7. Gradient and directional derivative.

xy

slope in the direction of our choice.

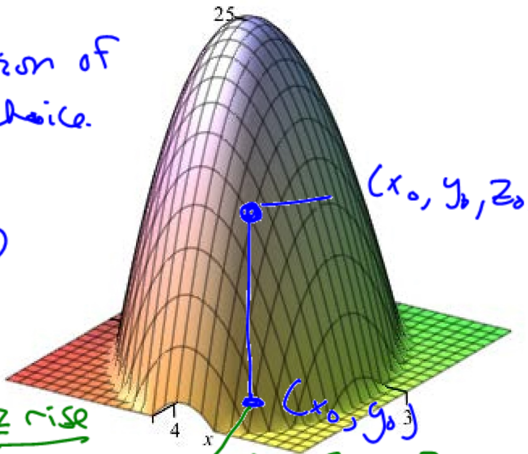
The directional derivative

We return to the paraboloid of Example 6.1.

$$z = \underline{f(x,y)} = 25 - x^2 + xy - 4y^2$$

Suppose you are standing on the surface at the point (x_0, y_0, z_0) and begin walking up the hill in the direction of the (horizontal) vector $[a, b]$. Use the multivariable chain rule to find a general expression for the slope of your path at that point.

down



for a parameterized curve $m = \dfrac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$ $\dfrac{z \text{ rise}}{xy \text{ run}}$

chain rule

also $\dot{z} = \dfrac{dz}{dt} = (f_x)\dot{x} + (f_y)\dot{y}$

Trajectory $x = x_0 + t \cdot a$

$y = y_0 + t \cdot b$

explicit dir'n !!

$\dot{x} = a$

$\dot{y} = b$

$$\dot{z} = (f_x) \cdot a + (f_y) b = [f_x, f_y] \cdot [a,b]$$

So our slope in $[a,b]$ dir'n is

$$m = \frac{[f_x, f_y] \cdot [a,b]}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{[f_x, f_y] \cdot [a,b]}{\sqrt{a^2+b^2}} = \underbrace{[f_x, f_y]}_{\text{gradient vector, } \nabla f} \cdot \overbrace{\frac{[a,b]}{\|[a,b]\|}}^{\text{length 1, unit vector}}$$

Definition. The *directional derivative* of $f(x, y)$ at the point (x_0, y_0) in the direction of the vector $[a, b]$ is defined as

surface

slope

dir'n

$$D_{[a,b]} f(x_0, y_0) = \frac{f_x(x_0, y_0)a + f_y(x_0, y_0)b}{\sqrt{a^2 + b^2}} = \frac{\nabla f \cdot \mathbf{u}}{\|\mathbf{u}\|}$$

point

This gives us the slope of the surface $z = f(x, y)$ at the point (x_0, y_0) in the direction of the vector $\mathbf{u} = [a, b]$.

x_0, y_0 $z_0 = f(x_0, y_0)$

Example 7.1. Find the slope of the paraboloid above at the point (3, 1, 15) in the direction [–2, –1]. [Answer: $m = 3\sqrt{5}$]

$[a, b]$

slope at (3,1) in dir'n $[-2,-1]$, $f(x,y) = 25 - x^2 + xy - 4y^2$

$\hookrightarrow f_x = -2x + y$

$\hookrightarrow f_y = x - 8y$

@ point (3,1)

$f_x(3,1) = -6 + 1 = -5$

$f_y(3,1) = 3 - 8 = -5$

so $\nabla f = [f_x, f_y]$

$\boxed{= [-5, -5]}$

$$D_{[-2,-1]} f(3,1) = \nabla f \cdot \frac{[-2,-1]}{\|[-2,-1]\|}$$

$$= [-5,-5] \cdot \frac{[-2,-1]}{\sqrt{5}} = \frac{+15}{\sqrt{5}} \text{ slope}$$

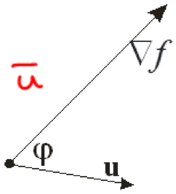
The Gradient

Recall that we have defined the gradient of a function f of x and y at the point (x_0, y_0) as the vector of its partial derivatives $\nabla f = [f_x \quad f_y]$, or more completely:

$$\nabla f(x_0, y_0) = [f_x(x_0, y_0) \quad f_y(x_0, y_0)]$$

This concept gives us a natural way of writing the multivariable chain rule and the directional derivative as a simple dot product.

