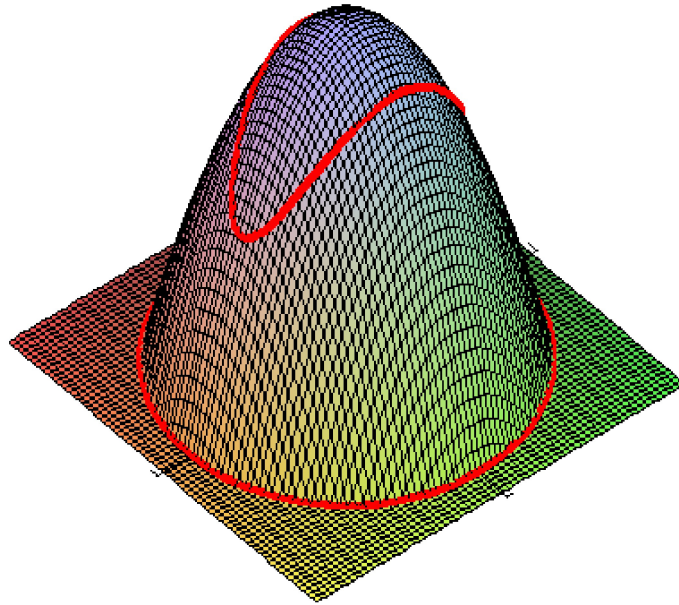


Chapter 1

Multivariable Differential Calculus



- 1. Lines and curves**
- 2. Planes in 3-space**
- 3. Functions of several variables.**
- 4. Partial derivatives.**
- 5. Tangent planes**
- 6. The multivariable chain rule and linear approximation**
- 7. Gradient and directional derivative**
- 8. Optimization--max and min**
- 9. Investigations**

1. Lines and curves

Equation of a line. An equation of the form

$$\vec{r} = \vec{v}_0 + t\vec{u}$$

is a vector parametric equation of a line. It passes through the point \vec{v}_0 and has direction vector \vec{u} . In coordinates, we can write the equation:

Physics

$$\vec{r}(t) = \vec{r}_0 + \vec{v} \cdot t$$

pos init pos velocity

$$\vec{x} = \vec{x}_0 + \vec{v}_0 t + \frac{1}{2} a t^2$$

$m \left(\frac{m}{s}\right)(s)$

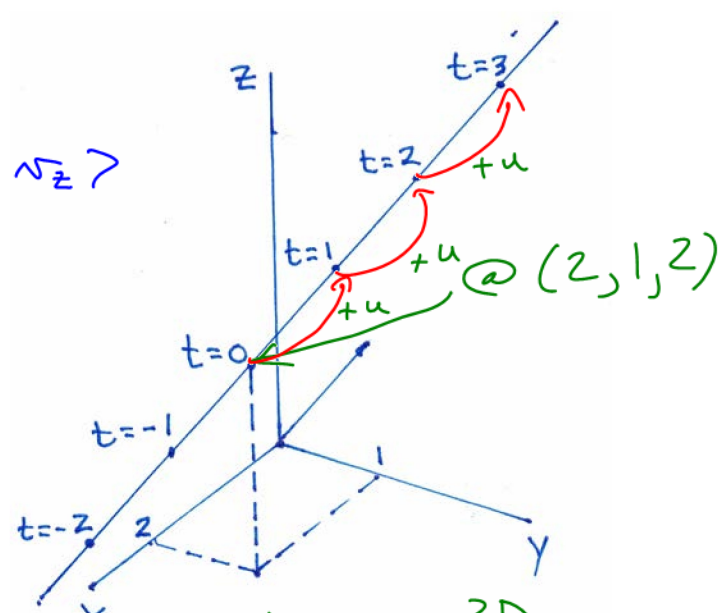
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

or

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}$$

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle v_x, v_y, v_z \rangle$$

Example 1.1. Find a parametric equation for the line that passes through the point $(2, 1, 2)$ in the direction of the vector $[-1, 1, 3]$. Make a sketch of the line and illustrate the parameterization. Interpret t as time and calculate the velocity and speed of the moving point.



