow-up: why?

Problem. If we replace a **small mass** with a **heavier mass**, do you expect the *period* of the oscillations to increase or decrease?

(a) The period will be **longer** if the mass is heavier.

(b) The period will be **shorter** if the mass is heavier.

Follow-up: why?



If we know k and m, and assume that friction is negligible, should we be able to determine the exact period of the oscillations?

Does anyone know the formula for the period of oscillations of a spring system?

The spring system is an excellent introduction to differential equations because

- we all know how it *should* work physically,
- the mathematics and physics are simple, **but**
- there's no obvious way to predict critical features (e.g. the period) from the given information.

We clearly need some new tools!

Analysis of Mass/Spring System

Problem. Use Newton's second law, $F = ma$, to construct an equation involving the position $x(t)$.

What are we solving for in this equation? I.e., what is the unknown?

Problem. To simplify matters temporarily, let us assume that both $k = 1$ N/m and $m = 1$ kg. Rewrite the previous differential equation.

This differential equation invites us to find a function $x(t)$ whose second derivative is its own negative. Try to think of such a function, or more than one!

We have found a set of solutions for the differential equation

$$\frac{d^2x}{dt^2} = -x.$$

Does this help us discover solutions for the differential equation

$$\boldsymbol{m}\frac{d^2x}{dt^2} = -\boldsymbol{k}x?$$

Problem. Find one or more solutions to this second differential equation.

$$m\frac{d^2x}{dt^2} = -kx$$

Problem. Find **the most general family** of solutions to this differential equation.

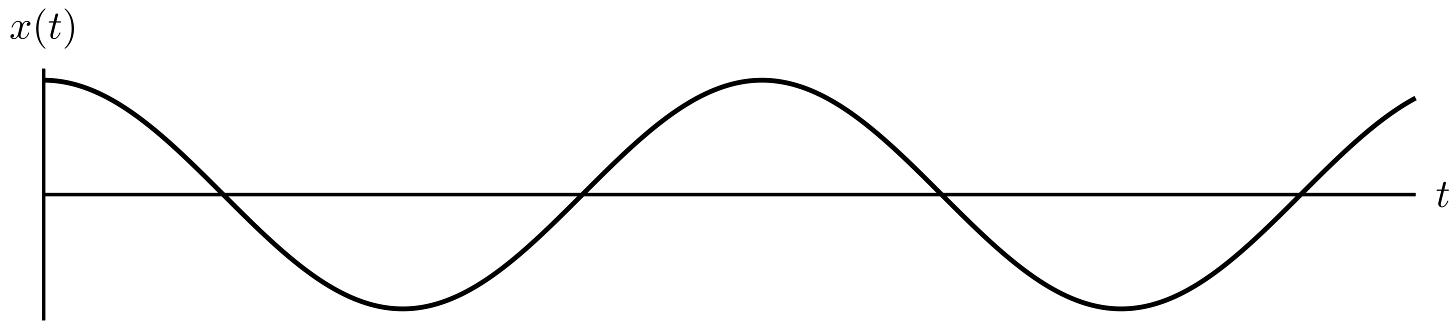
Spring System - Adding Friction

Problem. Re-draw the free body diagram for the mass in the spring system, and add a friction force (magnitude proportional to velocity).

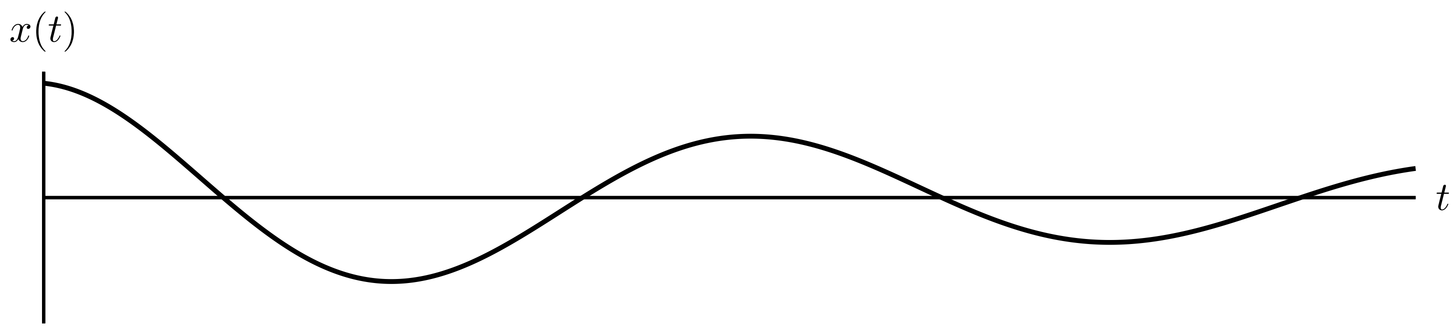
Starting with $F = ma$, write an equation that the position function $x(t)$ must satisfy.

