Geogebra 3D surface viewer
https://www.geogebra.org/m/jmTFk4eg

3. Functions of several variables

eg $f(x,y)$ or $z = f(x,y)$ surfaces.

Example 3.1. Find equations for the following surfaces.

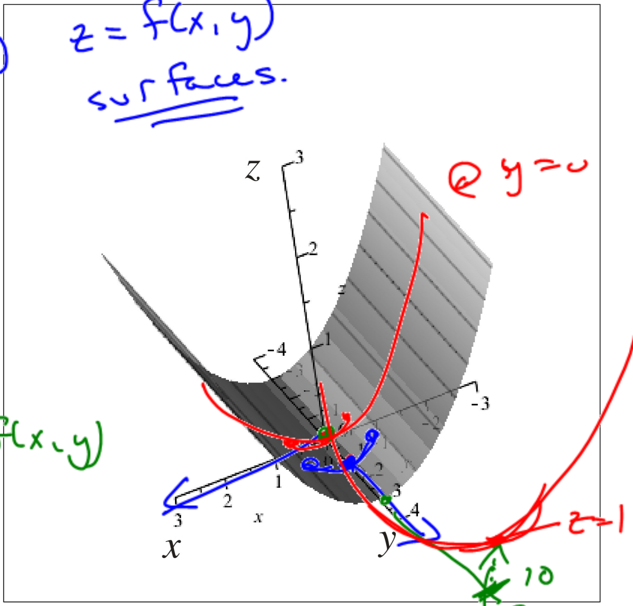
(a) A "parabolic cylinder" running along the *y*-axis.

↳ shape extended

- x, y are inputs
- y change → no change in z

$f: \mathbb{R}^2 \to \mathbb{R}$ ⇒ y does not appear in $z = f(x,y)$ formula.

e.g $z = f(x,y) = x^2$



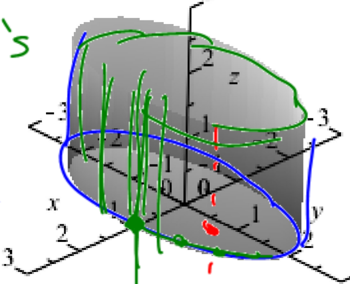
x	y	$z = x^2 + \frac{1}{10}y$
-1	0	+1
0	0	0
1	0	+1
-1	10	+2
0	10	1
1	10	+2

(b) Think of the cylinder as a gutter running along the roof of a house. Suppose I want to slope it so that it rises 1 unit for every 10 horizontal units. What would its equation be?

$$z = x^2 + \frac{1}{10} y$$

(c) An "elliptical cylinder" with *x* and *y* intercepts ±1 and ±2.

Is $z = f(x,y)$? - z is independent of x, y.
- multiple z's for a single (x,y)
(or no z's.



We need a relation here in $\mathbb{R}^3$,

ellipse: $x^2 + \frac{y^2}{2^2} = 1$

$x^2 + \frac{y^2}{4} = 1$

$(1,0)$ on ellipse ✓

$(0,2)$ " " ✓

; no z's.

→ infinite # z's @ this (x,y)

⇒ z is not a function of (x,y)

functions have at most 1 z for each (x,y)

z not mentioned
⇒ z can take on any value.

$(0, 2, z)$

topographic map / contour map / height map

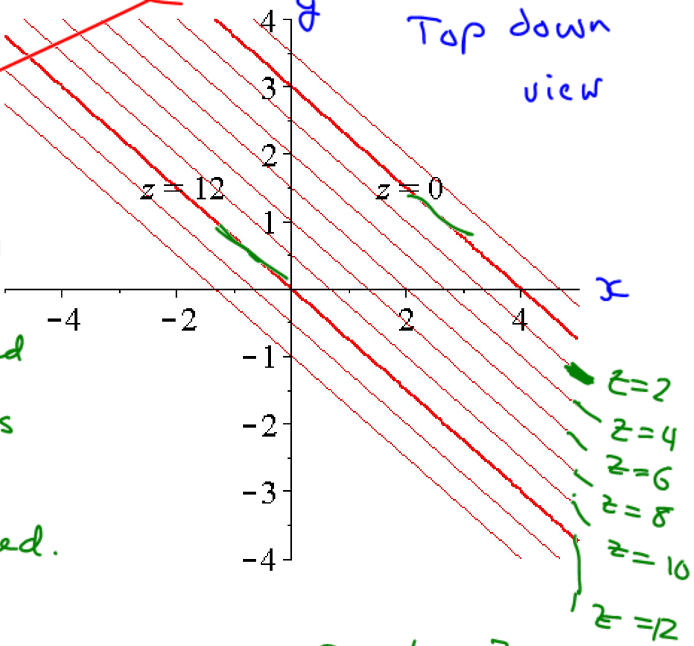
Example 3.2. A collection of level curves (equal z-intervals with $\Delta z = 2$) of the function $z = f(x, y)$ are drawn below, with the level curves z=12 and z=0 in bold. What is the function?