Example 7.2. How does the gradient interact with the graph of the function f? What's the "geometry" of the configuration?



pick $\bar{u}$

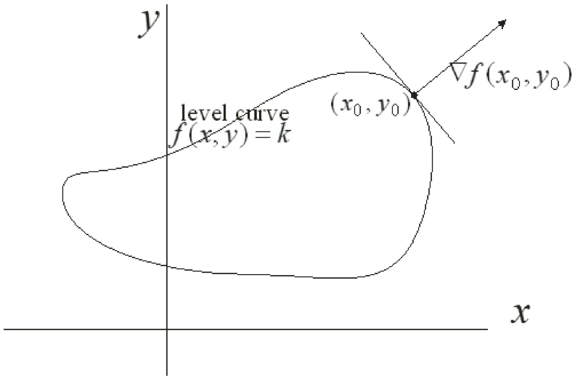
get to pick by picking $\bar{u}$

Consider the directional derivative of f in any direction **u**:

$$D_{\mathbf{u}}f = \frac{\nabla f \cdot \mathbf{u}}{\|\mathbf{u}\|} = \frac{\|\nabla f\|\,\|\mathbf{u}\|\cos(\varphi)}{\|\mathbf{u}\|} = \|\nabla f\|\cos(\varphi)$$

no control over

This equation shows that $D_{\mathbf{u}}f$ depends only on the length of the vector ∇f and the angle φ between ∇f and the direction vector **u**. We can deduce a couple of important things from that.



First, $D_{\mathbf{u}}f$ will be zero when φ = 90º. Now note that $D_{\mathbf{u}}f$ will also be zero in the direction of any level curve of f. We deduce that ∇f is orthogonal to the level curve of f passing through (x_0, y_0).

max slope

Secondly, $D_{\mathbf{u}}f$ will be a maximum when φ = 0°, that is when **u** points in the same direction as ∇f. Turning this around:

∇f points in the direction of maximum rate of increase of $f(x, y)$.

Further, this maximum rate of increase is:

$$D_{\mathbf{u}}f = \|\nabla f\|\cos(0) = \|\nabla f\|$$

which is the length of the gradient vector.

To Summarize:

* (1) ∇f is orthogonal to the level curve

(2) The maximum value that $D_{\mathbf{u}}f$ attains is the magnitude $\|\nabla f\|$ of the gradient. slope = $\|\nabla f\|$

* (3) This maximum is attained in the direction of the gradient ∇f.

$\nabla f = [f_x, f_y]$ if $f(x,y)$

(or $\nabla f = [f_x, f_y, f_z]$ if $f(x,y,z)$)

Example 7.3. We have defined the gradient for functions of two variables. But the notion extends to any number of variables. Let the temperature (ºC) at a point (x, y, z) in space be

$$T(x, y.z) = 4 + x^2 + 2y^3 + 3z^3$$

In which direction from the point (3, 2, 1) does the temperature increase most quickly, and what is this maximum rate of increase?

old

$T_x = 2x$
$T_y = 6y^2$
$T_z = 9z^2$

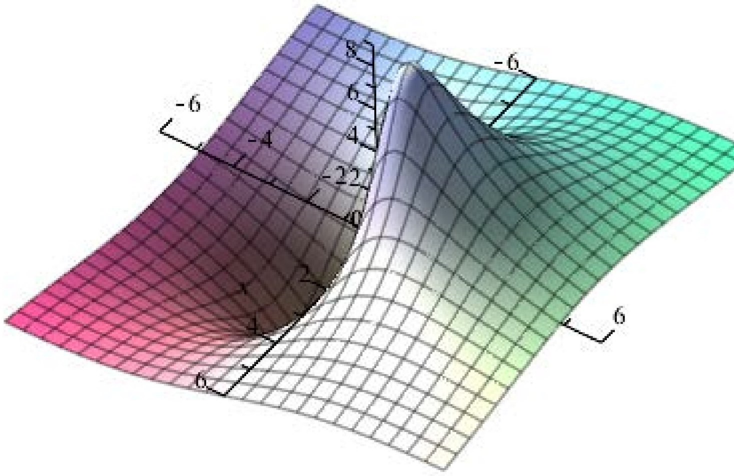
$T_x = 6$
$T_y = 6 \cdot 4 = 24$
$T_z = 9$
@(3, 2, 1)

new
∇T
$= [6, 24, 9]$

dir'n of the gradient.

rate of temp increase
$= \|\nabla T\| = \|[6, 24, 9]\|$
$\cong 26$ °C/m

Example 7.4. Below are diagramed a set of contour lines for the function $\boxed{f(x, y) = \dfrac{18y}{x^2 + y^2 + 1}}$ on the domain $y \geq 0$.