$$mx'' + cx' + kx = 0$$

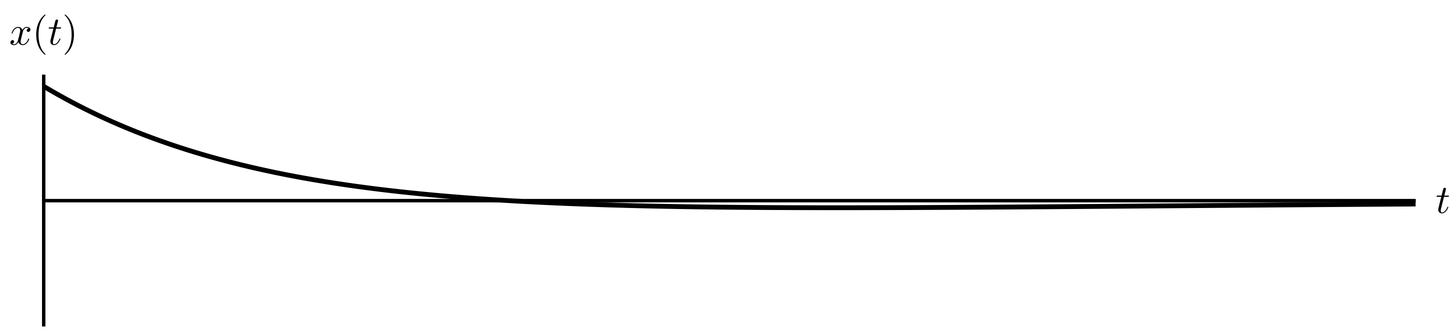
As we increase the friction coefficient c, the motion of the spring will be altered. Here are three plots of the position over time for a mass with increasing friction.



No friction



Light Friction



High Friction

$$mx'' + cx' + kx = 0$$

The new fun fact is that all of these graphs can be made of the same building blocks, a simple exponential, if we add a secret sauce: the "imaginary" number, $\sqrt{-1}$ or i.

Problem. To connect e^{it} or $e^{\sqrt{-1}\ t}$, let us return to the simpler no-friction system, $\frac{d^2x}{dt^2} = -x(t)$.

Show that $x(t) = \sin(t)$ satisfies this equation.

Show that $x(t) = e^{it}$ satisfies this equation, given that $i^2 = -1$.

This points to e^{it} somehow being related to $\sin(t)$: this is a major discovery! We need to investigate this value $i = \sqrt{-1}$ in more detail.

Introduction to Complex Numbers

A complex number is a number with two components, called the **real** and the **imaginary** components. They can be written in the form:

$$x = a + bi$$

where $i = \sqrt{-1}$ or $i^2 = -1$.

Problem. How are the *real numbers* you know already related to the set of all *complex numbers*?

Addition of Complex Numbers

Adding complex numbers works in a way analogous to vectors.

Problem.

(a) Compute $(3 + 2i) + (-1 + 5i)$.

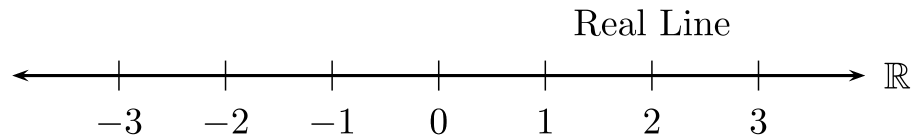
(b) Compute $(1 + 2i)$ - $(2 + i)$.

