

4. Taylor and Maclaurin Series

We have been using a power series $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$ to give us a systematic approximation of a given function $f(x)$ by a sequence of polynomials. We have pointed out that the interval of convergence of such a series is centered at $x = 0$. But sometimes we are interested in studying a function in a neighborhood of another point $x = a$. In that case we use a power series centered at $x = a$:

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

and its interval of convergence is centered at $x = a$.

The question we ask here is:

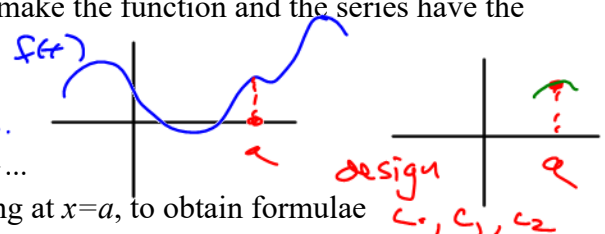
arbitrary "analytic" function \approx smooth, or C^∞

Given the function f and the point $x=a$ how do we find the coefficients c_n ?

There are different ways we might answer this question depending on where and in what way we want the approximation to be good. The approach we use is to make the function and the series have the same values of the derivatives of all orders at $x = a$.

Example 4.1: Write:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$



and repeatedly differentiate both sides, each time evaluating at $x=a$, to obtain formulae for the c_n .

Define $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$

so $f(a) = c_0$ (want)

and $f'(a) = c_1$ (want)

$f''(a) = 2c_2 \rightarrow$ want $c_2 = \frac{f''(a)}{2}$

$f'''(a) = 3 \cdot 2 \cdot c_3$

$c_3 = \frac{f'''(a)}{3 \cdot 2} = \frac{f'''(a)}{3!}$

$c_4 = \frac{f^{(4)}(a)}{4 \cdot 3 \cdot 2} \rightarrow 4!$

Taylor Series

If a power series centered at $x=a$ converges to a function $f(x)$ in some interval $(a-R, a+R)$ for $R > 0$:

$$f(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n c_k (x-a)^k \quad \text{for } a-R < x < a+R$$

then

$$c_k = \frac{f^{(k)}(a)}{k!}$$

kth deriv

and the polynomial

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + \frac{f'(a)}{1} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

is called the n th degree Taylor polynomial for f at the point $x=a$. The infinite series is simply called the Taylor series for f centered at $x=a$.

Note: the special case $a=0$ is sufficiently important it is given a name of its own and is called the Maclaurin series for f .

Example 4.2. (a) Find the Maclaurin series for $f(x) = e^x$.

n	$f^{(n)}(x)$	$x=0$	c_n
0	e^x	$e^0 = 1$	$\frac{1}{0!} = \frac{1}{1} = 1$
1	e^x	1	$\frac{1}{1!} = \frac{1}{1} = 1$
2	e^x	1	$\frac{1}{2!} = \frac{1}{2}$
3	e^x	1	$\frac{1}{3!} = \frac{1}{6}$
4	e^x	1	$\frac{1}{4!} = \frac{1}{24}$
5	e^x	1	$\frac{1}{5!}$

choose $a=0$
 $(x-0)$

$f(x)$
 $f'(x)$
 $f''(x)$

n th deriv
 @ $x=0$
 $\frac{f^{(n)}(0)}{n!}$

$0! = 1$

$$e^x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$= 1 + 1x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{5!}x^5 + \dots$$

hard to evaluate (terms) simple to evaluate (terms)

$e \approx 2.714$ $\pi = 3.1415$

for all x^2 's

(b) Set $x=1$ to obtain a famous series expansion for e .