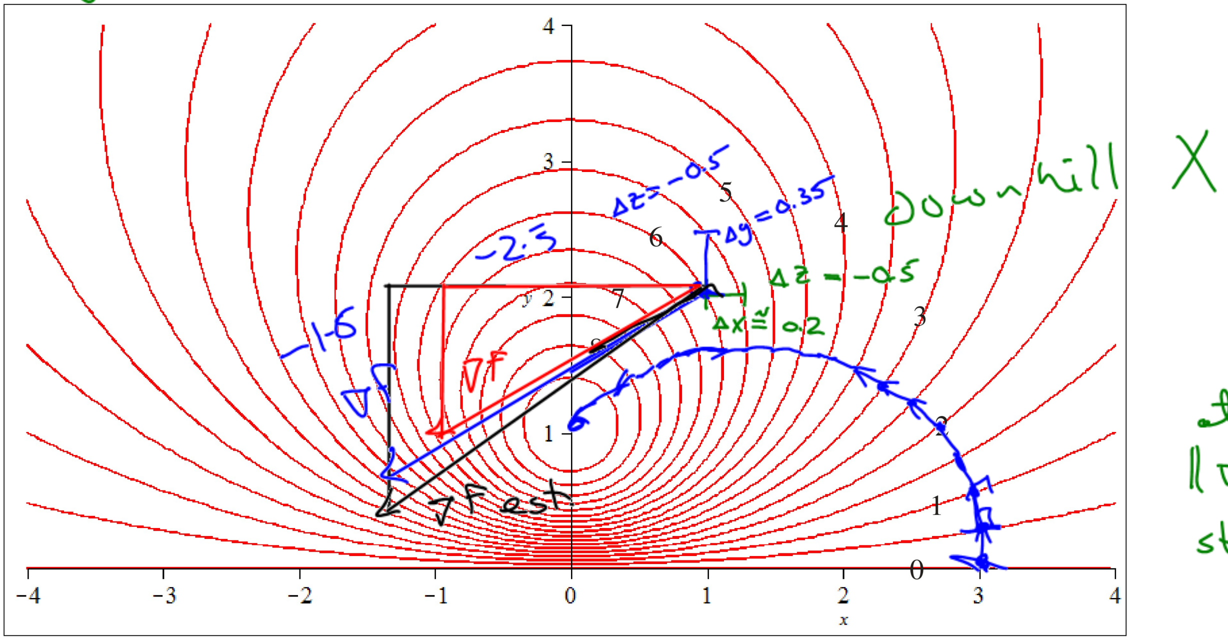


The zero level curve is the x-axis and as we move towards the top of the hill at (0, 1) the level curves are spaced at intervals of $\Delta z = 0.5$.

(a) Starting at (3, 0), sketch my path if I always ascend the hill most steeply. in tiny steps

(b) Starting at (3, 0), sketch a path that ascends the hill with a slope of 1.

$\frac{z}{xy}$



at (3, 0)
$\|\nabla f\| =$
steepest
slope availab
at (3, 0)

(c) Use the spacing of the contour lines to draw the gradient vector of f at the point (1, 2). Check your work using the given formula for f.

- dir'n of gradient is - uphill
- ⊥ to contours

– length of $\nabla f = \|\nabla f\| = \dfrac{\Delta z}{(\Delta x, y)}$ in dir'n of ∇f

small step

$= \dfrac{+0.5}{0.2}$ accurate / estimated $= 2.5$

Also from contour diagram

$\nabla f = [f_x \;\; f_y]$ at $(2,1)$

$\approx \left[\dfrac{\Delta z}{\Delta x}, \dfrac{\Delta z}{\Delta y}\right] = \left[\dfrac{-0.5}{0.2}, \dfrac{-0.5}{0.35}\right] \cong [-2.5, -1.5]$

for a small Δx step

for a small Δy step

Exact from formula $f(x,y) = \dfrac{18y}{x^2+y^2+1} = 18y\cdot(x^2+y^2+1)^{-1}$

so $f_x = 18y(-1)(x^2+y^2+1)^{-2}\cdot(2x)$

and $f_y = (18)(x^2+y^2+1)^{-1} + (18y)(-1)(x^2+y^2+1)^{-2}(2y)$

At $(\cancel{2,1})$ $(1,2)$ $\to (x^2+y^2+1) = 6 \to (\;)^{-1} = 1/6,\; (\;)^{-2} = 1/36$