- each line is a "contour" $z = f(x,y)$ or "level curve" or "level set"

$z = f(x,y)$

Planes $z=f(x,y)$

- equally spaced linear contours when z is equally spaced.



[Recall last week
plane w/ point + normal vector, $\vec{n} = [a,b,c]$
$P_0 = [x_0, y_0, z_0]$
was $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$ ← not $z=f(x,y)$ form.
$$\underbrace{\vec{n}}_{\text{vector normal to plane.}} \cdot \text{"}\big(\underbrace{(x,y,z) - P_0}_{\text{vector in plane}}\big)$$
]

Plane form for
$$z = f(x,y) = m_1 x + m_2 y + d$$
(like $y = mx + b$ in 1 variable)

To define the plane, we just need
- slopes m_1, m_2, and
- z-axis intercept d.

@ $x=0$, $y=0$ $\rightarrow$ $z = 0 + 0 + d = d$.

From contour diagram,
$z = f(0,0) = 12 = d$.

@ b/w $x=0$ and $x=2$,
$z = 12, \ldots\ z = 6$
$$m_1 = \frac{\text{rise}}{\text{run}} = \frac{\Delta z}{\Delta x} = \frac{6-12}{2-0} = \boxed{-3}$$

$$\boxed{\begin{aligned} z &= m_1 x + m_2 y + d \\ &= -3x - 4y + 12 \end{aligned}}$$

b/w $y=0$ and $y=1$

w/ $z=12$ and $z=8$

so $m_2 = \frac{\Delta z}{\Delta y} = \frac{8-12}{1-0} = \boxed{-4}$

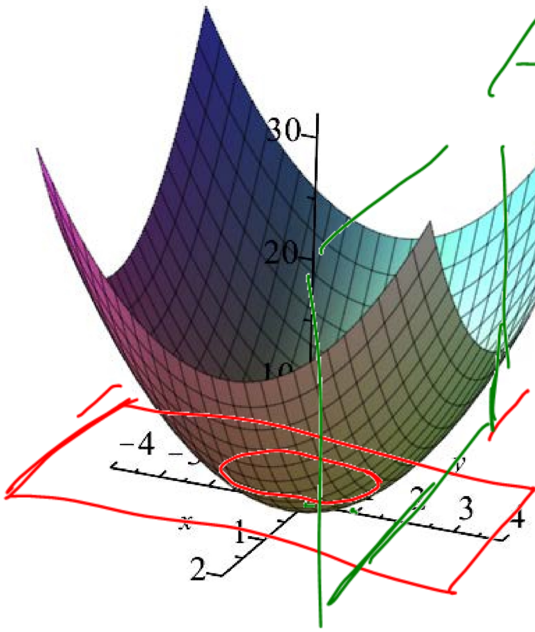
parabolas

Example 3.3. The graph of the elliptic paraboloid

ellipse

$$\boxed{z = 4x^2 + y^2}$$

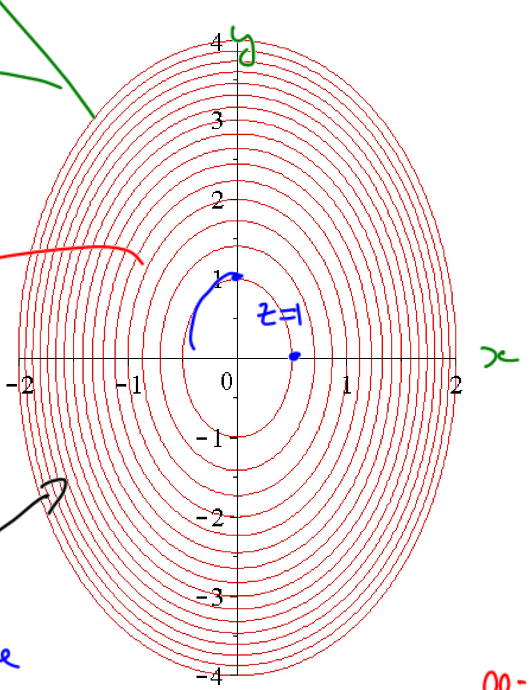
is drawn below and a collection of level curves is provided. The inner curve is z=1 and the outer curve is z=16. Find the parameters and study the parabolas obtained by the vertical cuts.