$$\vec{r}(t) = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \cdot t$$

init pos'n velocity / direction vector

Magnitude in 3D

$$\vec{w} = [a, b, c]$$

$$\|\vec{w}\| = \sqrt{a^2 + b^2 + c^2}$$

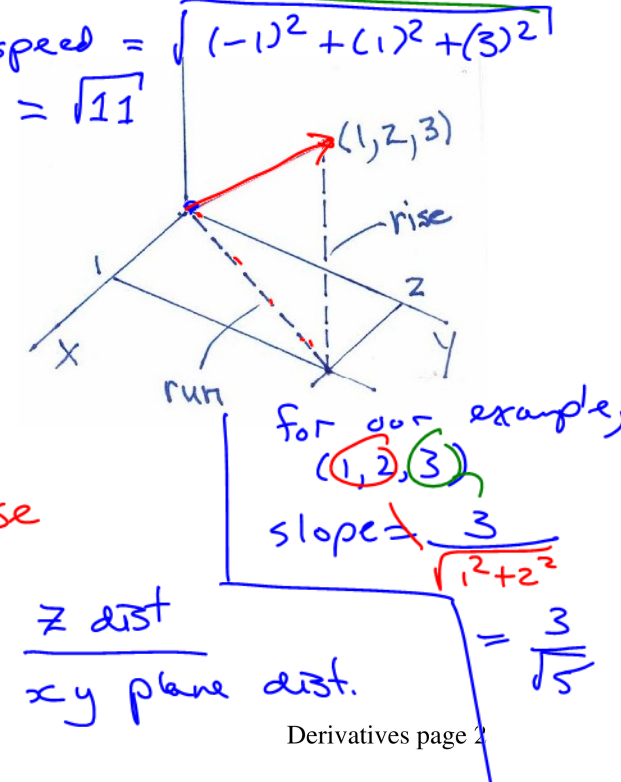
so

$$\vec{r}'(t) = \frac{d}{dt} \left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} t \right) = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

const

speed = $\sqrt{(-1)^2 + (1)^2 + (3)^2} = \sqrt{11}$

Example 1.2. A line is drawn from the origin to the point $(1, 2, 3)$. What is the slope of the line?
 Note: In a coordinate system, "slope" has no meaning unless we specify the vertical axis. In 3 dimensions we will always take this to be the z-axis.
 Answer: slope = $3/\sqrt{5}$



slope in a direction of $\vec{w} = [a, b, c]$

run in xy plane = $\sqrt{a^2 + b^2}$ rise

$$\text{slope} = \frac{c}{\sqrt{a^2 + b^2}} = \frac{\text{rise}}{\text{run}} = \frac{z \text{ dist}}{xy \text{ plane dist.}} = \frac{3}{\sqrt{5}}$$

Example 1.3. (a) Show that the circle centred at the origin with radius 3 can be parameterized by the equations

position: $x = 3 \cos(t)$
 $y = 3 \sin(t)$

$$v = \frac{D}{T} = \frac{\text{circumf}}{\text{time}} = \frac{2\pi \cdot (3)}{2\pi s}$$

(b) Using this parameterization, calculate the resulting speed in two different ways:

(i) using only geometry and trig.

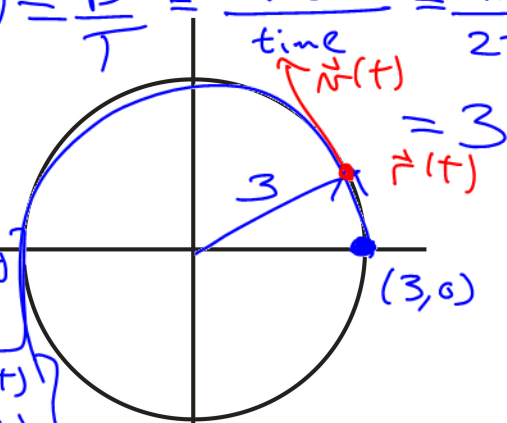
(ii) using calculus.

