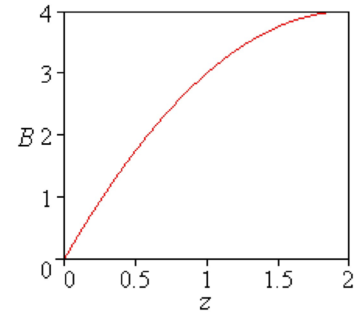
Game Theory Nash
A Beautiful Mind
– Mathematician

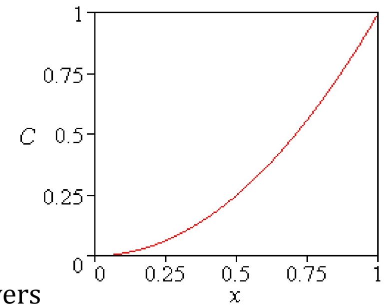
9. Investigations

Ex 9.1 *The parent game*

Suppose there is a task which requires an investment from two persons, a female A and a male B, and when carried out provides a benefit and a cost to each. Assume that the benefit B gained by each player depends on the total investment, but the cost C depends only on each player's investment. Thus the net payoff to a player investing x ($0 \le x \le 1$) with a partner investing y ($0 \le y \le 1$) has the form:

$$P(x, y) = B(x + y) - C(x)$$





$B(z) = z(4 - z)$
$C(x) = x^2$

$B'(z) = 4 - 2z$
$C'(x) = 2x$

The objective of the game is to get as large a payoff as possible. So the question for each player is, what's the best investment x, and of course, that will depend on your partner's y.

(a) *Cooperation.* Suppose first that they are a team with mutual trust, They decide that they will both make the same investment x. What is their optimal value of x?

(b) *Equilibrium.* Suppose we have a large well-mixed population with players encountering one another at random. By definitions, a strategy x^* is called a *Nash equilibrium* if when everyone is using it, a player that deviates from it can do no better. That is, for every x,

$$P(x, x^*) \le P(x^*, x^*).$$

Find x^*.

(a) Total investment of $\underset{\text{me}}{x} + \underset{\text{you}}{y} = z$

$\downarrow$ quality of offspring $B(x+y)$

My payoff $= B(x+y) - C(x)$

Your payoff $= B(x+y) - C(y)$

If we cooperate $\Rightarrow$ pick same investment $\Rightarrow x = y$ (we get same net payoff)

my payout $= P(x,x)$ Total investment $x + x = 2x = z$

$= \underbrace{B(2x)}_{\text{total invest}} - \underbrace{C(x)}_{\text{just my invest.}}$

$= (2x)(4-2x) - x^2$

tidy $= 8x - 4x^2 - x^2 = 8x - 5x^2 = P(x,x)$

Goal: find optimal cooperative x $\Rightarrow$ find c.p.s (1 variable)

$$P(x,x) = 8x - 5x^2$$

so $\frac{dP}{dx} = 8 - 10x \overset{\text{set}}{=} 0$

$8 = 10x$

$\boxed{x = \frac{8}{10} = 0.8}$ what we found graphically and w/ Excel!

Background calcs: assume x, y picked independently
me $\to x$, $y \leftarrow$ you

Scenario: I know/predict/spy your ~~state~~ investment y.
what's my best response? Best x, given y is known/const

My net payout $P(x,y) = B(x+y) - C(x)$ — Total invest $x+y=z$

$= (x+y)(4-(x+y)) - x^2$

tidy $= \left[4x + 4y - x^2 - xy - xy - y^2\right] - x^2$

tidy $= 4x + 4y - 2x^2 - 2xy - y^2$

want optimal x, given y const

$\frac{\partial P}{\partial x} = 4 - 4x - 2y \overset{\text{set}}{=} 0$

$\Rightarrow \frac{\partial P}{\partial x} = 0$ so my best x is (given y) $4x = 4-2y$ or $x = 1 - \frac{1}{2}y$

You pick $y^* =$ special y where I have to match you to get my optimum

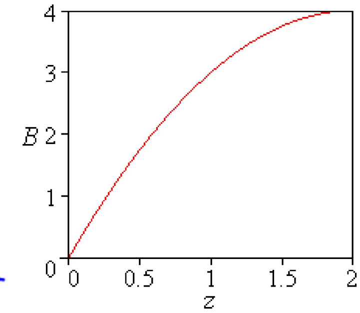
$\hookrightarrow y \to y^*$
my $x \to y^*$

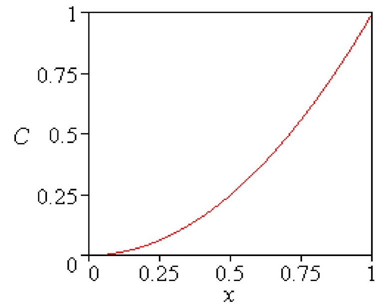
$y^* = 1 - \frac{1}{2}y^*$ equil

Solve for y^*: $\frac{3}{2}y^* = 1$

$\boxed{y^* = \frac{2}{3}}$ The Nash equilibrium

(c) *The asymmetric game.* Now suppose that the two players, a female A and a male B are independent, each acting to maximize his or her own payoff, with no regard for the partner's payoff. Suppose that the female A must "go first" so that B knows A's investment before he is required to choose his. Let A invest a and B invest b. Find the optimal values of a and b.