$f_x = 1\cancel{8(2)}(-1)\left(\dfrac{1}{\cancel{36}}\right)(2) = -2$

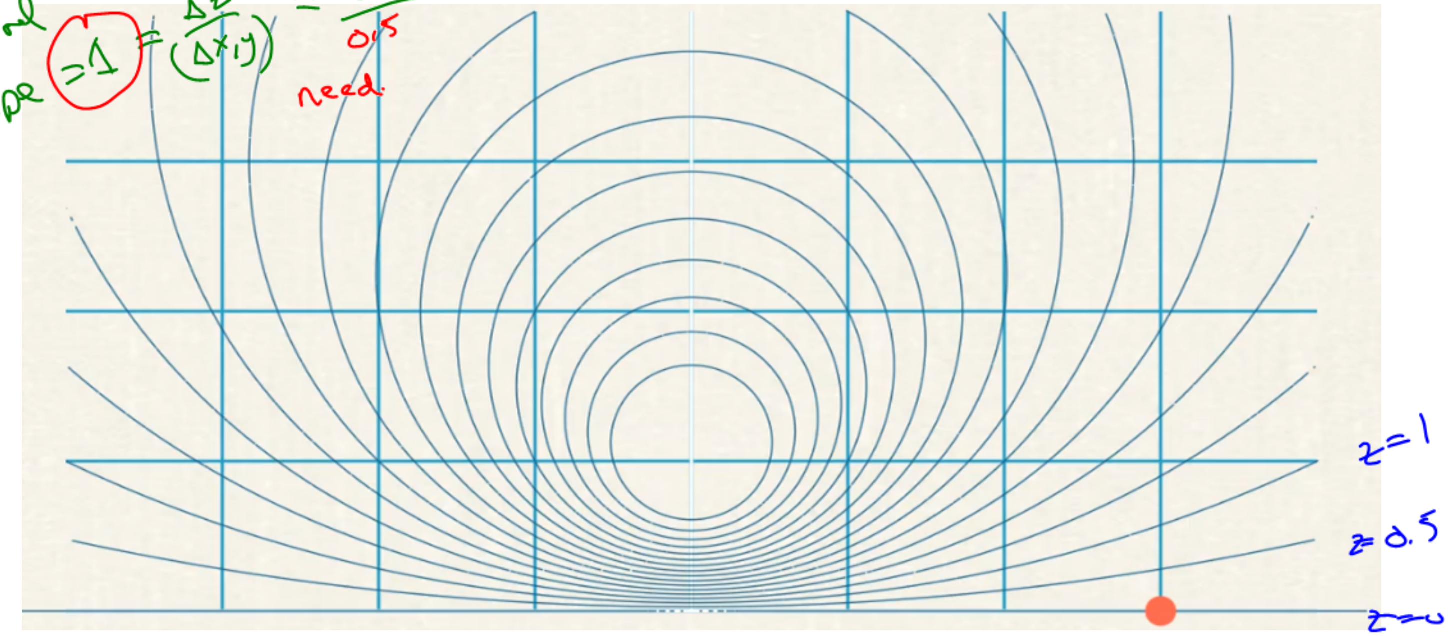
$f_y = (18)(1/6) + 1\cancel{8(2)}(-1)\left(\dfrac{1}{\cancel{36}}\right)(4)$