ii) speed = $\|\text{vel}\|$

$$\vec{r}(t) = \begin{bmatrix} 3 \cos t \\ 3 \sin t \end{bmatrix} \rightarrow \vec{v}(t) = \vec{r}'(t) = \begin{bmatrix} -3 \sin(t) \\ 3 \cos(t) \end{bmatrix} = 3 \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

$$\text{speed} = \left\| 3 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \right\| = 3 \sqrt{(-\sin t)^2 + (\cos t)^2} = 3 \cdot 1 = 3$$

$\sin^2 t + \cos^2 t = 1$



$$\dot{z} = \frac{d}{dt} z = 15z$$

Example 1.4 (a) Sketch the curve in R^3 with parametric equation:

$$\begin{aligned} x &= 2 \cos t \\ y &= 2 \sin t \\ z &= 3t \end{aligned}$$

x, y circle w/ radius 2
 z incr linearly w/ time

Here we use a special purpose slope formula:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{vert velocity}}{\text{horiz speed}} = \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

(b) Interpreting t as time, find the velocity and the speed of the moving point at any time t .

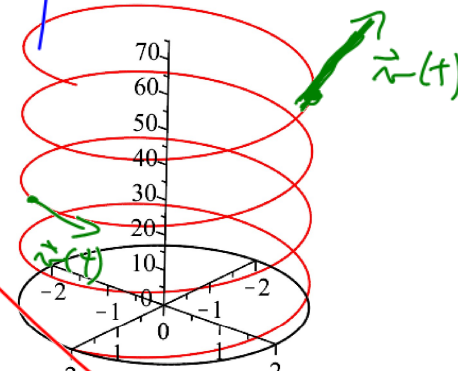
(c) Take the z -axis as vertical and calculate the slope m of the curve.

[Answer: $\mathbf{v} = [-2 \sin t, 2 \cos t, 3]$, speed = $\sqrt{13}$, slope = $3/2$]

$$\text{vel} = \frac{d}{dt} \begin{bmatrix} 2 \cos t \\ 2 \sin t \\ 3t \end{bmatrix} = \begin{bmatrix} -2 \sin t \\ 2 \cos t \\ 3 \end{bmatrix} \text{ m/s}$$

$$\begin{aligned} \text{speed} = \|\text{vel}\| &= \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + 3^2} \\ &= \sqrt{4(\sin^2 t + \cos^2 t) + 9} \\ &= \sqrt{4(1) + 9} = \sqrt{13} \text{ m/s} \end{aligned}$$

t	x	y	z
0	2	0	0
2π	2	0	6π



Slope = slope of tangent vector = slope of velocity vector = $\frac{15z}{\sqrt{15x^2 + 15y^2}} = \frac{3}{\sqrt{(2 \sin t)^2 + (2 \cos t)^2}} = \frac{3}{2}$ slope

Derivatives page 3

Every 2 steps in xy unitless \rightarrow 3 steps up in z

x	y	z
0	0	9

Example 1.5. Walking on the hill
I am walking on the hill

$z = f(x, y)$

$z = 9 - x^2 - y^2$

The projection of my path on the x-y plane follows the ellipse

no t's

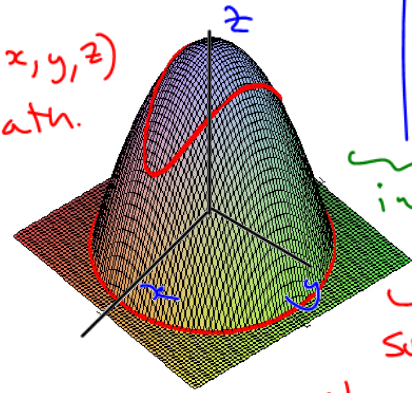
$x^2 + 4y^2 = 4$ slope

At what point is my path the steepest (going up or going down) and what is the slope of the climb at that point?

Answer: $x = \pm\sqrt{\frac{8}{3}}, y = \pm\sqrt{\frac{1}{3}}, z = 6, m = \pm 2$.

first z-variable function
graph is a surface

(x, y, z) path.