$B(z) = z(4-z)$
$C(x) = x^2$

Now, you are picking your y

Goal: to maximize your payout (and you'll tell me your plan)

Your payout $= B(x+y) - C(y)$

Only fact you need is that I am greedy → will always pick x to maximize my payout.

i.e. $x = 1 - y/2$ my best x, given your y.

Your payout $= B\left(\left(1 - y/2\right) + y\right) - C(y)$

$= B(1 + y/2) - C(y)$

$= \left[(1 + y/2)(4 - (1 + y/2))\right] - y^2$

expand $= \left[4 + 2y - 1 - y/2 - y/2 - y^2/4\right] - y^2$

tidy $= 3 + 2y - y - \frac{5}{4}y^2$

tidy $= 3 + y - 5/4\, y^2 = P(x,y)$ for you, if I'm always greedy.

For you to pick optimal y: take y deriv, set $= 0$

$$\frac{dP}{dy} = 1 - \frac{5}{2}y \overset{\text{set}}{=} 0 \to 1 = \frac{5}{2}y$$

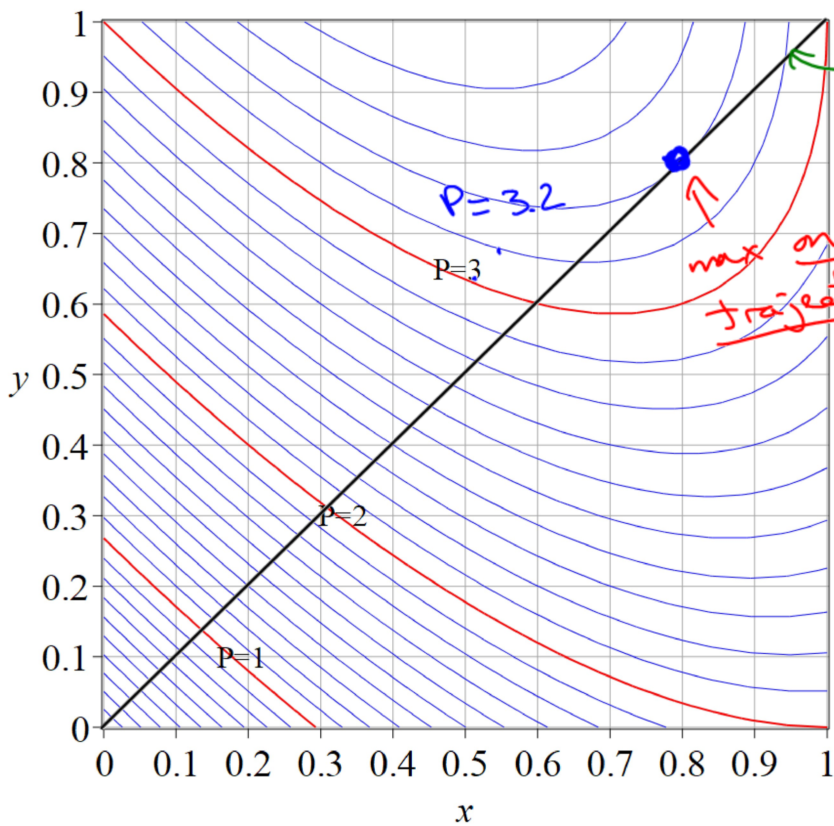
$$\boxed{y = \frac{2}{5} = 0.4}$$

You pick $y = 0.4$ → I'll pick $x = 1 - \frac{0.4}{2} = 0.8$ for my own benefit

your $P = 3.2$ my $P = 2.72$

Ex 9.2. Now solve the problems of Ex 9.1 graphically.
Recall that $P(x, y)$ is the payoff to the player investing x with partner y.

