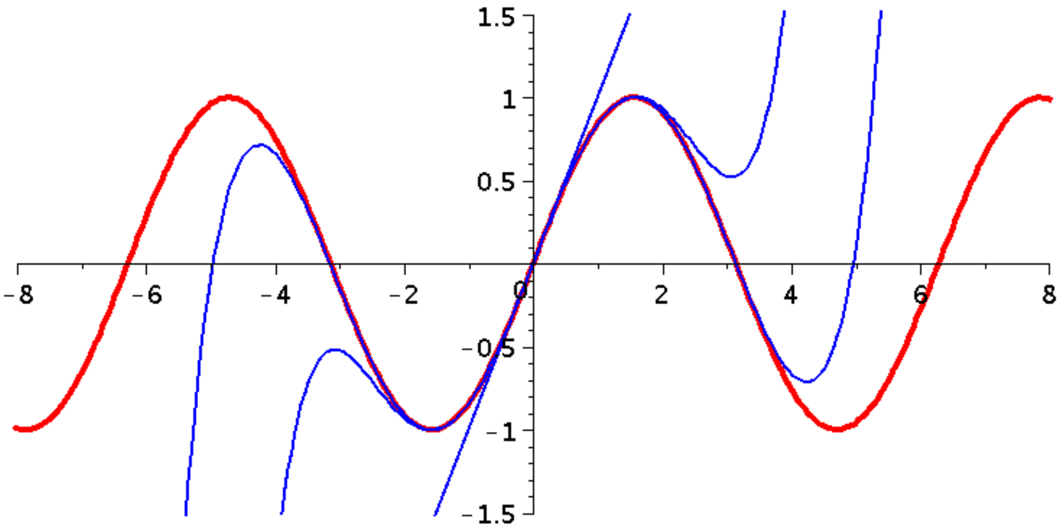
Chapter 2

Short: just 2 weeks

Infinite Series and Power Series



1. **Series: general concepts**
2. **Alternating series**
3. **Power series**
4. **Taylor and Maclaurin series**
5. **Approximating functions by polynomials**
6. **The great Leonhard Euler (1707 – 1783)**

1. Series: General concepts

An infinite series is an infinite sum of the form:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots .$$

Define its nth partial sum s_n as the sum of the first n terms:

$$s_n = \sum_{k=1}^{n} a_n = a_1 + a_2 + a_3 + \dots + a_n$$

If the sequence $\{s_n\}$ converges to a finite limit s as n approaches infinity, we say that the series *converges,* and that s is the sum of the series and write.

$$s = \sum_{n=1}^{\infty} a_k = \lim_{n\to\infty} s_n$$

If the series fails to converge, we say that it *diverges.*

Example 1.1. In each case does the series converge or diverge?

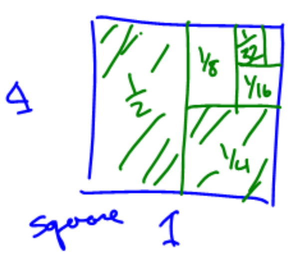
(a) $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \dots$

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + 1 - \dots$

Example 1.2. Find the partial sums of the series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots$$

Provide an intuitive argument that the series converges and find its sum.



$n=4 \quad S_4 = \cdots = \frac{15}{16} = 1 - \frac{1}{2^4}$

⋮

for any n^{th} partial sum,

$$S_n = 1 - \frac{1}{2^n}$$

e.g. sum of the first 20 terms $= S_{20}$

$$= 1 - \frac{1}{2^{20}}$$

formula for S_n

The infinite sum

$$= \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1 - 0 = \boxed{1.}$$

n^{th} partial sum

as $n \to \infty$, $\frac{1}{2^n} \to 0$

diverges

$r=2 \quad 3 + 6 + 12 + 24 + 48 + \ldots$

Example 1.3 A series in which each term is obtained from the previous term by multiplication by a constant r is called *geometric*. If we denote the first term by a, the series has the form:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$$

Show that the series converges if and only if $|r| < 1$, and in this case its sum is $s = \dfrac{a}{1-r}$.