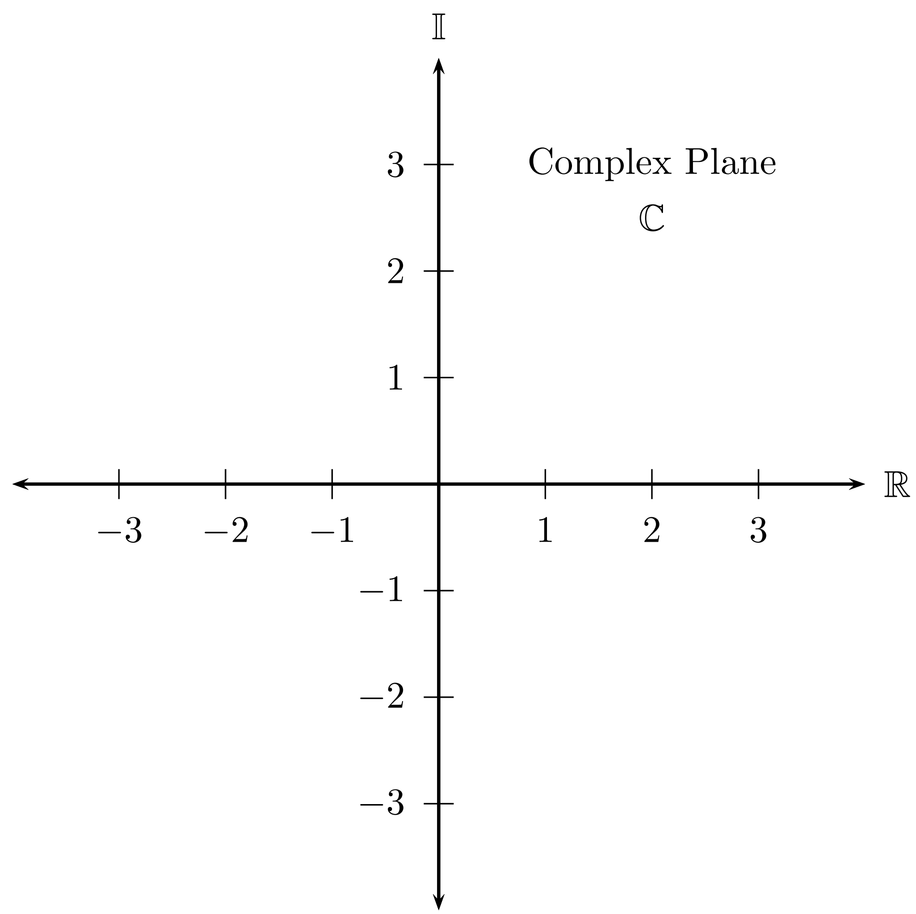
(c) Compute 3 + 7.

(d) Compute $6i + (-2 - 4i)$.

Graphical Addition of Complex Numbers

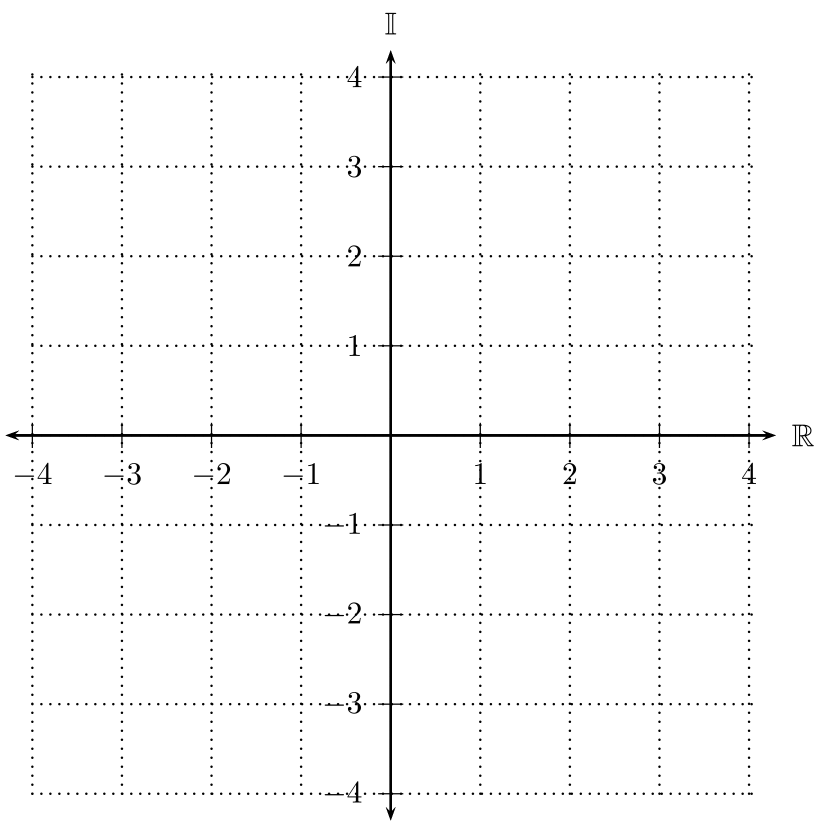
To visualize complex numbers, we can extend our traditional **real line** into the two-dimensional **complex plane**.