(a) Cooperation → both pick same investment level → $x = y$

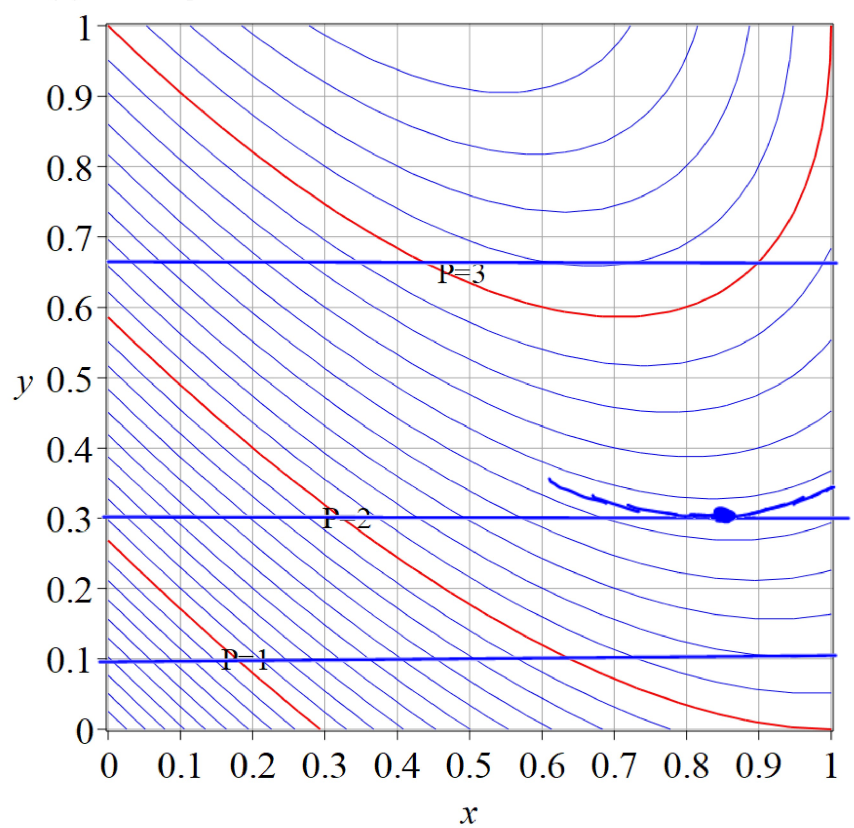


only allowed values (x, y) on this line

My payoff / x players payoff

max on trajectory occurs when trajectory and contour are tangent

(b) Nash Equilibrium



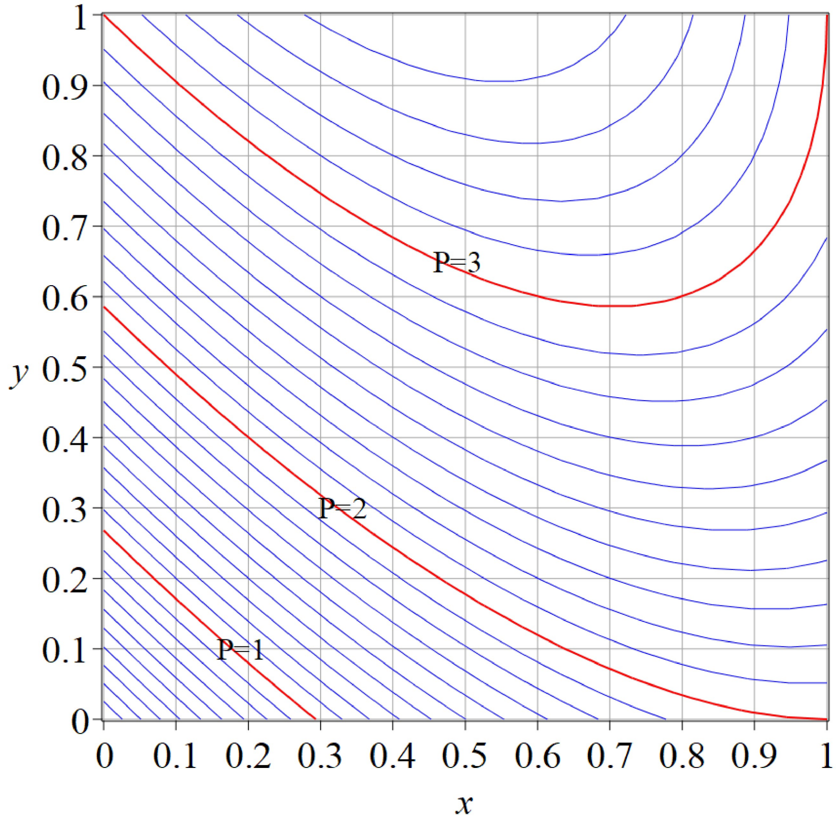
my best $x = 1 - y/2$

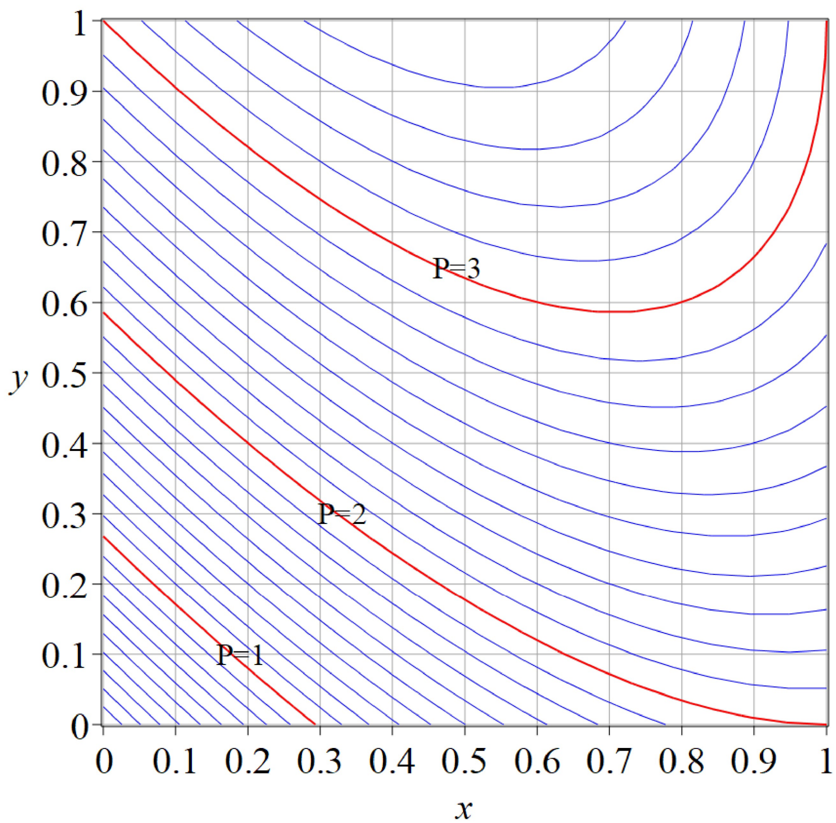
$y = 2/3$ → best $x =$ also $2/3$

$y = 0.3$ → best $x = 0.85$

$y = 0.1$ → best $x = 0.95$

(c) The Asymmetric Game: A goes first.





RESEARCH HIGHLIGHTS

CANCER BIOLOGY

Persistent problem

Cancer Cell **8**, 197–209 (2005)

The recurrence of a tumour after what intitially seemed successful treatment is the leading cause of death from breast cancer. A gene that plays a causal role in tumour recurrence has now been identified by Lewis Chodosh of the University of Pennsylvania School of Medicine and colleagues.

Working in mice, the team found that a gene called *Snail*, which normally helps cells change shape and migrate in embryos, was abnormally active in recurring breast tumours. Engineering tumours to express *Snail* strongly promoted their ability to recur.

Studying primary breast tumours in women, the team found that high levels of *Snail* expression were associated with an increased risk of a patient having a tumour reappear within five years. As well as being important for prognosis, *Snail* could be a useful target for cancer-fighting drugs.

GEOLOGY

Basalts flow slow

Geology **33**, 745–748 (2005)

Some 180 million years ago, the southern continent of Gondwana ruptured and the Indian Ocean began to form. Into this mighty rift flowed the massive Karoo–Ferrar flood basalts, which are preserved today in southern Africa, Antarctica, Australia and New Zealand.