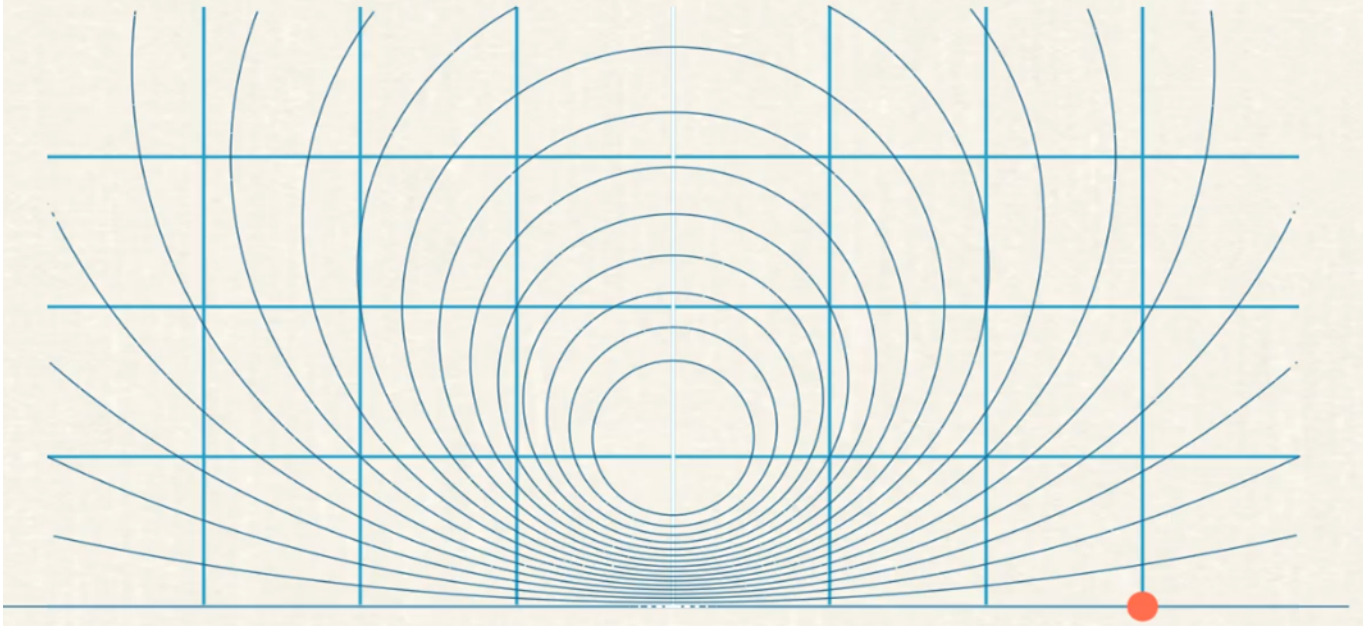
$= 3 - 4 = -1$

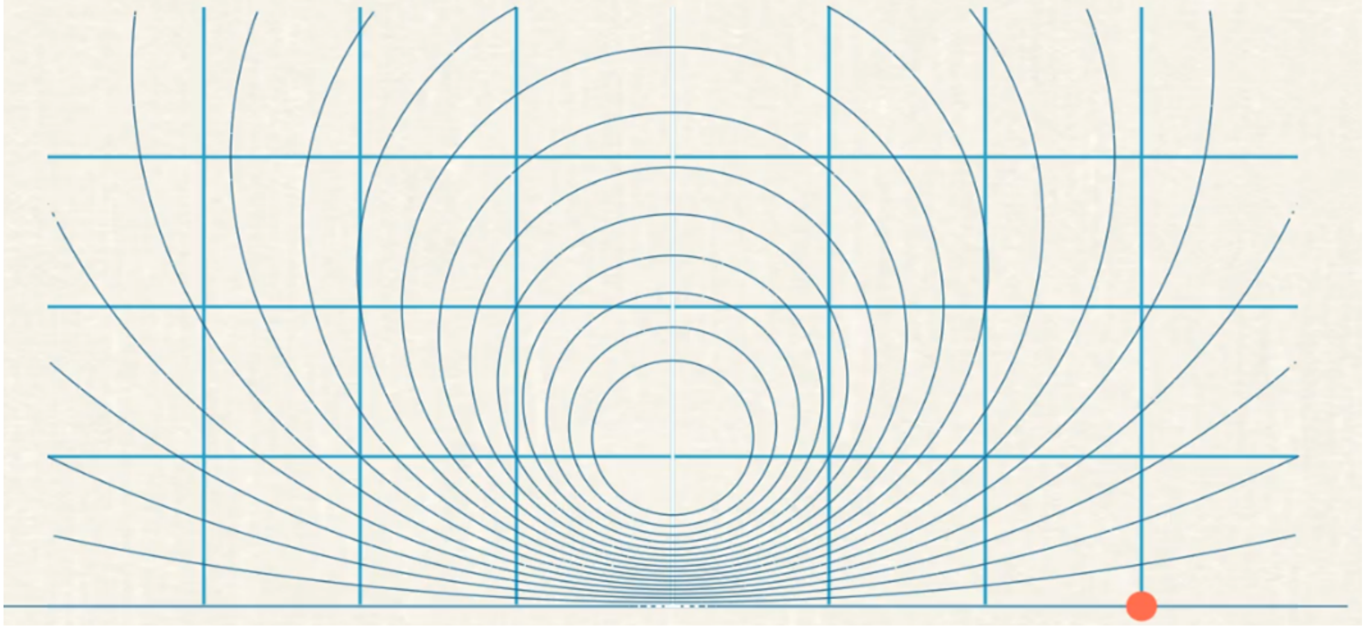
$\to \nabla f = [-2, -1]$