As an example, find the sum of the geometric series: $5 - \dfrac{10}{3} + \dfrac{20}{9} - \dfrac{40}{27} + \ldots$

Note. Most of our work here will be with series that have a strong geometric-series character, that is they have the form

$$\sum_{n=0}^{\infty} a_n r^n = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \ldots$$

for given coefficients a_n, and this will enable us to decide whether they converge and to find their sum.

An interesting example is the arithmetic-geometric series: $\sum_{n=1}^{\infty} nr^n = r + 2r^2 + 3r^3 + 4r^4 + \ldots$

This series converges for $|r| < 1$. Can you find the sum?

$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$

eg. $2 \to 2^n$, or $-2 \to (-2)^n$

2 major cases

$|r| > 1$, then $r^n \to \infty$

$\Rightarrow \frac{a(1-r^n)}{1-r}$ **diverges**

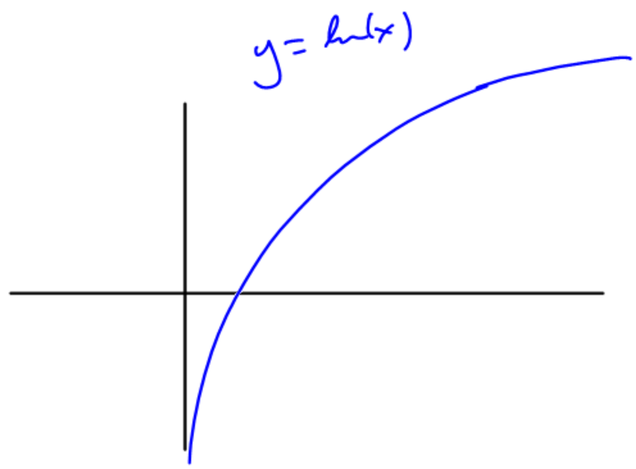
$|r| < 1$ eg $r = 1/2, -1/2$

then $r^n \to 0$

$\Rightarrow S = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$

single finite value, series converges.

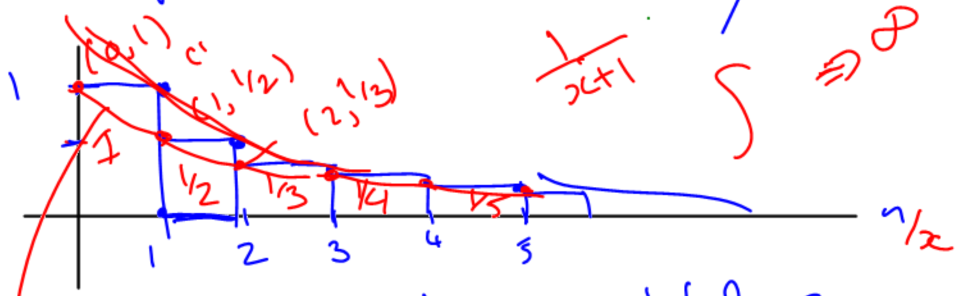
case $r=1$ must look at case by case. +1 −1 +1 −1 earlier is an example.



Example 1.4. Does the *harmonic* series $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ converge?

Harmonic Series
some partial sums:

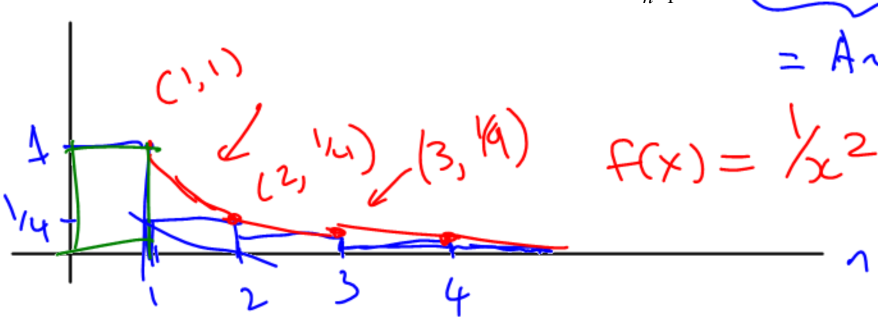
s_{10}	= 2.92
s_{100}	= 5.17
s_{1000}	= 7.48
$s_{10,000}$	= 9.87
$s_{100,000}$	= 12.09
$s_{1,000,000}$	= 14.39
$s_{10,000,000}$	= 16.70
$s_{100,000,000}$	= 19.00



Example 1.5 What's the story for the series $\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$?