Such flood basalts have been linked to mass extinction events. Earlier studies have proposed that the Karoo rocks flowed on to the Earth's surface in less than a million years — a short enough time to have constituted a massive environmental disruption.

But Fred Jourdan of the Berkeley Geochronology Center in California and colleagues think it took much longer. Using argon isotopes to date 38 rock samples taken from southern Africa, they have established that the basalts were put in place over eight million years — which might explain why there seems to be no associated mass extinction during this period.

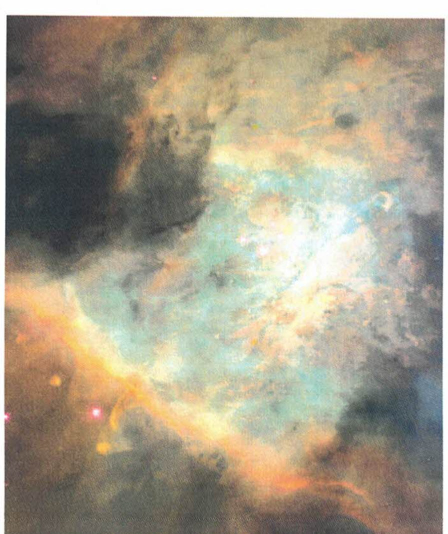
ASTROPHYSICS

Parting stars

Astrophys. J. (in the press); preprint at xxx.arxiv.org/abs/astro-ph/0509201 (2005)

Stars that are born together may not stay together, say Laura Gómez of the Centre for Radioastronomy and Astrophysics in Morelia, Mexico, and her co-workers.

The Trapezium is a star cluster embedded in the star-forming region known as the Orion nebula (pictured). By fixing a background frame of reference for the nebula, Gómez and her team find that three radio sources in the Trapezium seem to be moving away from the spot where they were all located about 500 years ago. These objects may be young stars emerging from a multiple-star system that tore itself apart.



NASA, C.R. O'DELL & S.K. WONG (RICE UNIVERSITY)

Queen's move

Proc. R. Soc. Lond. B doi:10.1098/rspb.2005.3234 (2005)

When a colony of social insects reproduces, queens are expected to want an equal number of sons and daughters — to maximize their genetic contribution to the next generation. But sterile female workers, being more closely related to the queen's daughters, want more females.

