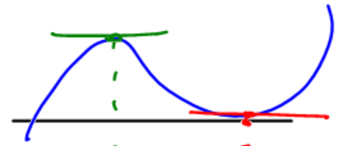
8. Optimization––max and min

One variable.
We start with a fast review of the one-variable situation.

1 variable functions (next page is multivariate)

A *critical point* of a function $f(x)$ is a point $x = a$ at which the derivative is zero:

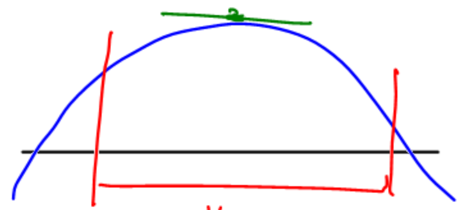
$$f'(a) = 0.$$



a_1, a_2 are both critical points

At such a point, the tangent line to the graph $y = f(x)$ is horizontal.

A standard result is that an *interior* local maximum (hilltop) or minimum (valley bottom) of $f(x)$ will always be a critical point. This is geometrically quite intuitive, but it is rewarding to review how a rigorous argument might go. Of course it will depend on the definition of the derivative.

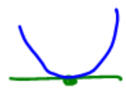


allow only this interval

The interesting question goes the other way. Suppose you have a critical point. How do we tell if it's a local max? Or a local min? And what other behaviours are possible?

The 2nd derivative test. If $x = a$ is a *critical point* of $f(x)$ then:

most common

- if $f''(a) > 0$, then a is a local min → conc up

- if $f''(a) < 0$, then a is a local max conc down


rare

- if $f''(a) = 0$, then the test fails

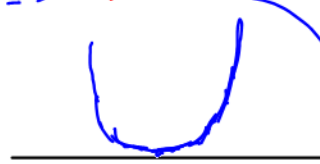
rare case study

Example 8.1 Give examples of a critical point a for which $f''(a) = 0$ and which is

and at a c.p. so $f'(a) = 0$ too

- a local min
- a local max
- neither

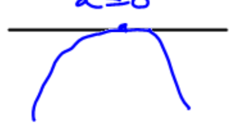
$y = x^4$



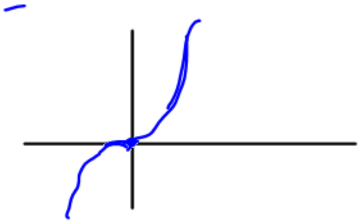
$y' = 4x^3 \to y'(0) = 0$
and $y'' = 12x^2 \to y''(0) = 0$ too

2nd deriv test fails

$y = -x^4$



$y = x^3, x^5$



or or all

Two variables.

A *critical point* of a function $f(x, y)$ is a point (a, b) at which both partial derivatives vanish:

$$f_x(a, b) = 0 \;\text{ and }\; f_y(a, b) = 0 \,.$$

$\Leftrightarrow \nabla f = [0, 0]$ is gradient at a c.p.

At a critical point, the tangent plane to the graph $z = f(x, y)$ is horizontal. at a c.p

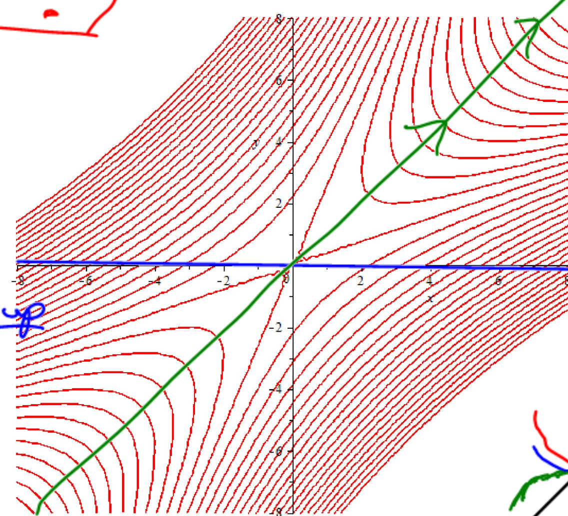
$z = z_0 + (f_x)(x - x_0) + (f_y)(y - y_0)$

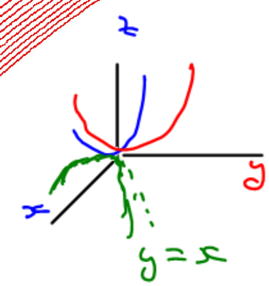
$z = z_0$ as our plane

Max and min and saddle

By analogy to the one-variable case (and with a similar argument), a local maximum (hilltop) or minimum (valley bottom) of $f(x, y)$ will always be a critical point. But how does the converse go now? Suppose we have a critical point. How do we tell if it's a local max or min? And what other behaviours are possible?