"Euler" Series
some partial sums:

s_{10}	= 1.54977
s_{100}	= 1.63498
s_{1000}	= 1.64393
$s_{10,000}$	= 1.64483
$s_{100,000}$	= 1.64492
$s_{1,000,000}$	= 1.64493
$s_{10,000,000}$	= 1.64493



Example 1.6. Find the sum of the arithmetic-geometric series:

$$\sum_{n=1}^{\infty} nr^n = r + 2r^2 + 3r^3 + 4r^4 + \cdots$$

r					
r^2	r^2				
r^3	r^3	r^3			
r^4	r^4	r^4	r^4		
r^5	r^5	r^5	r^5	r^5	
.	.	.	.	.	
.	.	.	.	.	
.	.	.	.	.	
					sum of all entries

The arithmetic-geometric series:

$$r + 2r^2 + 3r^3 + 4r^4 + \cdots = \frac{r}{(1-r)^2}$$

2. Alternating series.

Recall that

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the nth partial sum of the series

$$S = \sum_{k=1}^{\infty} a_k$$

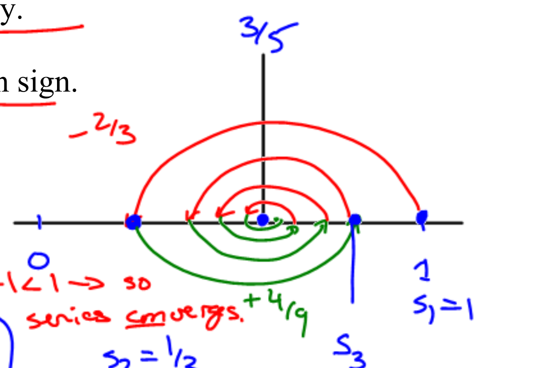
We say that S converges to the limit s if the partial sums s_n converge to s as n approaches infinity.

A series is *alternating* if the terms alternate in sign.

For example, the geometric series:

$$1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \ldots$$

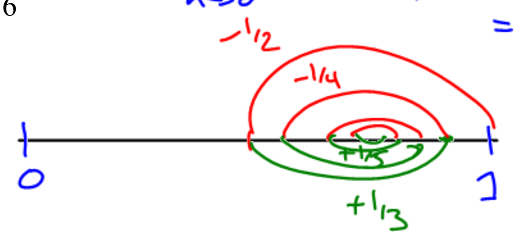
is alternating and converges to 3/5.



Example 2.1. An important example is the *alternating harmonic series*:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

Does it converge or diverge?



Convergence test for alternating series.

Consider the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots \qquad (b_n > 0)$$

If

(i) the b_n decrease with n

(ii) and approach zero $\left(\lim_{n \to \infty} b_n = 0 \right)$

then the series converges.

If a series converges, then its partial sums s_n converge to its sum s. Thus, by taking s_n for a large enough n, we should get a good estimate of s. The trouble is, we need to know how big to take n (i.e. how many terms to take) to obtain a desired measure of closeness.

> ***Error estimation for alternating series.***
>
> Suppose that an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ ($b_n > 0$) satisfies (i) and (ii) of the convergence test. Then the "error" in approximating its sum s with the nth partial sum s_n is less than the absolute value of the first omitted term:
>
> $$|s - s_n| \le b_{n+1}$$

Example 2.2: How many terms of the alternating harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

do we need to take to get within $1/100^{\text{th}}$ of the sum?

Example 2.3. Find the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$ correct to three decimal places (i.e., with error less than 0.0005).

'Til now:

eg $\sum_{n=1}^{\infty} \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} ar^{n-1}$

numbers only!

3. Power Series

A *power series* is an infinite series of the form:

$$\sum_{n=0}^{\infty} c_n x^n \quad \text{or} \quad \sum_{n=0}^{\infty} c_n (x-a)^n$$

whose terms are ascending powers of x or more generally of $(x–a)$. When such a series converges, its sum will be a function of x. The question we will be asking here is the reverse—given a function $f(x)$, can we find a power series that converges to it in some x-interval?