$z = 1$

$\overset{20}{\cancel{z}} = 4x^2 + y^2$

just in xy



Take a slice through surface by setting one variable = a constant.

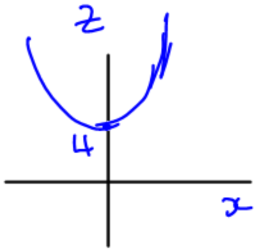
Take a vertical slice

eg. $y = 2$ (get xz plane)

$\downarrow \quad z = 4x^2 + y^2$

$z = 4x^2 + 2^2$

or $z = 4x^2 + 4$



same ellipses, radii change.

Eg. Set $z = 1, 2, 3, \ldots 10$.

$\downarrow \quad z = 4x^2 + y^2$ surface

$1, 2, 3, \ldots 10 = 4x^2 + y^2$

xy eq'n/curve.

get y radius by setting $x=0$

$1 = 4 \cdot 0^2 + y^2 \rightarrow y^2 = 1$

$\boxed{y = \pm 1}$

get x radius by setting $y=0$

$1 = 4x^2 + 0^2$

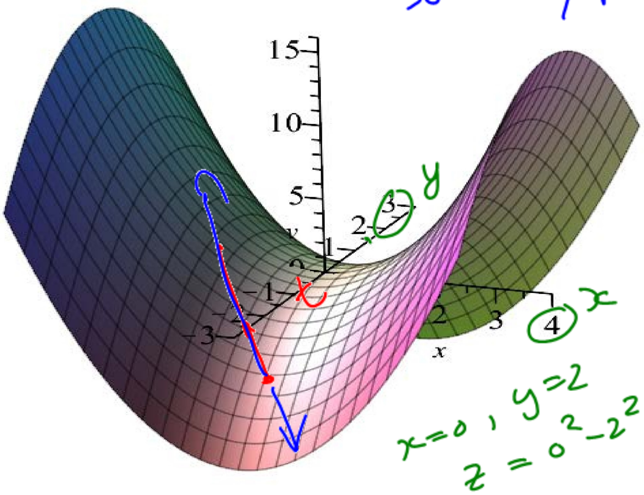
$\boxed{x = \pm 0.5}$

mountain pass

Example 3.4. The graph on the left belongs to the surface $z = x^2 - y^2$, and the level curves at the right are spaced at intervals of $z = 1$.

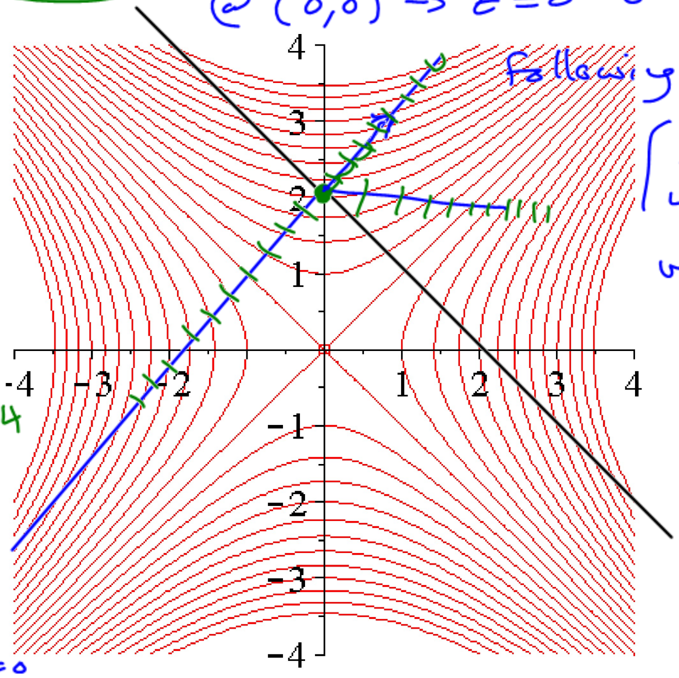
saddle / pass

@ $(0,0) \rightarrow z = 0^2 - 0^2 = 0$



$x = 0,\ y = 2$

$z = 0^2 - 2^2 = -4$



following $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ vector

From the contour map it is clear that the surface contains two straight lines: $y = x$ and $y = -x$. Find a line passing through the point $(0, 2, -4)$ that lives entirely on the surface.
Draw its projection on the *x*-*y* plane.