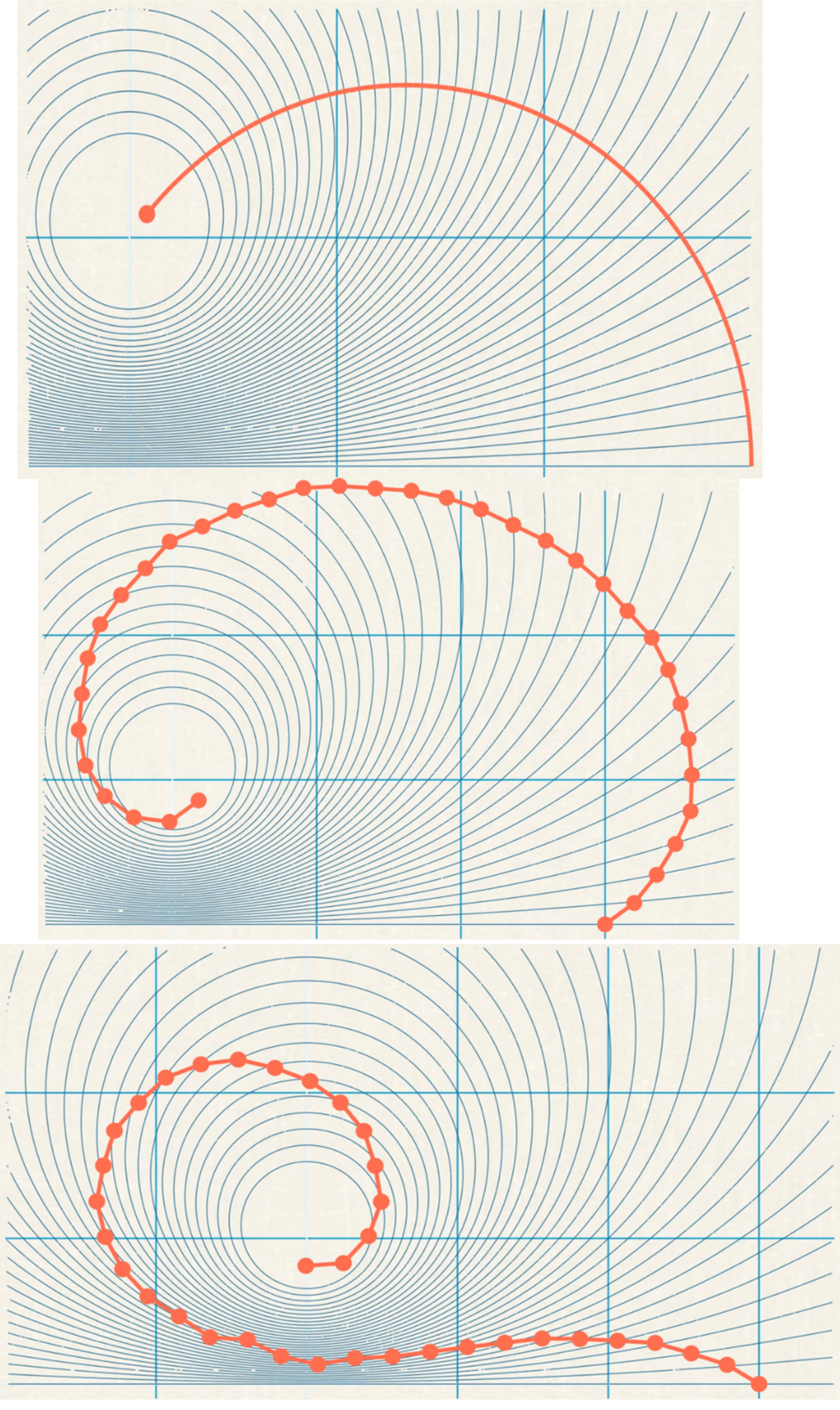
goal slope $\boxed{=1} = \frac{\Delta z}{(\Delta x, y)} = \frac{0.5}{0.5}$