For example, the function $f(x) = \frac{1}{1-x}$ is recognized as the sum of a geometric series and that gives us a power series expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

that converges for $|x| < 1$. The *interval of convergence* of a power series is the set of x for which the series converges.

The interval of convergence is always centered at $x = a$—except that we may or may not have convergence at the endpoints. The radius of the interval is called the *radius of convergence* of the series.

Example 3.1. Express $\frac{1}{1+x^2}$ as the sum of a power series in x and find the interval of convergence.

Example 3.2. Express $\dfrac{x^3}{x+2}$ as the sum of a power series in x and find the interval of convergence.

Differentiation and integration.

It turns out that under standard assumptions of continuity, differentiability and integrability, we can get new power series representations out of old by differentiating and integrating. Also it can be shown that, except for the end points (which we will treat as a special case) this process leaves the interval of convergence unchanged.

Example 3.3 Obtain a power series for $\dfrac{1}{(1-x)^2}$ by recognizing it as a derivative.

$$= \sum_{n=0}^{\infty} (n+1)x^{n} = \sum_{n=1}^{\infty} n\, x^{n-1}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x|<1$$

$\times \frac{1}{2}$:

$$\frac{1}{2-2x} = \frac{1}{2}\left(1 + x + x^2 + x^3 + \cdots\right) \quad \text{for } |x|<1$$

subs $x \to -x/2$ $\qquad$ $x \to -\frac{x}{2}$

$$\frac{1}{2-2\left(-\frac{x}{2}\right)} = \frac{1}{2}\left(1 + \left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 + \cdots\right)$$

for $\left|-\frac{x}{2}\right| < 1$

$$= \frac{1}{2+x}$$

$\times x^3$

power series for $\frac{x^3}{2+x}$

$$\frac{x^3}{2+x} = x^3 \cdot \frac{1}{2}\left(1 + \left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 + \cdots\right)$$

for $|x| < 2$

$$= \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

$n=0 \qquad n=1 \qquad n=2 \qquad n=3$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{n+1}} x^{n+3}$$

Example 3.4. Find a power series representation for ln(1+*x*).

relate to $\frac{1}{1-x}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

subs $x \to (-x)$ ↓ subs $x \to -x$

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4$$

integrate ↓ integrate

$$\int \frac{1}{1+x}\,dx = \ln(1+x) + C = \int 1 - x + x^2 - x^3 + x^4 - \dots \,dx$$

term by term.

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$$

In an equation, if we integrate both sides, the two sides can differ by up to a constant

⇒ just one +C needed

Find +C checking at a simple x: $x = 0$

$$LHS = \ln(1+0) + C = RHS = 0$$

$\ln(1+0) = 0$, $C = 0$ ✓

So

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

($n=0$, $n=1$, $n=2$)

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\text{or} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

old $\ln(1+x)$

↓

$\frac{1}{1+x}$ ↓ $x \to -x$

$$\frac{1}{1-x} = 1 + x + x^2$$

Example 3.5. Find a power series representation for $\tan^{-1}x$. or $\arctan(x)$

Recall $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \rightarrow \int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

Know $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ for $|x|<1$

↳ subs $x \to (-x^2)$ ↓ subs $x \to -x^2$

$$\frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots$$

$|-x^2|<1$ equivalent $|x|<1$

↓ tidy ↓ tidy

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

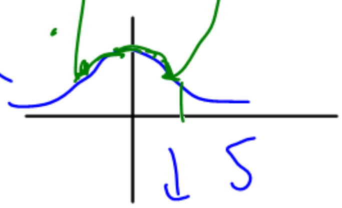
↓ integrate $\int dx$ term by term.

integrate ↓ $\int dx$

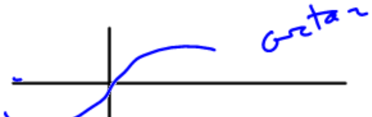
$$\arctan(x) + C = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$n=0$ $n=1$ $n=2$ $n=3$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$ for $|x|<1$.