$z=0$

point + surface

want a line.

point $(0,2,-4)$

vector? $[a, b, c]$

line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

want to lie on / touching surface at all points

set = to

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ x^2 - y^2 \end{bmatrix}$

@ $z = 0$

$z = x^2 - y^2$

$0 = x^2 - y^2$

$y^2 = x^2$

$y = \pm x$

set of xy points

through (0,0)

$\hookrightarrow$ line $\quad y = x \leftarrow$ slope 1

or $\quad y = -x \leftarrow$ slope -1

$$\left.\begin{aligned} x &= at \\ y &= 2 + bt \\ z &= -4 + ct \end{aligned}\right\} \rightarrow z = x^2 - y^2$$

$(-4 + ct) = (at)^2 - (2 + bt)^2$

$\cancel{-4} + ct = a^2t^2 - (\cancel{4} + 4bt + b^2t^2)$

$0 = (a^2 - b^2)t^2 - ct - 4bt$

$0 = \cancel{t}\left[(a^2 - b^2)t - c - 4b\right]$

must be true for all t's

want a, b, c $\cancel{\neq}$ 0

$\rightarrow$ vector dir'n.

Pick $c = 1$ $\longrightarrow$ $0 = (a^2 - b^2)t - 1 - 4b$

always $= 0$ for all t

pick: $t = 1$ $0 = (a^2 - b^2) - 1 - 4b$

① $a^2 = b^2 + 1 + 4b$

$t = 2$ $0 = (a^2 - b^2) \cdot 2 - 1 - 4b$

② $2a^2 = 2b^2 + 1 + 4b$

② − ① $a^2 = b^2 + 0 + 0$

$a = \pm b$ $a = +b$