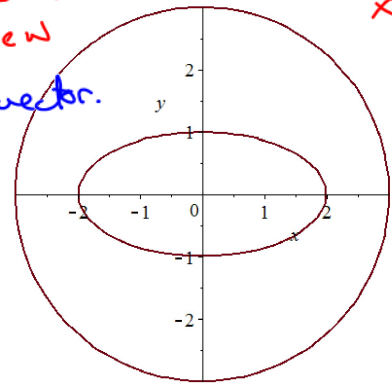
inp out put
Surface above/below xy plane

parameterize path / add time

need slope → need tjt lines → velocity vector is tjt
get slope of vel vector.

Top down view

variable tree



Build parameterization:

$x = 2 \cos t$

$y = 1 \sin t$

$z = 9 - x^2 - y^2$



d/dt ↓ but we need velocity for tjt vectors

$\dot{x} = -2 \sin t$
 $\dot{y} = \cos t$

$\dot{z} = -2x \dot{x} - 2y \dot{y} = -2(2 \cos t)(-2 \sin t) - 2(\sin t)(\cos t)$
 $= 8 \sin t \cos t - 2 \sin t \cos t = 6 \sin t \cos t$

slope $m = \frac{\text{rise in } z}{\text{run in } xy} = \frac{6 \cos t \sin t}{\sqrt{(-2 \sin t)^2 + (\cos t)^2}}$
 $m(t) = 6 \frac{\sin t \cos t}{\sqrt{4 \sin^2 t + \cos^2 t}}$

To maximize m, find crit points → take deriv of m wrt time, set = 0
can instead minimize (m²)

Example 1.5 (cont'd)

Recall : $\sin^2 t + \cos^2 t = 1$
 so $\cos^2 t = 1 - \sin^2 t = 1 - s$
 Chapter 1. Differential Calculus

like subs. in integrals

$$m^2(t) = 36 \frac{\sin^2 t \cdot \cos^2 t}{4 \sin^2 t + \cos^2 t}$$

Let $s = \sin^2 t$

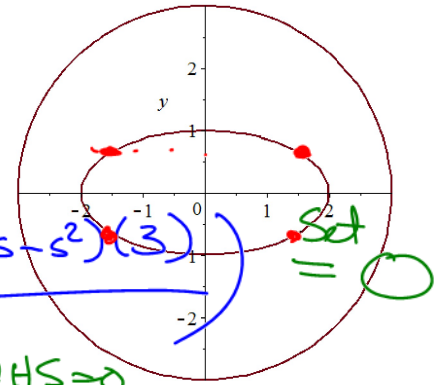
$$m^2(s) = 36 \frac{s \cdot \cos^2 t (1-s)}{4s + \cos^2 t (1-s)}$$

tidy = $36 \frac{s(1-s)}{4s + 1-s} = 36 \frac{s-s^2}{3s+1}$

Find crit points of $m^2(s)$ \downarrow $\frac{d}{ds}$

$$\frac{d}{ds} (m^2(s)) = 36 \left(\frac{(1-2s)(3s+1) - (s-s^2)(3)}{(3s+1)^2} \right)$$

Set = 0
 RHS = 0



so $(1-2s)(3s+1) - 3(s-s^2) = 0$

$$-6s^2 + s + 1 - 3s + 3s^2 = 0$$

$$-3s^2 - 2s + 1 = 0$$

or $-(3s^2 + 2s - 1) = 0$

$$-(s+1)(3s-1) = 0$$

$s = -1$ $3s = 1$ or $s = \frac{1}{3}$

@ $s = -1, = 0$
 $\Rightarrow (s - (-1))$ or $(s+1)$ is a factor

Where are these points?

$\sin^2 t = -1$

all real \rightarrow no solution!