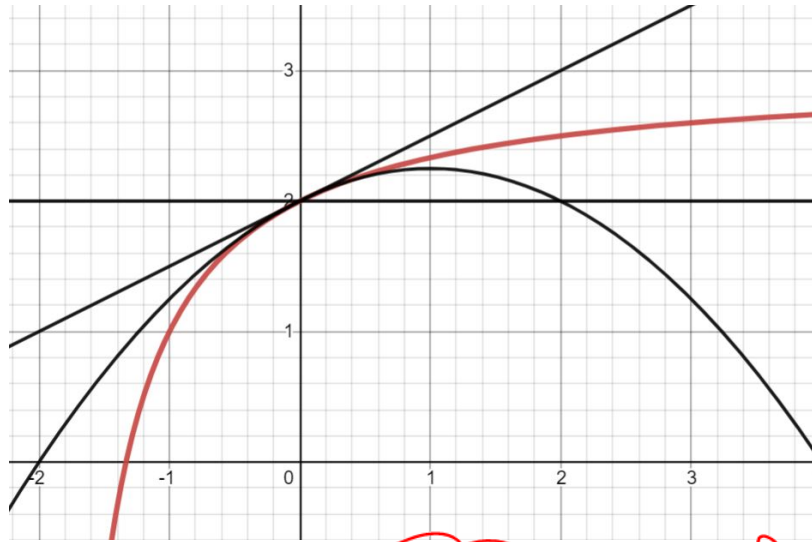
$$e = e^1 = 1 + 1 + \frac{1}{2} \cdot 1^2 + \frac{1}{6} \cdot 1^3 + \frac{1}{24} \cdot 1^4 + \frac{1}{5!} \cdot 1^5 + \dots$$

$e \approx 2.7$

Summary: The n th Taylor polynomial

$$T_n(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots + c_nx^n$$

$$T_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$



Describe the graphical significance of $T_n(x)$ for $n=0, 1$ and 2 . (a=0) point of reference/center.

Given

$$1 + 1x + \frac{1}{2}x^2 + \frac{x^3}{3!} + \dots = \text{some } f(x)$$

$c_0 = f(a)$

$c_1 = f'(a)$

$c_2 = \frac{1}{2} = \frac{f''(a)}{2}$

$f(0) = 1$

$f'(0) = 1$

$f''(0) = 1$

Height of $f(x)$ at $x=0$ is 1

slope of $f(x)$ at $x=0$ is 1

$f''(0) = 1$

Concavity/curvature of $f(x)$ at $x=0$ is 1.

Example 4.3. Find the Maclaurin series for $f(x) = \sin x$. Use technology to draw graphs of successive Maclaurin polynomials and observe the progress of the convergence.