A mathematical model from Ido Pen of the University of Groningen, the Netherlands, and Peter Taylor of Queen's University in Ontario considers the outcome if queens and workers execute their sex-control strategies simultaneously and independently. In such a case, the ratio of males is midway between the 50% favoured by queens and the 25% favoured by workers.

But the ratio in real colonies can be nearer the queen's ideal. This is explained by a second model in which the queen acts first and the workers observe her decision. The queen places the workers in a bind by announcing her intent — once she declares that she will produce mostly males, the workers must simply favour females as much as they can. This means the queen can declare an initial position that ends up with her preferred ratio.

C. KÖNIG

The queen ant must outwit her diminutive workers to produce enough sons.

BIOCHEMISTRY

Predictable proteins

Science **309**, 1868–1871 (2005)

Predicting a protein's structure from its amino-acid sequence is not an easy task. Still, scientists can now calculate the structure of small proteins accurately enough to get all atoms — including amino-acid side chains — in the right places.

The predicted structures are less than 2 angstroms off the native protein structure. This is a marked improvement on previous attempts.

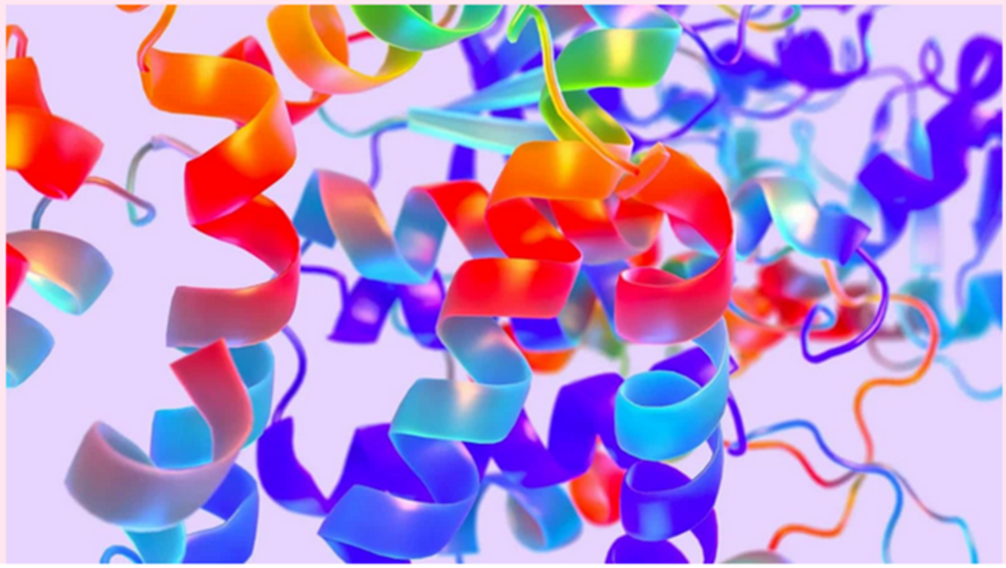
To get this close to the real structure, researchers led by David Baker of the University of Washington, Seattle, calculated approximate structures for the protein in which they were interested and for related proteins. The team then had enough candidate structures to start detailed atomic modelling.

So far, their method only works for proteins smaller than 100 amino acids. But with more computer power, the researchers hope to accurately predict the structures of entire protein domains.

Research

AlphaFold: Using AI for scientific discovery

January 15, 2020



UPDATE: In July 2022, we released AlphaFold protein structure predictions for nearly all catalogued proteins known to science. Read the latest blog here.

In our study published in Nature, we demonstrate how artificial intelligence research can drive and accelerate new scientific discoveries. We've built a dedicated, interdisciplinary team in hopes of using AI to push basic research forward: bringing together experts from the fields of structural biology, physics, and machine learning to apply cutting-edge techniques to predict the 3D structure of a protein based solely on its genetic sequence.

Article

Improved protein structure prediction using potentials from deep learning

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Protein structure prediction can be used to determine the three-dimensional shape of a protein from its amino acid sequence[1]. This problem is of fundamental importance as the structure of a protein largely determines its function[2]; however, protein structures can be difficult to determine experimentally. Considerable progress has recently been made by leveraging genetic information. It is possible to infer which amino acid residues are in contact by analysing covariation in homologous sequences, which aids in the prediction of protein structures[3]. Here we show that we can train a neural network to make accurate predictions of the distances between pairs of residues, which convey more information about the structure than contact predictions. Using this information, we construct a potential of mean force[4] that can accurately describe the shape of a protein. We find that the resulting potential can be optimized by a simple gradient descent algorithm to generate structures without complex sampling procedures. The resulting system, named AlphaFold, achieves high accuracy, even for sequences with fewer homologous sequences. In the recent Critical Assessment of Protein Structure Prediction[5] (CASP13)—a blind assessment of the state