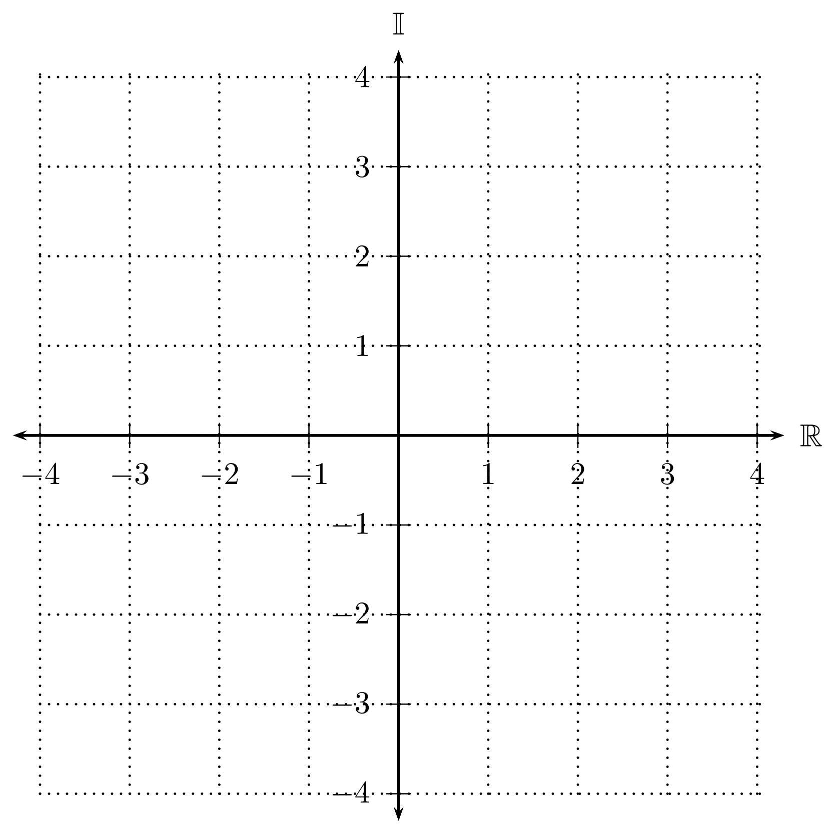


Problem. Draw the following complex numbers on the plane, then compute and draw their sum.

$z_1 = 2 + 3i$, $z_2 = 1 - 2i$



$z_3 = -1 + i$, $z_4 = 4$



Multiplication of Complex Numbers

Addition of complex numbers can make it feel like complex numbers are just another way to represent vectors. However, **multiplication** of complex numbers will show that we are in new territory.

(a) Compute $(3 + 2i) \cdot (-1 + 5i)$.

(b) Compute $(1 + 2i) \cdot (1 - 2i)$.