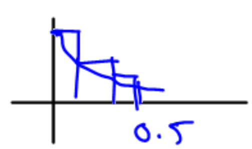
↓ $\int$



Example 3.6. Use a power series representation to approximate $\int_0^{0.5} \frac{1}{1+x^7}\,dx$ correct to within 10^{-7}.

~~opt 1)~~ integrate $\frac{1}{1+x^7}\Big|_0^{0.5}$ using Fund'l Thm of Calculus. no simple, non-series anti deriv for $\frac{1}{1+x^7}$

opt 2)



estimate integral using LH sum or Trapezoid rule etc.

opt 3) get power series for $\frac{1}{1+x^7}$, then integrate series, eval first n terms opt 3 looks like.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{for } |x|<1$$

subs $x \to (-x^7)$

$$\frac{1}{1-(-x^7)} = 1 + (-x^7) + (-x^7)^2 + (-x^7)^3 + \dots$$

tidy

$$\frac{1}{1+x^7} = 1 - x^7 + x^{14} - x^{21} + x^{28} - \dots$$

note $[0, 0.5]$ is all in $|x|<1$ ✓

$\int_0^{0.5} \cdot \, dx$

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \left. x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \dots \right|_{x=0}^{x=0.5}$$

sub in limits

$$= \left[(0.5) - \frac{(0.5)^8}{8} + \frac{(0.5)^{15}}{15} - \frac{(0.5)^{22}}{22} + \dots\right]$$

$$- \left[0 - \frac{0^8}{8} + \frac{0^{15}}{15} - \dots\right] = 0 \quad \text{yay!}$$

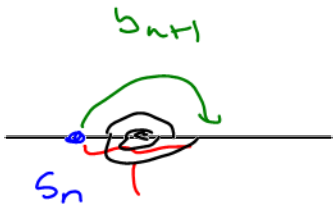
so

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = 0.5 - \frac{(0.5)^8}{8} + \frac{(0.5)^{15}}{15} - \frac{(0.5)^{22}}{22} + \dots$$

exact (+) (−) (+) (−)

cannot evaluate the infinite sum "

↑ alternating series.



From last week, error in a truncated alternating series is less than the magnitude of the first omitted term

Goal: error < first omitted term $< 10^{-7}$

: guess & check

$$\frac{(0.5)^{22}}{22} \approx 10^{-8} \checkmark$$

so $\int_0^{.5} \frac{1}{1+x^7}\,dx \approx 0.5 - \frac{0.5^8}{8} + \frac{(0.5)^{15}}{15}$, within $\pm 10^{-7}$.

Midterm Tue 6:30 PM.

Rooms: On Q

Power series $\rightarrow$ "infinite polynomials"

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

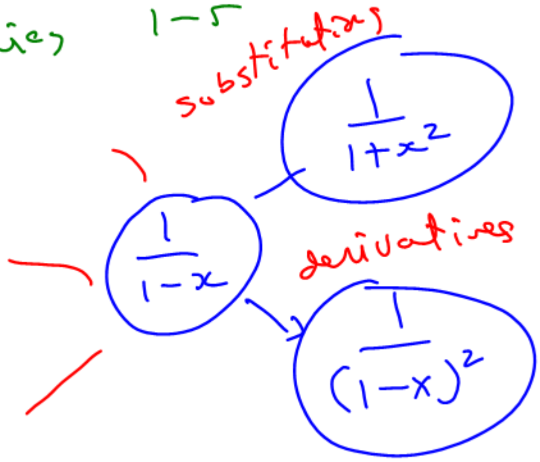
Geometric power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \qquad \text{for } |x|<1$$

$a=1$, $r=x$

Geometric series

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + \dots \qquad |r|<1$$



3.7 General summary of section 3—finding power series.

The "Go To" example is always the geometric series

$$1 + r + r^2 + \cdots = \frac{1}{1-r} \qquad \text{(converges for } |r| < 1\text{)}$$

1. Use algebra:

$$f(x) = \frac{1}{x+2}$$

2. Use derivative:

$$f(x) = \frac{1}{(1-x)^2}$$

3. Use integral:

$$f(x) = \ln(1+x)$$

4. Substitution or derivative.

$$f(x) = \ln(1+x^2)$$