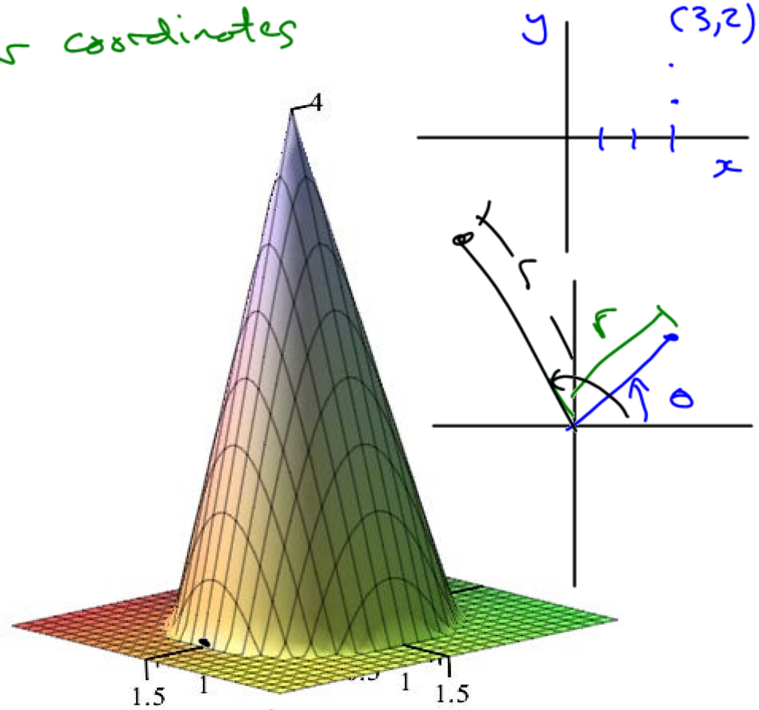
9.3 *Climbing Cone Mountain*

circularity
– polar coordinates

The well-known Cone Mountain has equation

$$\boxed{z = 4\left(1 - \sqrt{x^2+y^2}\right)} = 4(1 - r)$$

The graph is a cone with height 4 and base radius 1.



A bug starting at the foot of the mountain at (1, 0, 0) climbs the mountain taking a spiral road winding counterclockwise around the mountain always with slope 1. Your job is to find an equation for the bug's journey up the mountain.

(a) On the contour diagram below sketch a rough spiral graph of the bug's path. Note that the contours are equally spaced at z-intervals of size 0.2. On the other hand, since there are 20 intervals, the spacing along the *x*-axis will be at *x*-intervals of size 0.05, and the same for *y*.

slope $= \dfrac{\Delta z}{(\Delta x, y)} = \dfrac{z \text{ rise}}{xy \text{ run}}$