need









$z_0 = f(2,1) = 2$

Example 7.5. (a) In what direction in the *x*-*y* plane does

gradient dir'n $z = f(x, y) = x^2y^2 - xy^3$

have its maximum rate of change at the point (2, 1)? What is this maximum rate?

Answ: $\nabla f = [3, 2]$ $||\nabla f|| = \sqrt{13}$ at (2, 1) point

$f_x = 2xy^2 - y^3$ $f_x = 2 \cdot 2 - 1^3 = 3$

$f_y = 2x^2y - 3xy^2$ $f_y = 2 \cdot 4 - 3 \cdot 2 = 2$

so $\nabla f = [3,2]$ is dir'n of max incr of $f(x,y)$

Also max rate of change is $\| \nabla f \| = \| [3,2] \| = \sqrt{13}$

(b) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point (2, 1).

Answ: $z = 2 + 3(x - 2) + 2(y - 1)$

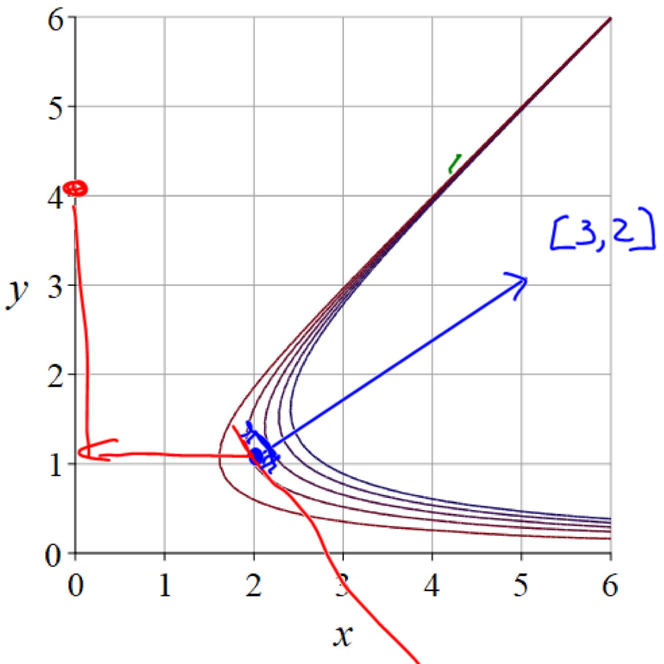
Easy after (a)

$z = z_0 + (f_x)(x - x_0) + (f_y)(y - y_0)$

$z = 2 + 3(x-2) + 2(y-1)$ is the tgt plane.

(c) The diagram at the right displays the contour lines of *f*(*x*, *y*) for ~~z = 1, 2,3, 4, 5.~~ Draw the gradient vector of *f* based at the point (2, 1).

$\hookrightarrow \nabla f(2,1) = [3,2]$



(d) Now draw the contour lines of the tangent plane passing through the point (2, 1) and through the point

$$(2,1) + \nabla f(2,1).$$

[That is, the lines should pass through the tail and the head of the gradient vector you drew in (c)]
Label both lines with their *z*-value. Do this in two different ways, working with the graph, and working with the equations.

$z = 2 + 3(x-2) + 2(y-1)$

pick z, drawing x,y relationship

z = 2 b/ match height at (2,1)

$\cancel{2} = \cancel{2} + 3(x-2) + 2(y-1)$ $\longrightarrow$ $2(y-1) = -3(x-2)$

solve for y
know to graph $y = \ldots$ formulas

$y - 1 = -\frac{3}{2}(x-2)$

$y = -\frac{3}{2}(x-2) + 1$

x,y relation we can graph.