① $a^2 = \cancel{b^2} = \cancel{b^2} + 1 + 4b$

$0 = 1 + 4b$

$4b = -1$ $b = -\frac{1}{4}$

$a = -1/4$

so $\vec{v} = [a, b, c]$

$= \underline{[-1/4, -1/4, 1]}$

so one line always in surface

$\therefore$ $\begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

+

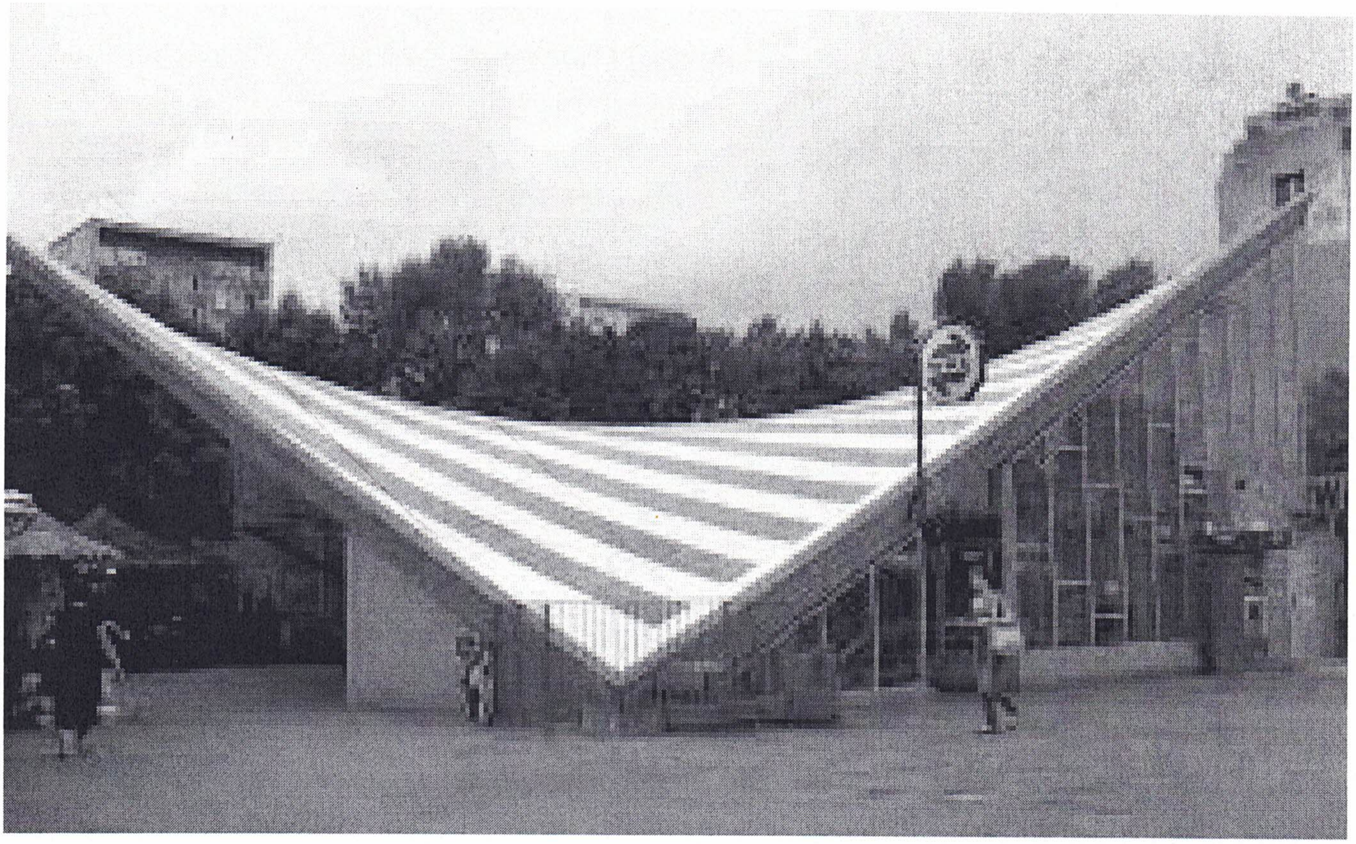
second solution

$\begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

$c = 4$

⋮

$[-1, -1, 4]$ ✓



The Calgary saddledome is a hyperbolic paraboloid—
built out of straight planks.

more than 1 = slopes

4. Partial derivatives.

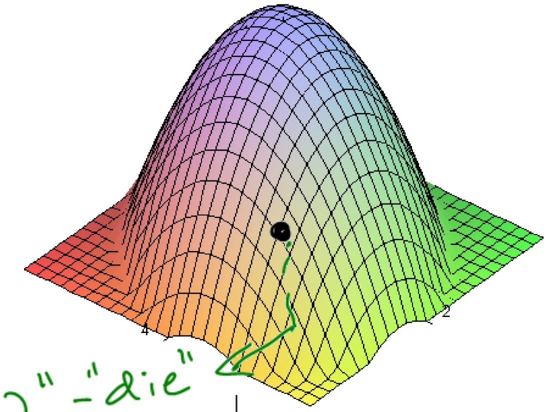
Example 4.1. The graph of the function

$$z = 16 - x^2 + xy - 4y^2$$

is drawn at the right.

z=6

Suppose you are standing on the hill at the point $(x, y, z) = (3, 1, 6)$ and start walking in the positive x-direction. What is the slope of your path? Same question for the positive y-direction.



"∂" - "die"

Start @ (3,1) = (x,y)

Go in pos x dir'n

∂

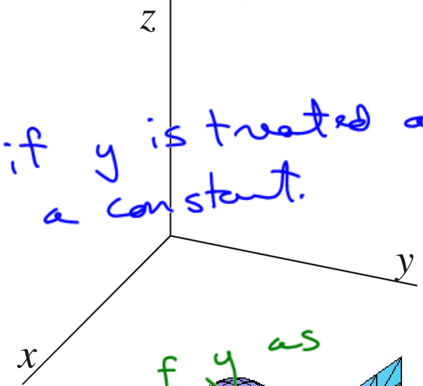
y's stay constant.

$$\frac{\partial z}{\partial x} = \frac{d}{dx}\left(16 - x^2 + x\overset{1}{y} - 4\overset{1}{y}^2\right)$$

if y is treated as a constant.