$1 = (\Delta z = 0.2)$

need $\Delta xy = 0.2$

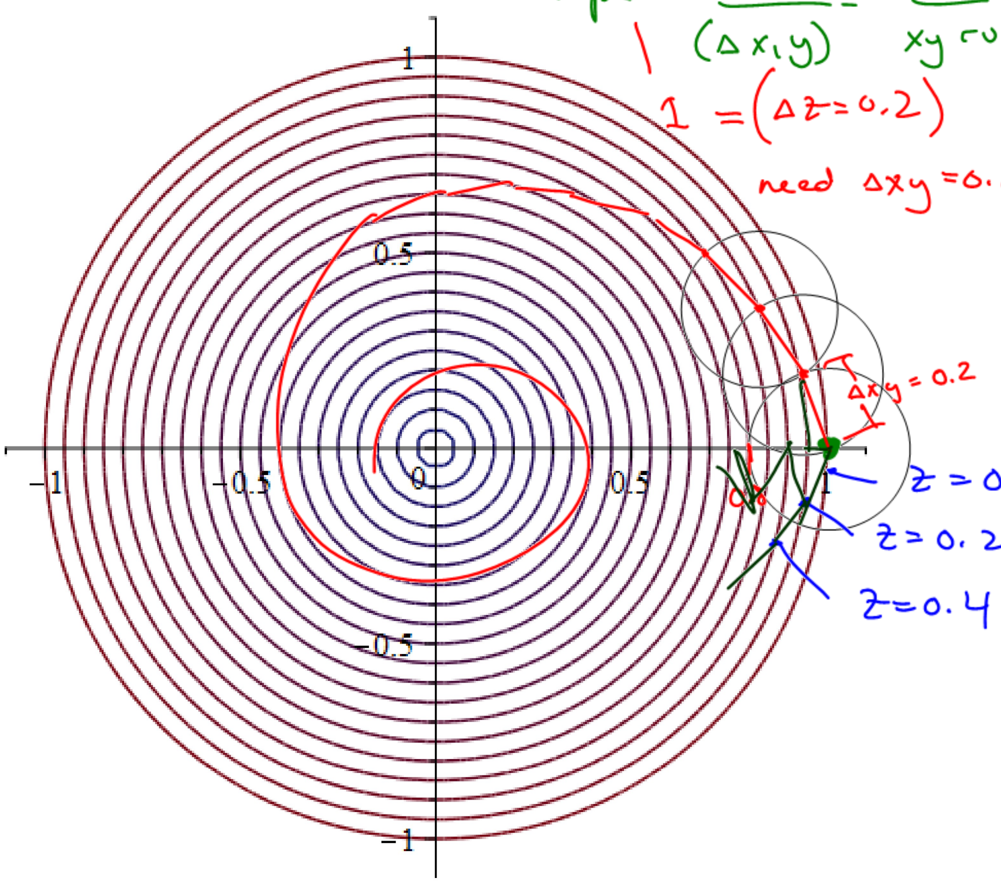
$x = r\cos\theta$

$y = r\sin\theta$

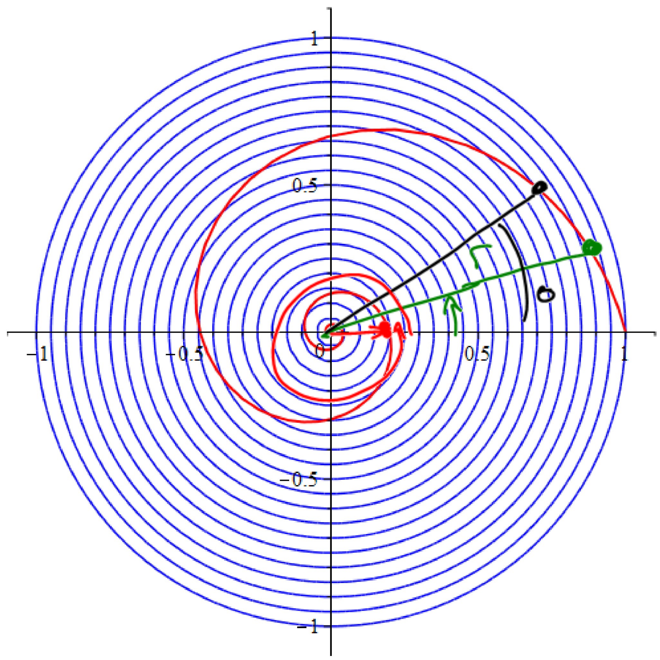
$\Updownarrow$

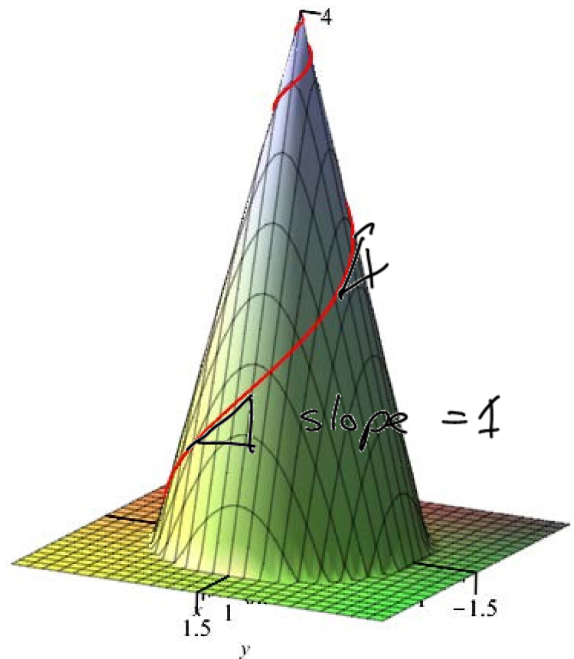
$r = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$



(b) Find an equation in polar coordinates that describes the path on the contour diagram.





We use polar coordinates so we are looking for a function:

$$r = f(\theta)$$ instead of $y = f(x)$

That gives us the distance of the curve from the origin at any angle θ.

z us map

Slope $= \dfrac{\Delta z}{\Delta xy \text{ step}} \overset{\text{Set}}{=} 1$

Δz: know $z = 4\left(\underbrace{1-\sqrt{x^2+y^2}}_{\text{gross!}}\right) = 4\underbrace{(1-r)}_{\text{nice!}}$

small change in height ↓

small change ↓ in

$z = 4(1-r)$ want a relation b/w Δz, Δr

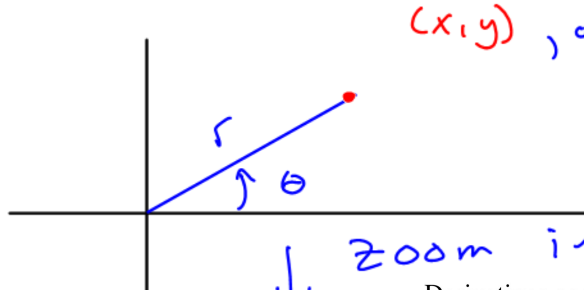
r position

calculus ↓ take d/dr both sides

$\dfrac{\Delta z}{\Delta r} \approx \dfrac{dz}{dr} = -4 \rightarrow \boxed{\Delta z \approx -4\,\Delta r}$ (1)