use $a=0$ as center

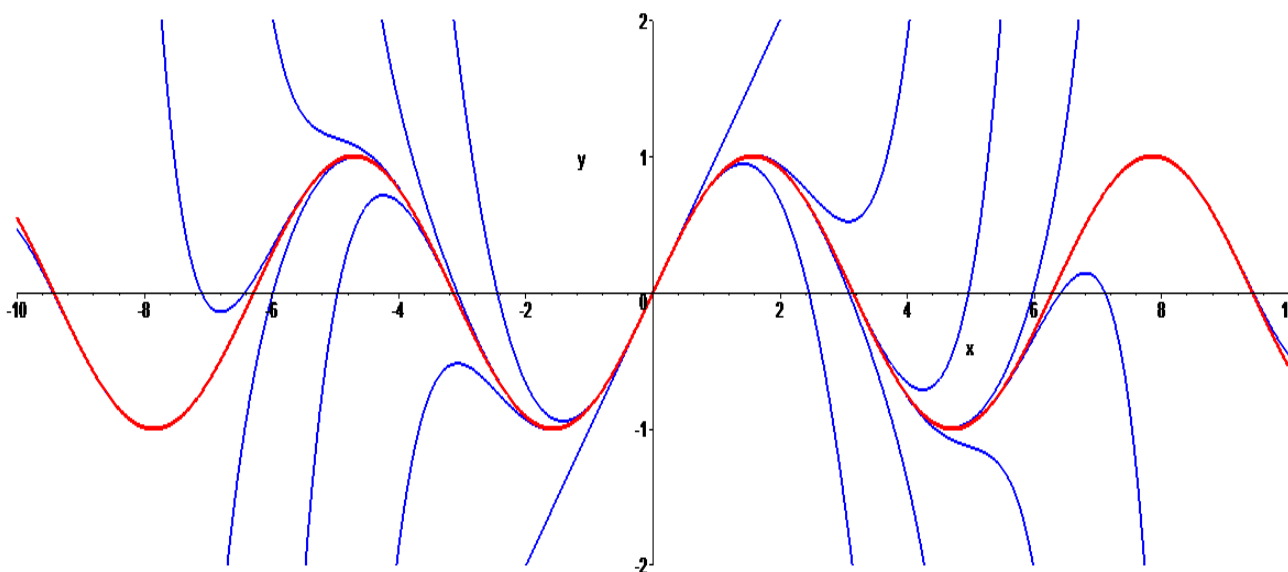
$f^{(n)}(0)$
 $\frac{f^{(n)}(0)}{n!}$

n	$f^{(n)}(x)$	$x=0$	c_n
0	$\sin(x)$	0	0
1	$\cos(x)$	1	1
2	$-\sin(x)$	0	0
3	$-\cos(x)$	-1	$-\frac{1}{3!}$
4	$\sin(x)$	0	0
5	$\cos(x)$	1	$\frac{1}{5!}$
6	$-\sin(x)$	0	0
7	$-\cos(x)$	-1	$-\frac{1}{7!}$

$\frac{d}{dx}$ ↪

$\sin(x)$
 $= 1x + \left(\frac{-1}{3!}\right)x^3 + \left(\frac{1}{5!}\right)x^5 + \dots$
 $n=1$ $n=3$ $n=5$

skip 0 terms



Example 4.4. Find the Maclaurin series for $f(x) = x \cos x$.

$x \cdot \cos(x)$

[Note: Use the simple idea of working with the series for \sin which we already have. Of course, this indirect approach leaves open the question of how we know that what we got is in fact the Maclaurin series. There is a theorem which tells us that such indirect methods always give us the genuine article.]

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

deriv ↪

$\cos(x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \dots$

↪ $\frac{d}{dx}$

$\frac{3}{3!} = \frac{3}{3 \cdot 2 \cdot 1} = \frac{1}{2}$ $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

↪ tidy

$\frac{5}{5!} = \frac{5}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{4!}$