$$= -2x + 1 + 0 = -2x + 1$$

const const

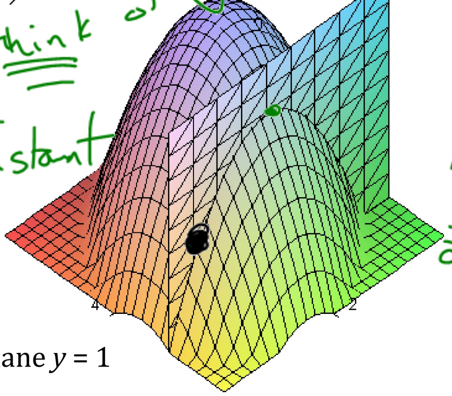


OR

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(16 - x^2 + xy - (4y^2)\right)$$

$$= 0 - 2x + y \cdot 1 + 0$$

Think of y as a constant



The plane $y = 1$

$$\frac{d}{dx} 3x = 3$$

$$\frac{\partial}{\partial x} y \cdot x = y$$

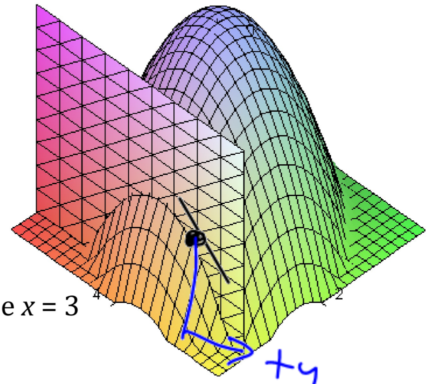
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(16 - x^2 + xy - 4y^2\right)$$

treat x as a constant

$$= 0 - 0 + x - 8y$$

@ (3,1)

$$\frac{\partial z}{\partial y}(3,1) = 3 - 8(1) = -5$$



The plane $x = 3$

If $z = f(x, y)$ is a function of x and y, and we hold y fixed at y_0, then $z = f(x, y_0)$ becomes a function of the single variable x. The derivative of this function at the point $x = x_0$ is called *the partial derivative of f with respect to x* and is written

$$f_x(x_0, y_0) \quad \text{or} \quad \frac{\partial z}{\partial x}(x_0, y_0)$$

no f'

primes!

$$\frac{d}{dx} = 2^3 \cdot x = 2^3$$

Geometrically, it is the slope of the graph in the x-direction.

Similarly, if we hold x fixed at x_0, then $z = f(x_0, y)$ becomes a function of the single variable y and the derivative of this function at the point $y = y_0$ is called *the partial derivative of f with respect to y* and is written

$$f_y(x_0, y_0) \quad \text{or} \quad \frac{\partial z}{\partial y}(x_0, y_0)$$

Geometrically, it is the slope of the graph in the y-direction.

treat y as const.

Example 4.2. Let $f(x, y) = (x^3 - xy^3)^2$

$$f_x = \frac{\partial z}{\partial x} = 2(x^3 - xy^3)^1 \cdot \frac{\partial}{\partial x}(x^3 - xy^3)$$

chain rule

Calculate $f_x(2,1)$ and $f_y(2,1)$.

$\partial/\partial y$

$$= 2(x^3 - xy^3)(3x^2 - y^3)$$

[Answ: $f_x(2,1) = 132$ and $f_y(2,1) = -72$

$$@(2,1),\ f_x(2,1) = 2(2^3 - 2\cdot 1^3)(3\cdot 2^2 - 1^3)$$
$$= 2(6)(11) = 132$$

$$f_y = \frac{\partial z}{\partial y}$$

const

$$= 2(x^3 - xy^3)^1 \frac{\partial}{\partial y}(x^3 - xy^3)$$

chain rule

$$= 2(x^3 - xy^3)(\overset{0}{\cancel{3x^2}} - 3xy^2)$$

@ (2,1)

$$f_y(2,1) = 2(2^3 - 2\cdot 1^3)(-3\cdot 2\cdot 1^2)$$
$$= 2(6)(-6)$$
$$= -72$$

Example 4.3. Let

$$f(x, y) = \ln\left(\frac{x^2 + y}{1 + xy}\right)$$

$\ln(ab) = \ln(a) + \ln(b$

$\ln(a/b) = \ln(a) - \ln(b)$

Calculate f_x and f_y.

rewrite first

$$f(x,y) = \ln(x^2 + y) - \ln(1 + xy)$$

$$f_x = \frac{2x}{x^2+y} - \frac{y}{1+xy}$$

$$\frac{d}{dx}\ln(\text{—}) = \frac{1}{(\text{—})} \times \ldots$$

chain

$$\begin{tikzcd} x \arrow[dr] & \\ & z = f(x,y) \\ y \arrow[ur] & \end{tikzcd}$$

$$f_y = \frac{1}{x^2+y} \cdot 1 - \frac{1}{1+xy} \cdot x$$

Shape $x^2+y^2+z^2=1$ is a sphere but not a function.