Example 8.2. Construct a function $f(x, y)$ which is a strict local min at the origin when restricted to x (i.e. along the line $y = 0$) and is a strict local min at the origin when restricted to y (i.e. along the line $x = 0$), but is a strict local *max* at the origin when restricted to the line $x = y$.





$z = x^2 + y^2 - 3xy$

Along $y = 0 \Rightarrow z = x^2$ conc ave up

Along $x = 0 \Rightarrow z = y^2$ conc up too.

But along $y = x$ line let $x = t$ and $y = t$

$z = t^2 + t^2 - 3t \cdot t$ needed this >2 to get concave down.

$= -t^2$

so conc down along this line.

The two-variable second derivative test.

Suppose that (a, b) is a critical point of the function $f(x, y)$, that is, both partial derivatives are zero. Define the quantity:

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

Recall $f_{xy} = f_{yx}$ always

Then there are three cases:

almost all cases.

1. $D > 0$. In this case note that f_{xx} and f_{yy} cannot be zero and must have the same sign at (a, b)— otherwise D would certainly be negative. If they are positive then the surface is concave up and we have a local minimum. If they are negative then the surface is concave down and we have a local maximum.

local min

local max

2. $D < 0$. In this case we have a saddle point, that is, we have a local minimum in some directions and a local max in others.

3. $D = 0$. In this case the test fails and we need to look at the function in more detail.

very rare

Rational for 2^{nd} deriv test

- scalar $f(x,y)$
- 1^{st} derivs is a vector gradient $\nabla f = [f_x, f_y]$
- 2^{nd} " " a 2x2 matrix

Hessian matrix $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \xrightarrow{\text{rotate axes } u,v} \begin{pmatrix} f_{uu} & 0 \\ 0 & f_{vv} \end{pmatrix}$

our D for 2^{nd} deriv test is $\underbrace{(f_{xx})(f_{yy})}_{\text{mult diagonal elements}} - \underbrace{(f_{xy})(f_{yx})}_{= (f_{xy})^2 \text{ mult of off-diagonal elements}}$ = Determinant of 2^{nd} deriv matrix

(week 10 in APSC 174)

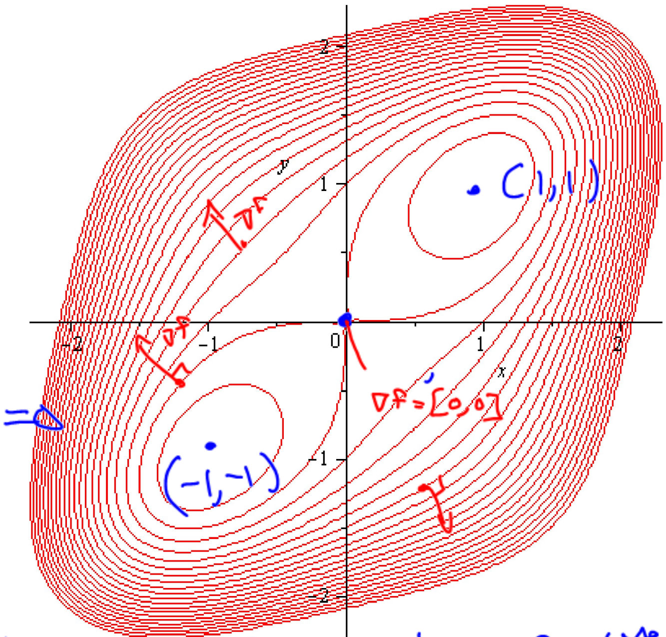
D = prod of 2 concavities in key directions (= eigen vectors in week 11 of APSC 174)

2 conc up dir'ns $\to$ $\cup$ local min $\to$ $D = (+)(+) = (+)$ overall

2 conc down dir'ns $\to$ $\cap$ local max $\to$ $D = (-)(-) = (+)$ overall too

1 conc down
1 conc up $\to$ saddle $\to$ $D = (+)(-) = (-)$ overall

Example 8.3. Find and classify the critical points of $f(x,y) = x^4 + y^4 - 4xy + 1$.