↪ mult by x

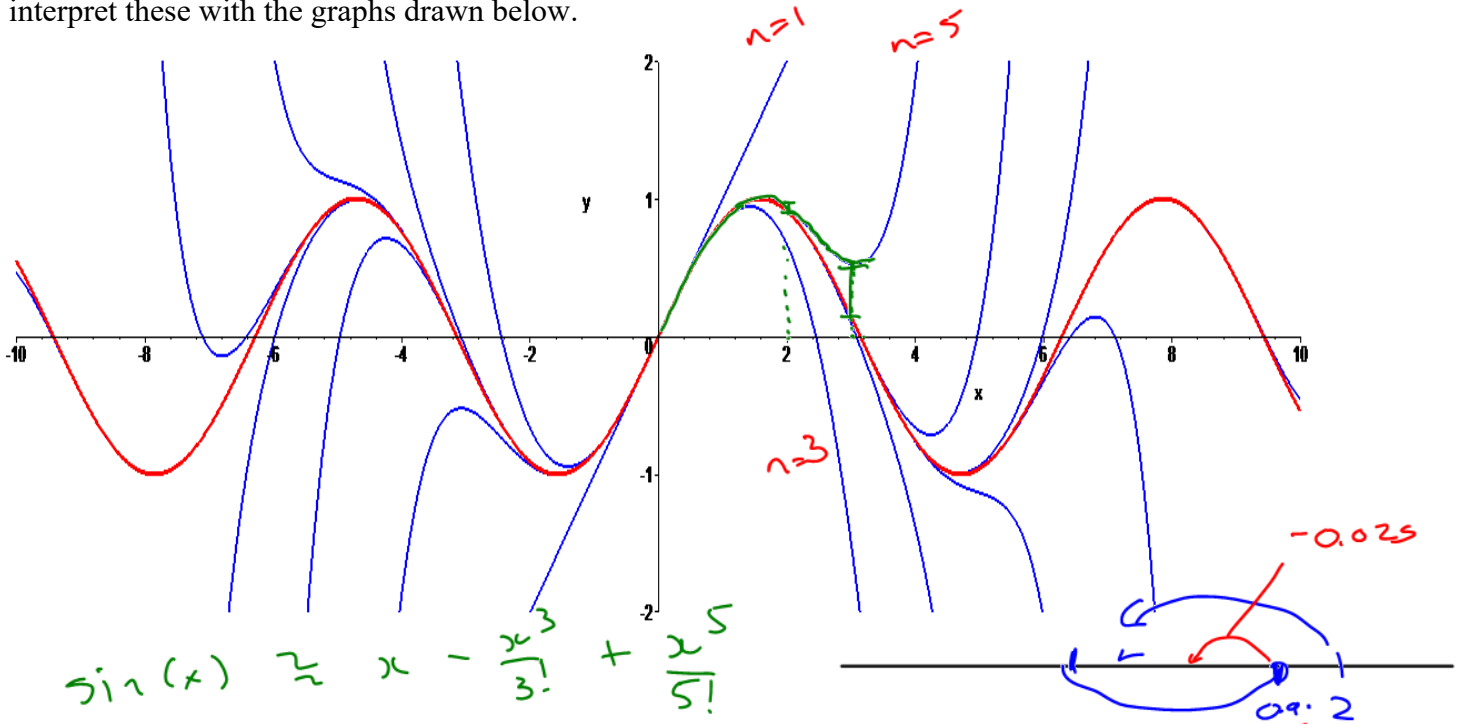
$x \cdot \cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \dots$

Example 4.5. Find an upper bound on the error in the approximation

$T_5 = \text{up to } x^5 \text{ term.}$

$$\sin x \approx T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

at $x=2$ and $x=3$. Check your results by comparing the values of \sin and T_5 at $x=2$ and $x=3$ and interpret these with the graphs drawn below.



$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

checking
0.90932

$$\sin(2) \approx 2 - \frac{2^3}{3!} + \frac{2^5}{5!} = 0.933 \pm 0.025$$

Recall: error in alt'g series < mag of first omitted term.

$$\frac{2^7}{7!} = 0.025$$

$$\sin(3) \approx 3 - \frac{3^3}{3!} + \frac{3^5}{5!} \approx 0.525 \pm 0.43$$

error < $\frac{3^7}{7!} = 0.43$

Example 4.6. By calculating successive derivatives, find the Maclaurin series for $f(x) = (1+x)^k$ where k is any real number.

$e \ a=0$
 $or \ x=0$

eg. $(1+x)^2$
or $(1+x)^{3/2}$
or $\frac{1}{\sqrt{1+x}}$

n	$f^{(n)}(x)$	$x = 0$
0	$(1+x)^k$	1
1	$k(1+x)^{k-1}$	k
2	$k(k-1)(1+x)^{k-2}$	$k(k-1)$
3	$k(k-1)(k-2)(1+x)^{k-3}$	$k(k-1)(k-2)$

This gives us the series:

$$(1+x)^k = 1 + \frac{k}{1}x + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

Note that if k is an integer this will terminate, and we get the familiar binomial theorem. Otherwise, it goes on forever and we get an infinite series which converges for $|x| < 1$.

$\frac{1}{\sqrt{1+x}}$... subs, $\times \frac{1}{4}$

Example 4.7. Find the Maclaurin series for $f(x) = \frac{1}{\sqrt{4-x}}$. Give an explicit formula for the n th term.

The first thing we need to do is turn the 4 into a 1, and we do that by taking it out of the square root.