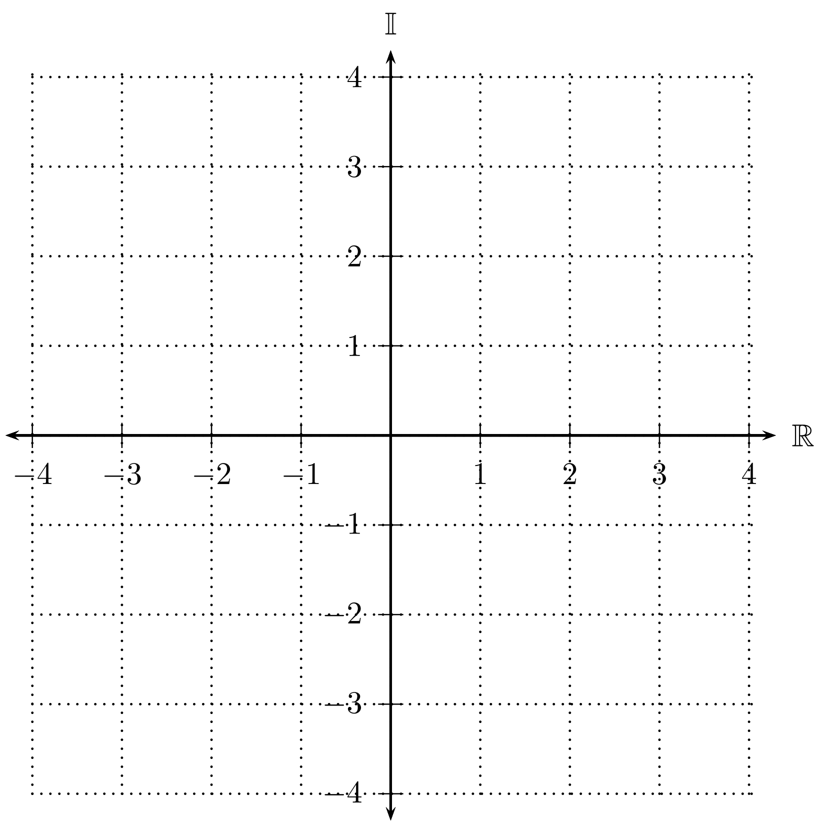
(c) Compute $(3 + 0i) \cdot (7 + 0i)$.

(d) Compute $6i \cdot (2 + 4i)$.

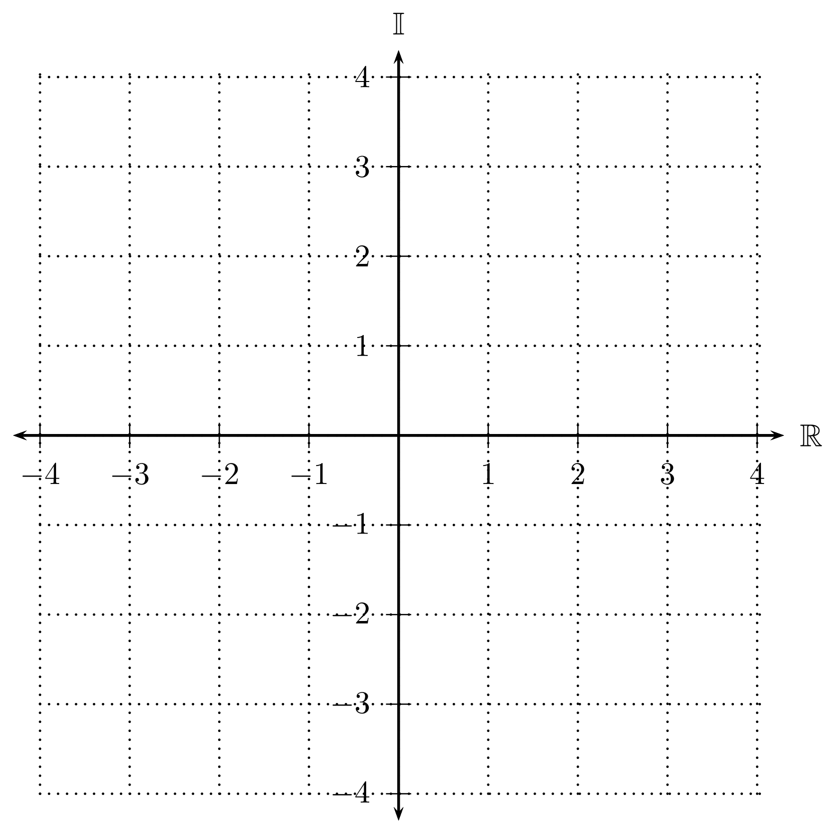
At first glance, there may appear to be little structure to the multiplication of complex numbers. However, a graphical perspective can help us see patterns in what is happening.