$s = -1$

$s = \sin^2 t$

$s = \frac{1}{3} \rightarrow \sin^2 t = \frac{1}{3}$

or $\sin(t) = +\frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

$y = +\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

$x = \cos(t)$

or $x^2 + 4y^2 = 4$

$$y = \pm \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$$

$$x^2 = 4 - 4\left(\frac{1}{3}\right)$$

$$x^2 = \frac{8}{3}$$

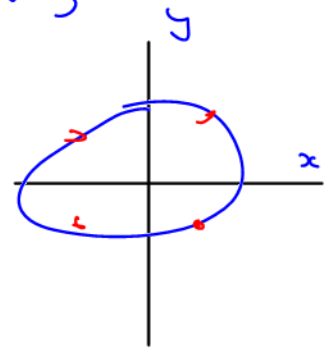
$$x = \pm \sqrt{\frac{8}{3}}$$

max slopes

@ (x, y)

$$= \left(\sqrt{\frac{8}{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{\sqrt{8}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\left(\sqrt{\frac{8}{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\sqrt{\frac{8}{3}}, -\frac{1}{\sqrt{3}}\right)$$



The slope here

is $m = \pm 2$

2. Planes in 3-space

The *dot product* of two vectors $\mathbf{u} = [a, b, c]$ and $\mathbf{v} = [x, y, z]$ is defined as:

$$\mathbf{u} \cdot \mathbf{v} = [a, b, c] \cdot [x, y, z] = ax + by + cz.$$

I have given this definition in R^3 but it can be formulated in any R^n . Thus

$$[1, 2, 3] \cdot [4, -1, 1] = 4 - 2 + 3 = 5$$

$$[1, 2, 3, 4] \cdot [4, 3, 2, 1] = 4 + 6 + 6 + 4 = 20$$

$$[1, 2] \cdot [4, -2] = 4 - 4 = 0$$

Note that the dot product is a scalar, a real number.

The dot product is called a “product” so you might suspect that it distributes over addition (as any self-respecting “product” ought to do). And indeed it does:

$$\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v}.$$

It is also useful to observe that it is commutative: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

The *length* of a vector \mathbf{u} is defined as

$$\|\mathbf{u}\| = \|[a, b, c]\| = \sqrt{a^2 + b^2 + c^2}.$$

Thus the vector $[1, -2, 3]$ has length: $\sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$

Note that the dot product of a vector with itself is the square of its length:

$$\mathbf{u} \cdot \mathbf{u} = [a, b, c] \cdot [a, b, c] = a^2 + b^2 + c^2 = \|\mathbf{u}\|^2.$$

The distance between two points is the length of their vector difference.

For example, the distance between the points $[1, 2, 3]$ and $[4, -2, 1]$ is:

$$\text{dist}([1, 2, 3], [4, -2, 1]) = \|[1-4, 2+2, 3-1]\|$$

$$= \|[-3, 4, 2]\|$$

$$= \sqrt{(-3)^2 + 4^2 + 2^2}$$

$$= \sqrt{29}$$

$$u \cdot u =$$

$$\vec{u} = [1, 2, 0]$$

$$\|\vec{u}\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$$

compare to

$$\vec{u} \cdot \vec{u} = 1^2 + 2^2 + 0^2 = \|\vec{u}\|^2$$

components

$[1, 2, 3] \cdot [4, 5, 6] = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6$

Example 2.1. The geometry of the dot product.

$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta$

where θ is the angle between the two vectors.

We'll prove this in a moment, but a significant (and immediate) consequence is:

non-zero

The Orthogonality Theorem. Two vectors are orthogonal (perpendicular) if and only if their dot product is zero:

$\mathbf{u} \perp \mathbf{v} \Leftrightarrow \mathbf{u} \cdot \mathbf{v} = 0$

Proof of the Geometry Result: The trick is to express the length of the vector difference $\mathbf{u} - \mathbf{v}$ in two different ways. The first is algebraic and writes it as the dot product of the vector with itself, and then expands:

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \end{aligned}$$

The second is geometric and uses the cosine law for the triangle formed by the vectors \mathbf{u} , \mathbf{v} and $\mathbf{u} - \mathbf{v}$:

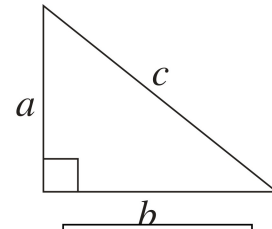
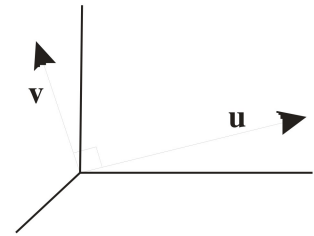
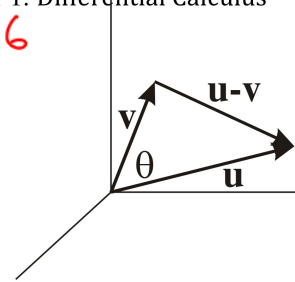
$c^2 = a^2 + b^2 - 2ab \cos \theta$

$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$

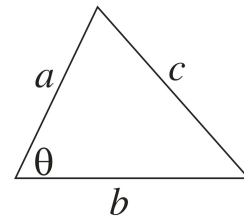
Now compare the two right sides to get

$2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta = 2\mathbf{u} \cdot \mathbf{v}$

and the result follows.



$c^2 = a^2 + b^2$



$c^2 = a^2 + b^2 - 2ab \cos \theta$

Find a non-zero vector that is orthogonal to both $\mathbf{u} = [1, 1, 1]$ and $\mathbf{v} = [1, -1, 2]$.

Solution: The vector $\mathbf{w} = [a, b, c]$ must satisfy the equations:

$\mathbf{w} \cdot \mathbf{u} = a + b + c = 0$
 $\mathbf{w} \cdot \mathbf{v} = a - b + 2c = 0$

We have two equations in three unknowns so there will be a "degree of freedom" in the solution. We expect that as on geometric grounds the solution will be determined only up to a constant. To eliminate one variable add the equations:

$2a + 3c = 0$

Using the degree of freedom, set $a = 3$. Then $c = -2$. And then from either of the original equations, $b = -1$. The answer is $[3, -1, -2]$ or any of its multiples.

$f(x)$
 $L(x) = f(x_0) + f'(x_0)(x - x_0)$

P is any point on plane.

n known

An equation of the form

$$\mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_0) = 0$$

$P - P_0 = \text{vector in plane}$

$$[a, b, c] \cdot ([x, y, z] - [x_0, y_0, z_0]) = 0$$

variables

describes a plane which passes through the point $P_0 = [x_0, y_0, z_0]$ and is orthogonal to the normal vector $\mathbf{n} = [a, b, c]$.

This equation can be written in two other useful forms:

The point-normal form

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

The standard form

$$ax + by + cz = d$$

$d = +ax_0 + by_0 + cz_0$

where $d = ax_0 + by_0 + cz_0$.

known single point on plane

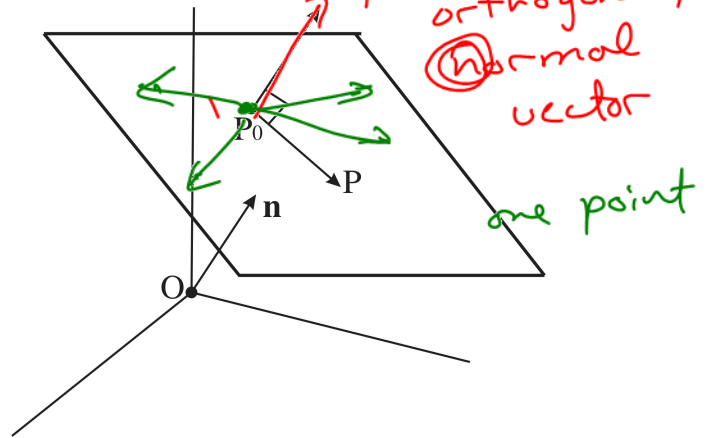
$[a, b, c]$ from normal vector

x_0, y_0, z_0 from P_0

perpendicular/orthogonal/
normal vector

in plane

$$3(x-2) \dots = 0$$



Example 2.2. Find the equation of a plane that passes through the point $(2, 3, 2)$ with normal vector in the direction of the vector $[1, -2, 4]$.