Find c.p.s

Need 1st derivs:

$f_x = 4x^3 - 4y$

$f_y = 4y^3 - 4x$

To find c.p's, set all partial derivs $= 0$

① $\cancel{4}x^3 - \cancel{4}y = 0 \rightarrow y = x^3$ ③

② $\cancel{4}y^3 - \cancel{4}x = 0$

③ → ② $(x^3)^3 - x = 0$

tidy $x^9 - x = 0$

factor $x(x^8 - 1) = 0$

$x = 0$ → $y = 0$; $x = 1$ → $y = 1$; $x = -1$ → $y = -1$

③ $y = x^3$

So we found 3 c.p.s

(0,0), (1,1), (-1,-1)

To classify each c.p, use 2nd deriv test

Need

$f_{xx} = 12x^2$

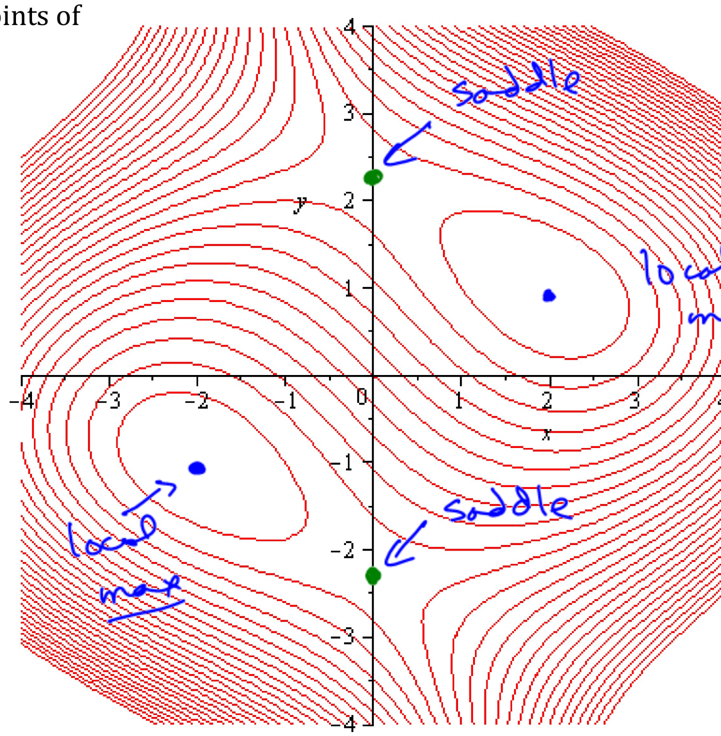
$f_{yy} = 12y^2$

$f_{xy} = -4$

x	y	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx} f_{yy} - f_{xy}^2$	Signs / type
0	0	0	0	-4	$0-(-4)^2 = -16$	$(-) = (-)(+)$ → saddle
1	1	+12	+12	-4	$144-(-4)^2 = +128$	$(+) = (+)(+)$ conc up
-1	-1	+12	+12	-4	+128	$(+) = (+)(+)$ local min

also local min

Example 8.4 Find and classify the critical points of
$f(x, y) = x^3 + y^3 - 15(x + y) + 3xy^2$.



Find c.p.s
Need

$$\boxed{\begin{aligned} f_x &= 3x^2 - 15 + 3y^2 \\ \text{and } f_y &= 3y^2 - 15 + 6xy \end{aligned}}$$

Set both $= 0$, solve for x, y

$3x^2 - 15 + 3y^2 = 0$ (1)

$3y^2 - 15 + 6xy = 0$ (2)

(1)/3 $\quad x^2 - 5 + y^2 = 0$ (3)

(2)/3 $\quad y^2 - 5 + 2xy = 0$ (4)

(3)−(4) $\quad x^2 - \cancel{y^2} + \cancel{y^2} - 2xy = 0$

tidy $\quad x^2 - 2xy = 0$

factor $\quad x(x - 2y) = 0$

$x = 0$ or $x = 2y$ (5)

$\to$ (3) $\quad 0^2 - 5 + y^2 = 0$

$y^2 = 5$

$x = 0, \; y = \pm\sqrt{5}$

$\to$ (3) $\quad (2y)^2 - 5 + y^2 = 0$

$\cancel{5}y^2 = \cancel{5}1 \quad y = \pm 1$

$y = 1 \to$ (5) $\; x = 2 \cdot 1 = 2$

or $\; y = -1 \to$ (5) $\; x = -2$

so $(-2, -1)$ and $(2, 1)$

$f_{xx} = 6x$

$f_{yy} = 6y + 6x$

$f_{xy} = 6y$

x	y	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx} f_{yy} - f_{xy}^2$	
0	$+\sqrt{5}$	0	$6\sqrt{5}$	$6\sqrt{5}$	$0 - (6\sqrt{5})^2 = (-)$	saddle
0	$-\sqrt{5}$	0	$-6\sqrt{5}$	$-6\sqrt{5}$	$0 - (-6\sqrt{5})^2 = (-)$	saddle
−2	−1	−12	−18	−6	$(-12)(-18) - (-6)^2 \to (+)$	local max
2	1	+12	+18	6	(+)	local min

D

(+) $\to$ f_{xx}, f_{yy} both (+) $\to$ conc up $\cup$ local min

(+) $\to$ f_{xx}, f_{yy} both (−) $\cap$ local max

(−) $\to$ (−)(+) concavities saddle

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

on closed, bounded domain will occur at either
• an interior c.p.
or • on the boundary.