- line
- slope $-3/2$
- through (2,1)

Example 7.6. These diagrams depict a surface in R³. We have two ways of thinking about this surface
1) as the graph $z = f(x, y)$ of a function f of two variables
2) as a level surface $T(x, y, z) = k$ of a function T of three variables.
In this case take $T(x, y, z) = z - f(x, y)$

Using the diagrams, depict the gradient vector of each function and find an equation for the plane tangent to the surface in terms of each function. Recall that the gradient of a function f at a point is the vector of partial derivatives at that point and it points in the direction of the maximum rate of increase of f.

$z = f(x,y) \rightarrow \nabla f = [f_x, f_y]$ in xy plane

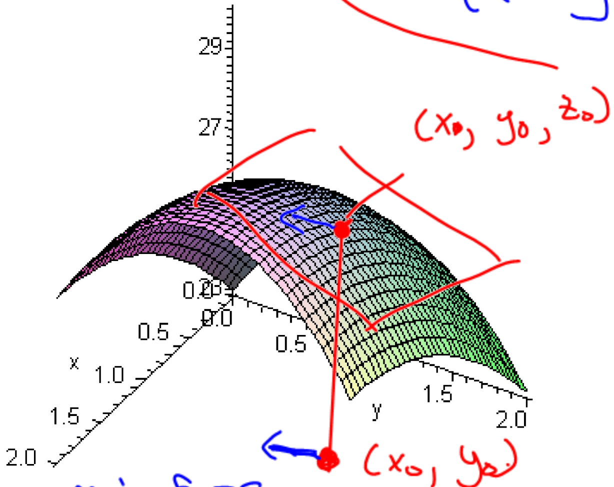
$\longleftrightarrow$ rewrite

$0 = z - f(x,y)$

$\overset{\shortparallel}{k} = T(x, y, z)$

$\nabla T = [T_x, T_y, T_z]$ in $\underline{\underline{3}}$D

Points in dir'n of max temp increase



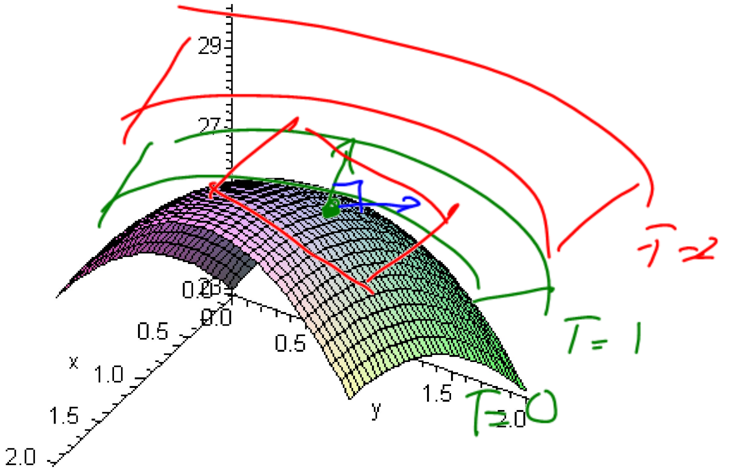
dir'n of ∇f

$\shortparallel$

direction of steepest $\underline{z}$ incr for any small xy change

Tgt plane

① $z = z_0 + (f_x)(x - x_0) + (f_y)(y - y_0)$



Tgt plane

we have point (x_0, y_0, z_0)

and $\vec{n} = \nabla T = [T_x, T_y, T_z]$

a, b, c

$\vec{n} \cdot [x - x_0, y - y_0, z - z_0] = 0$

② $(T_x)(x - x_0) + (T_y)(y - y_0) + (T_z)(z - z_0) = 0$

Prove ① = ② starting w/ $0 = \underbrace{z - f(x,y)}_{T(x,y,z)}$

so $T(x,y,z) = z - f(x,y)$

so $T_x = -f_x$

and $T_y = -f_y$

$T_z = 1$

so ② plane becomes $(-f_x)(x-x_0) + (-f_y)(y-y_0) + (1)(z-z_0) = 0$

Isolate z: $z - z_0 = (f_x)(x-x_0) + (f_y)(y-y_0)$

or $z = z_0 + (f_x)(x-x_0) + (f_y)(y-y_0)$

which is ① ✓