$z=f(x,y)$.

Implicit differentiation

Example 4.4. Find $\partial z/\partial x$ and $\partial z/\partial y$ at the point $(1, 2, 1)$ if z is defined implicitly as a function of x and y by the equation

$$xz^3 + yz - xy + y^2 = 5$$

$x \to z$

$y \to z$

Take $\frac{\partial}{\partial x}$ of both sides of the eq'n

$$\frac{\partial}{\partial x}(x\cdot z^3 + yz - xy + y^2) = \frac{\partial}{\partial x}(5)$$

y const, z as a func of x

↓ prod

$$1\cdot z^3 + (x)\left(3z^2\cdot\frac{\partial z}{\partial x}\right) + y\cdot\frac{\partial z}{\partial x} - y + 0 = 0$$

chain

isolated terms

@ (1,2,1) → sub in, $\frac{\partial z}{\partial x}=\ldots$

$\frac{\partial}{\partial x}y=0$

$\frac{\partial}{\partial x}x=1$

$\frac{\partial}{\partial x}z=\frac{\partial z}{\partial x}$

Take $\partial/\partial y$ of both sides

$$\frac{\partial}{\partial y}(xz^3 + yz - xy + y^2) = \frac{\partial}{\partial y}(5)$$

$$x\left(3z^2\frac{\partial z}{\partial y}\right) + 1\cdot z + y\cdot\frac{\partial z}{\partial y} - x + 2y = 0$$

$\frac{\partial}{\partial y}y=1$

$\frac{\partial}{\partial y}x=0$

$\partial/\partial y\, z=\frac{\partial z}{\partial y}$

Second derivatives.

Example 4.5. Find the second partial derivatives of

$$z = f(x, y) = x^3 + 2x^2y^3 - 2y^2$$

$f(x,y)$

For example f_{xy}, sometimes written as

$\partial/\partial x$

$$\frac{\partial^2 z}{\partial x \partial y},$$

means differentiate first with respect to x and then with respect to y.

3rd deriv

$f_{xyy}=f_{yxy}$

$=f_{yyx}$

$\partial/\partial x$

$f_x = 3x^2+4xy^3$

$f_{xx} = 6x+4y^3$

$\partial/\partial y$

$f_{xy} = 12xy^2$

$f_{xy}=f_{yx}$

always

$\partial/\partial y$ and

$\partial/\partial x$

$f_y = 6x^2y^2-4y$

$f_{yx} = 12xy^2$

$\partial/\partial y$

$f_{yy} = 12x^2y-4$

Note that in the Example above $f_{xy} = f_{yx}$. This is not a coincidence but is always the case (if the derivatives exist and are continuous). More generally if you are calculating any mixed higher order partial derivative, it doesn't matter in what order you take the variables. This is known as *Clairaut's Theorem*.

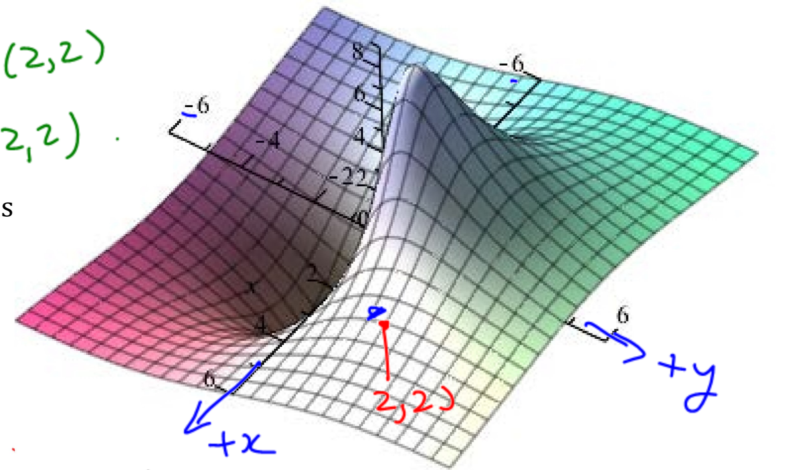
Example 4.6. Below are diagramed a set of contour lines for the function