Δxy step $\Leftrightarrow$ $\Delta r, \Delta\theta$ step

(x, y), or (r, θ)

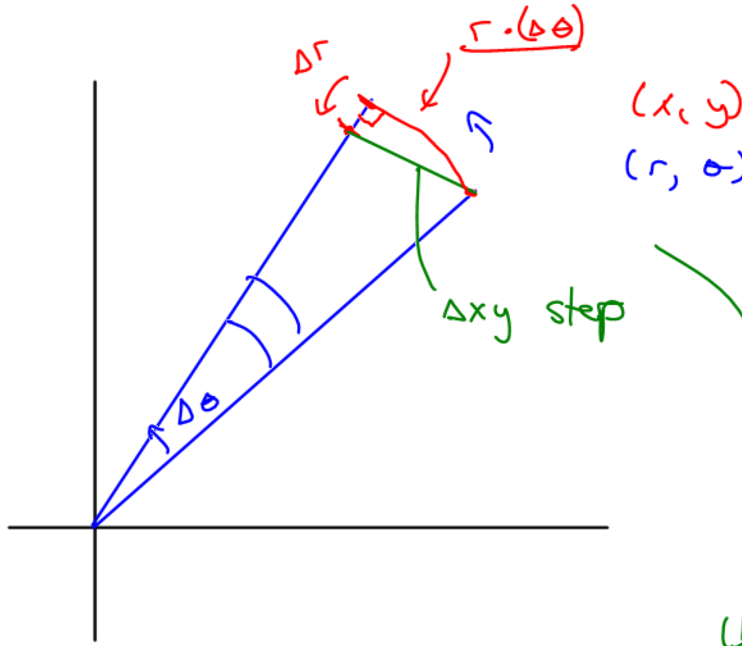


↓ zoom in

How long is that
Δxy step?

small angle
approximation:

arc length
$\approx$
straight line
if $\Delta\theta$ is
small



$$\Delta xy \text{ step} = \sqrt{(\Delta r)^2 + (r\ \Delta\theta)^2} \quad ②$$

Pythagoras!

Back slope

want

$$\frac{\Delta z}{\Delta xy \text{ step}} = 1 \rightarrow \frac{-4\,\Delta r}{\sqrt{(\Delta r)^2 + (r\ \Delta\theta)^2}} = 1 \quad ③$$

① ②

Goal is to relate $r \to \theta$
(first $\Delta r \to \Delta\theta$)

tidy
③

$$-4(\Delta r) = \sqrt{(\Delta r)^2 + r^2(\Delta\theta)^2}$$

$(\ )^2$ both sides

$$16(\Delta r)^2 = (\Delta r)^2 + r^2(\Delta\theta)^2$$

$$15(\Delta r)^2 = r^2(\Delta\theta)^2$$

$\sqrt{\ }$ both sides

$$(\Delta r)^2 = \frac{r^2(\Delta\theta)^2}{15} \rightarrow \Delta r = \pm\frac{r\ \Delta\theta}{\sqrt{15}}$$

Δr is negative/
r decr.
as θ incr.

rearrang

$$\frac{\Delta r}{\Delta\theta} = \frac{-r}{\sqrt{15}}$$

$\lim_{\Delta\theta \to 0}$

$$\frac{dr}{d\theta} = -\frac{1}{\sqrt{15}} r$$

$$\boxed{r = a e^{-\frac{1}{\sqrt{15}}\theta}}$$

Aside

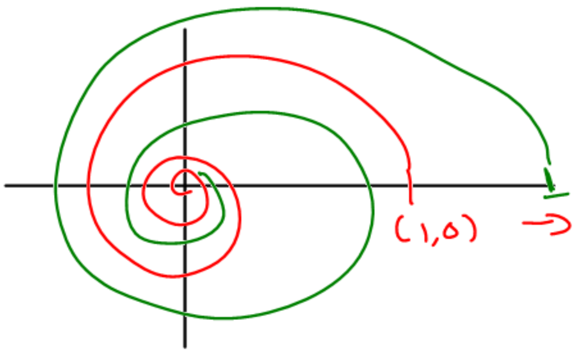
a) $\frac{dy}{dx} = y$

$\downarrow$

$y = \cancel{a}e^{x}$

b) $\frac{dy}{dx} = \frac{1}{2}y$

$y = ae^{\frac{1}{2}x}$



$(1,0) \to r=1,\ \theta=0 \to a=1$

$r=2,\ \theta=0 \to a=2$

$$\frac{dr}{d\theta} = \frac{-1}{\sqrt{15}} r$$

if $r = e^{-\frac{1}{\sqrt{15}}\theta}$

$\downarrow$

$$\frac{dr}{d\theta} = \frac{-1}{\sqrt{15}} e^{-\frac{1}{\sqrt{15}}\theta}$$

$$= \frac{-1}{\sqrt{15}} r$$

(c) Discuss the asymptotic behavior of the bug's trajectory (as the bug approaches the vertex).