[Answer: $x - 2y + 4z = 4$.]

$$\vec{n} = [1, -2, 4]$$

$$= [a, b, c]$$

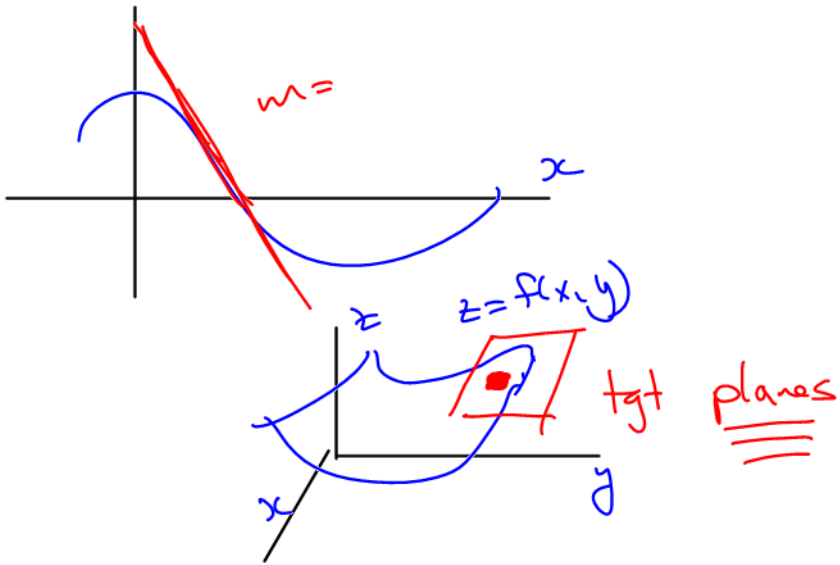
$(x_0, y_0, z_0) = (2, 3, 2)$

all points (x, y, z) on this plane satisfy

$$\vec{n} \cdot [x - x_0, y - y_0, z - z_0] = 0$$

for perp'r

$$1(x-2) + (-2)(y-3) + 4(z-2) = 0$$



Example 2.3 Reflection in a plane. Given the point $A(2, 4, -6)$ and the plane

$$x - 2y + z = 6$$

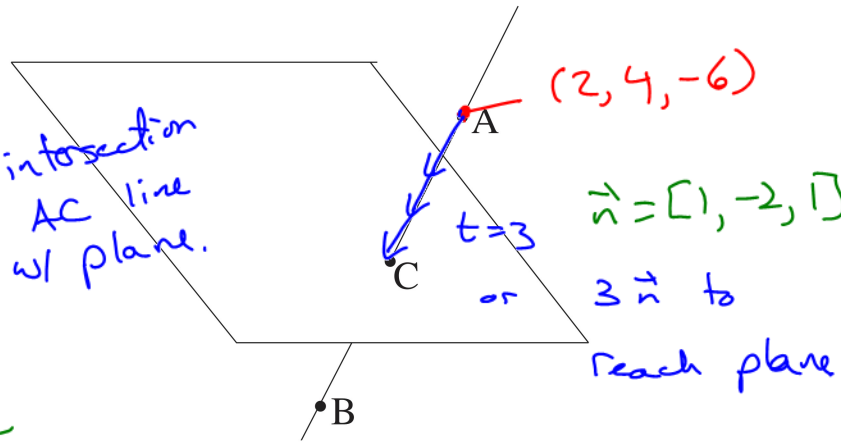
find the reflection B of A in the plane.

That is, if the plane is a mirror, B is the image of A .