Problem. Draw the following complex numbers on the plane, then compute and draw their **product**.

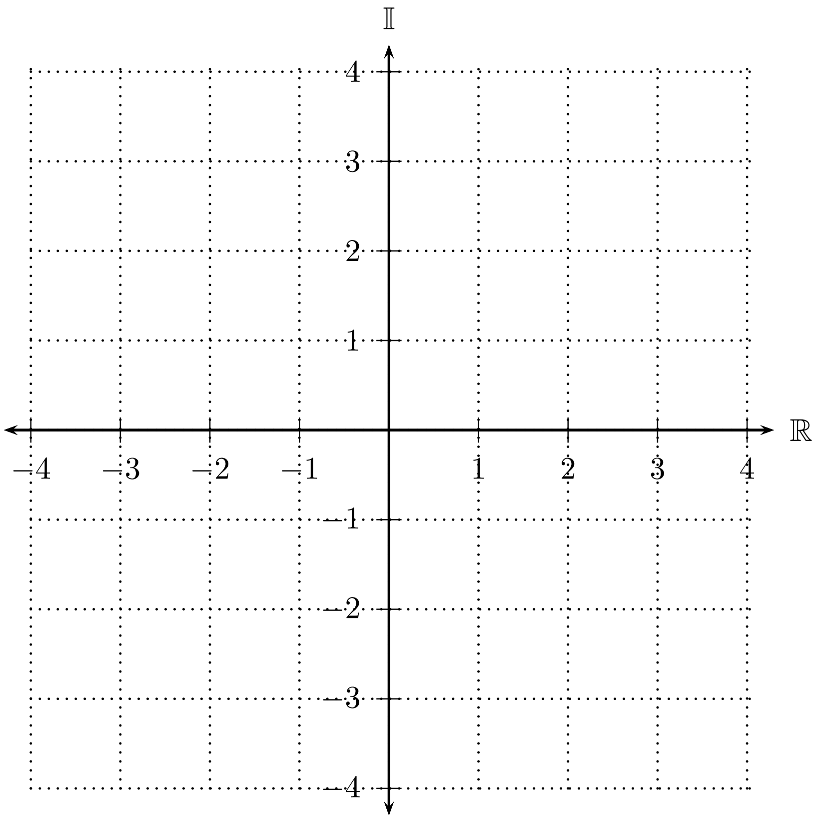
$z_1 = 1 + 2i$, $z_2 = 2$



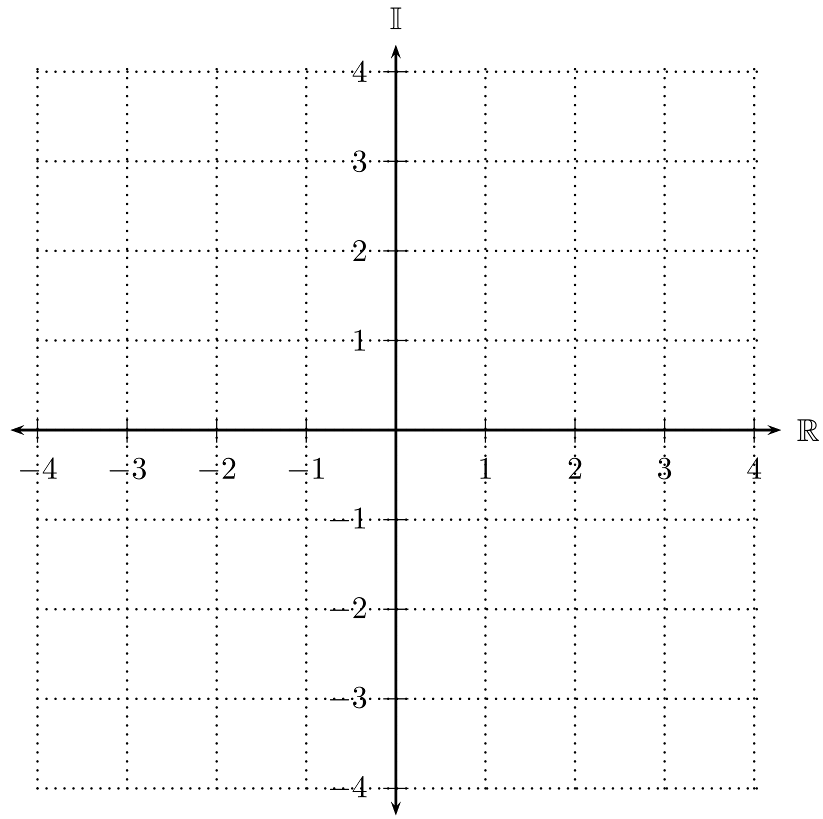
$z_3 = 1 + 2i$, $z_4 = 2i$



$z_1 = 1 + i$, $z_2 = 1 - 1i$



$z_3 = -2 + 2i$, $z_4 = -2 + 2i$



Problem. What patterns do you notice in the multiplication of complex numbers?

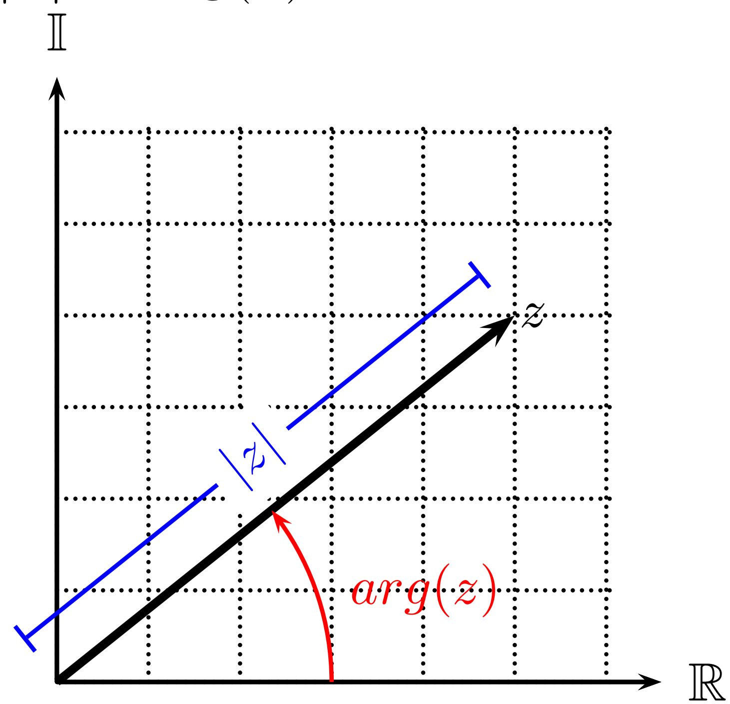
Polar form of Complex Numbers

Computing products of complex numbers can be made easier if we write our complex numbers z in **polar form**:

- A **length** or **magnitude** of the number, $|z|$, and
- An **angle** in radians, measured counter-clockwise from the positive $\mathbb{R}$ axis, $\arg(z)$.

We can then write complex numbers as either

$z = a + bi$ **or** $z = |z| \angle \arg(z)$.



Problem. Find the polar form of the following complex numbers. Write the result using the notation $(|z| \angle \phi)$.

(a) $z_1 = -2 + 2i$

(b) $z_2 = -i$

(c) $z_3 = -2$

Problem. Use the polar forms to compute the products of the following complex numbers. You can leave the results in polar form.

(a) $z_1 = (-2 + 2i) = (\sqrt{8}\ \angle 3\pi/4)$,
$z_2 = -i = (1\ \angle - \pi/2)$

(b) $z_1 = (-2 + 2i) = (\sqrt{8}\ \angle 3\pi/4)$,
$z_3 = -2 = (2\ \angle \pi)$

Foreshadowing

When we **multiply** complex numbers, the actual process involves the **addition** of the angular components of the numbers.

In high school, you saw a similar situation: a computation where multiplication is actually calculated by using an addition operation.

Can you think of the situation where you used that approach?