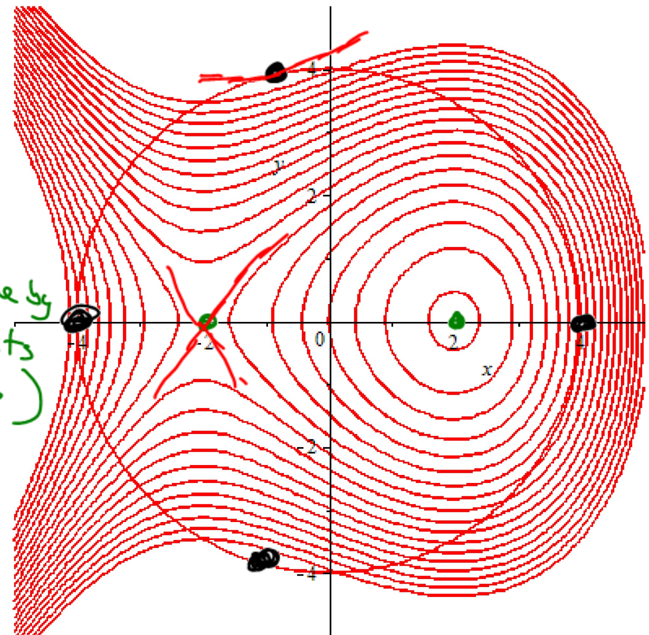
we just this ← sort-of new

Example 8.5

Find the global maximum and minimum values of the function

$$f(x,y) = 2x^3 + 9y^2 - 24x + 40$$

in the disk $x^2 + y^2 \le 16$



Look for (interior) c.p.s

Need

$f_x = 6x^2 - 24$

and $f_y = 18y$

Set both $= 0$ to find c.p.s (divide by consts too)

(÷6) $x^2 - 4 = 0$ ①

(÷18) $y = 0$ ②

① → $(x-2)(x+2) = 0$

$x = -2, +2$

so c.p.s are $(-2,0)$ and $(2,0)$

Check: both are inside region $x^2+y^2 \le 16$ / circle of radius 4. ✓

Look for local extrema on the boundary.

Strategy: parameterize the border

$\left\{ \begin{array}{l} x = 4\cos t \\ y = 4\sin t \end{array} \right\} \to z(t)$ single variable func.

→ find c.p.s in 1 variable

c.p. for border is at $\frac{dz}{dt} = 0$; $\boxed{z = 2x^3 + 9y^2 - 24x + 40}$

$\frac{dx}{dt} = -4\sin t = -y$

$\frac{dy}{dt} = 4\cos t = x$

Critical Points			
x	*y*	*z*	
Interior			
−2	0	72	
2	0	8	
Boundary			
4	0	72	
−4	0	8	
−1	$\sqrt{15}$	197	
−1	$-\sqrt{15}$	197	

Global min value is 8 tie @ (2,0) (−4,0)

Global max value is 197 @ tie over two points $(-1,\sqrt{15})$ $(-1,-\sqrt{15})$

Find multi. variate chain rule

$$\begin{tikzcd} & x \arrow[dr] & \\ t \arrow[ur] \arrow[dr] & & z \\ & y \arrow[ur] & \end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{dt} = (6x^2 - 24)\frac{dx}{dt} + (18y)\frac{dy}{dt}$$

could insert cos, sines but it gets ugly

$$\frac{dz}{dt} = (6x^2-24)(-y) + 18y(x) = 0$$

Set $= 0$ for c.p.

on y on 4-radius circle

$x^2+y^2=16$

tidy $(\div 6)$ $\quad -x^2y + 4y + 3xy = 0$

factor $\quad y(-x^2+3x+4) = 0 \quad \times(-1)$

$\times(-1)$

$$y(x^2-3x-4) = 0$$

$$y(x-4)(x+1) = 0 \quad ①$$

$$② \quad x^2+y^2=16$$

$y=0$ or $x=4$ or $x=-1$

$y=0 \to ②$ $\quad x^2+0=16 \to x=\pm 4 \to (4,0)$ and $(-4,0)$

$x=4 \to ②$ $\quad \cancel{4^2}+y^2=\cancel{16} \to y^2=0 \to y=0 \quad (4,0)$ same

$x=-1 \to ②$ $\quad (-1)^2+y^2=16 \to y^2=15 \to y=\pm\sqrt{15} \quad (-1,\sqrt{15})$ and $(-1,-\sqrt{15})$