[Answer: $B = (8, -8, 0)$]

$$\vec{n} = [1, -2, 1]$$

Goal: 1) Find C - find intersection of AC line w/ plane.
 2) know \vec{AC}
 $\vec{AB} = 2 \times \vec{AC}$



AC line: point $(2, 4, -6)$, direction $\vec{n} = [1, -2, 1]$
 so line is $A + t \cdot \vec{n}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Intersects plane
 shared/equal
 x, y, z

$$x - 2y + z = 6$$

$$(2+t \cdot 1) - 2(4-2t) + (-6+t) = 6$$

legit, just t

$$2+t - 8+4t - 6+t = 6$$

$$6t - 12 = 6$$

$$6t = 18$$

$$t = 3$$

$$\text{so } B = A + 2(3\vec{n})$$

reflection \Rightarrow go twice as far as plane

$$= (2, 4, -6) + 6[1, -2, 1] = [8, -8, 0]$$

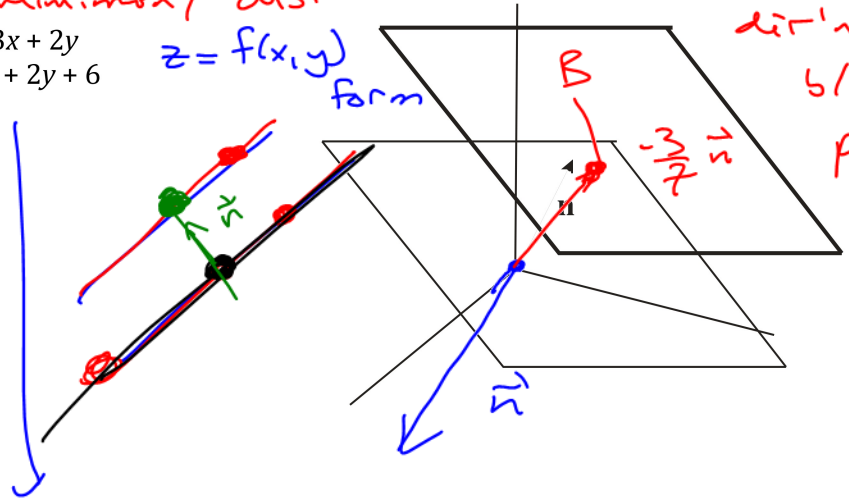
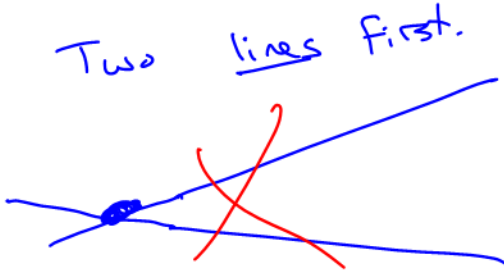
is reflection of A
 in the plane.

most be parallel

Example 2.4. How far apart are the following two planes?

- ① $z = 3x + 2y$
- ② $z = 3x + 2y + 6$

at a minimum / distance measured along a perp'n dir'n b/w planes



to get \vec{n} , need x, y, z 's all on one side of the eq'n

$$3(x-x_0) + 2(y-y_0) - (z-z_0) = 0$$

P_0 of $(0,0,0) \rightarrow$

- ① $3x + 2y - z = 0$
 - ② $3x + 2y - z = -6$
- $\vec{n} = [3, 2, -1]$

1) Build line through $(0,0,0)$ (on 1st plane)
dir'n $\vec{n} = [3, 2, -1]$

2) Find intersection point of line & 2nd plane $\rightarrow B$

3) distance = dist from $(0,0,0)$ to B .

Line: $[x, y, z] = [0, 0, 0] + t[3, 2, -1]$

Plane ② $3x + 2y - z = -6$

\swarrow sub in \searrow

$$3(3t) + 2(2t) - (-t) = -6$$

$$9t + 4t + t = -6$$

tidy

$$14t = -6$$

$$t = \frac{-6}{14} = \frac{-3}{7}$$

so $B = [0, 0, 0] + \left(\frac{-3}{7}\right)[3, 2, -1] = \frac{1}{7}[-9, -6, 3]$

$$\text{Dist b/w planes} = \text{dist b/w } (0,0,0) \text{ and } \frac{1}{7} [-9, -6, 3]$$

$$\text{or} = \left\| \frac{1}{7} [-9, -6, 3] \right\|$$

$$= \frac{1}{7} \sqrt{(-9)^2 + (-6)^2 + 3^2}$$

$$= \frac{1}{7} \sqrt{126} \quad m$$