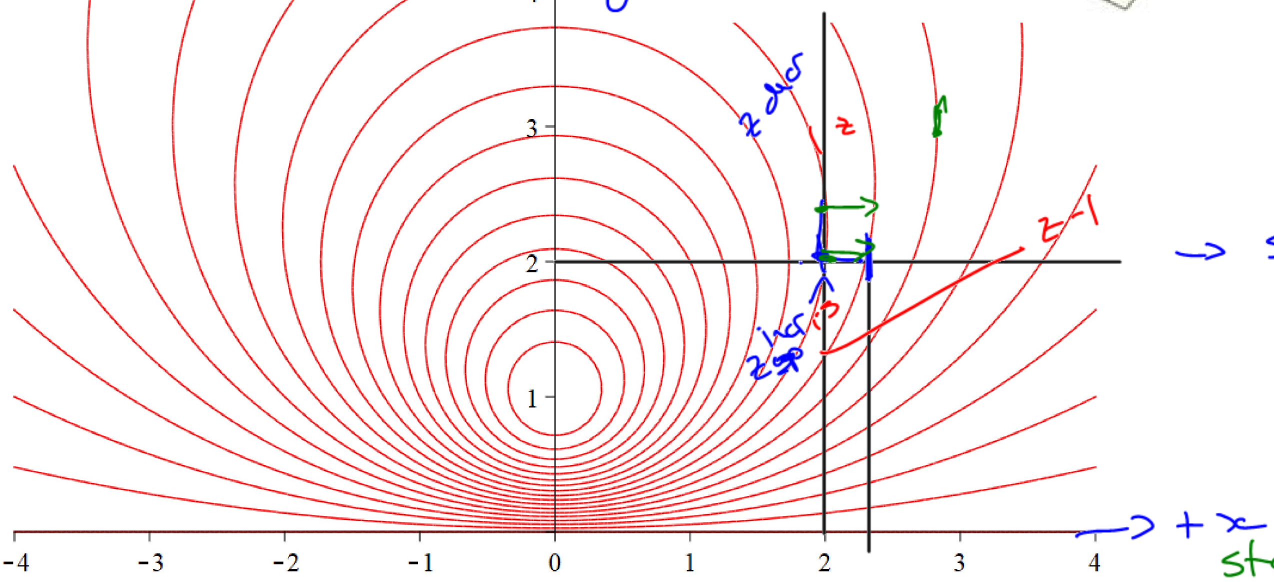
$$f(x,y) = \frac{18y}{x^2+y^2+1}$$

→ f_x, (2,2)

→ f_y (2,2)

on the domain $y \geq 0$. The zero level curve is the x-axis and as we move towards the top of the hill at (0, 1) the level curves are spaced at intervals of $\Delta z = 0.5$.





→ step in +x dir'n

step to nearest contour

(a) Use the diagram to estimate the partial derivatives

$f_x(2,2)$ and $f_y(2,2)$

$$f_x = \frac{\partial z}{\partial x} \simeq \frac{\Delta z}{\Delta x} = \frac{-0.5}{0.3} = -1.66$$

for a small Δx (exact $= -1.7$)

$$f_y(2,2) = \frac{\partial z}{\partial y} \approx \frac{\Delta z}{\Delta y} = \frac{0}{\text{small step in } y} = 0$$

(exact $= 0.01$)

tangent to contour ⇒ 0 slope

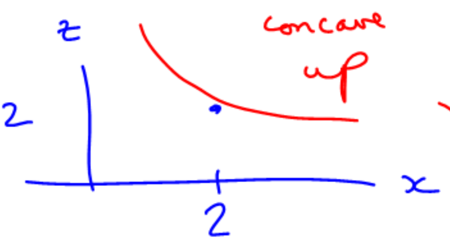
(b) Use the diagram to find the sign of the second-order partial derivatives

$f_{xx}(2,2)$ → lik 1 variable

$f_{yy}(2,2)$

$f_{xy}(2,2)$

along $y=2$



concave up ⇒ f_{xx} pos

z along $x=2$

concave down f_{yy} neg

$$f_{xy} = \frac{\partial}{\partial y}(f_x)$$

How does f_x change if y incr by just a bit

x slopes got less steep / negative

slope → $-1 \,..\, -0.5$

incr in value

f_{xy} +ve.