This gives us

$$\frac{1}{2} \left(1 - \frac{x}{4}\right)^{-1/2} =$$

$$= \frac{1}{2} \left(1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!}\left(-\frac{x}{4}\right)^4 + \dots \right)$$

$$= \frac{1}{2} \left(1 + \frac{1}{2}\left(\frac{x}{4}\right) + \frac{1 \cdot 3}{2^2 2!}\left(\frac{x}{4}\right)^2 + \frac{1 \cdot 3 \cdot 5}{2^3 3!}\left(\frac{x}{4}\right)^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 4!}\left(\frac{x}{4}\right)^4 + \dots \right)$$

$$= \frac{1}{2} \left(1 + \frac{1}{8}x + \frac{1 \cdot 3}{8^2 2!}x^2 + \frac{1 \cdot 3 \cdot 5}{8^3 3!}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{8^4 4!}x^4 + \dots \right)$$

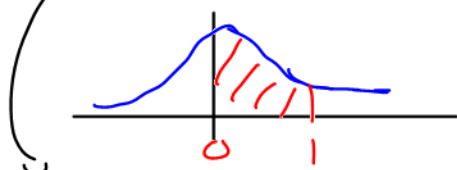
The n th term is $= \frac{1}{2} \left(\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{8^n n!} \right) x^n$.

error

Example 4.8 Use a Maclaurin series to approximate the value of $\int_0^1 e^{-x^2} dx$ with an accuracy of 0.001.

opt 1)
exact
anti-deriv,
Fund'l theorem of
calculus.
 $x e^{-x^2}$ has
no simple (non-series)
anti deriv.

opt 2)
LH sum
TRAP,
 x we have
no error
estimates
for those



opt 3): use a series
approx'n.

Best class
know

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

subs
 $x \rightarrow -x^2$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3!} + \dots$$

$$\int_0^1 e^{-x^2} dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots \right) dx$$

$$= \left(x - \frac{x^3}{3} + \frac{1}{2} \frac{x^5}{5} - \frac{1}{3!} \frac{x^7}{7} + \frac{1}{4!} \frac{x^9}{9} - \dots \right) \Big|_0^1$$

$$\int_0^1 e^{-x^2} dx = 1 - \frac{1}{3} + \frac{1}{2} \frac{1}{5} - \frac{1}{3!} \frac{1}{7} + \frac{1}{4!} \frac{1}{9} - \frac{1}{5!} \frac{1}{11} + \frac{1}{6!} \frac{1}{13} - \dots$$

This is an alternating
series
infinite / exact sum??

Fact: error of a partial sum $s_n <$ mag of first omitted term, b_{n+1}

Find a term w/
mag < 0.001

i.e. error < b_{n+1} < 0.001

try &
error.

$$\frac{1}{5!} \cdot \frac{1}{11} \approx 0.0007 \checkmark$$

this will be our b_{n+1}

our estimate will

$$\text{be } 1 - \frac{1}{3} + \frac{1}{2 \cdot 5}$$

$$- \frac{1}{3!} \cdot \frac{1}{7} + \frac{1}{4!} \cdot \frac{1}{9}$$

$$\dots - \frac{1}{5!} \cdot \frac{1}{11}$$

$$= 0.7474$$

ans 0.7474 ± 0.0007



5. Approximating functions by polynomials.

We can use a power series to get a polynomial approximation to a function on a given interval. The question is how many terms of the power series we need to get a desired accuracy. We already have one way of answering this question—if the series is alternating and satisfies the assumptions of the alternating series test, then the error is less than the size of the first omitted term. This is a very simple test and, in the examples below, we will use it whenever we can. But if the power series is not alternating, this result won't help us. Here's a theorem that will:

Taylor's Inequality.

For any value of x , the error in the Taylor polynomial approximation:

$$f(x) \approx T_n(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is no greater than

$$E_n = \frac{M}{(n+1)!} d^{n+1}$$

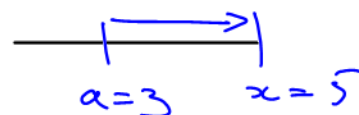
where $d = |x - a|$ and M is an upper bound of $|f^{(n+1)}(x)|$ on the interval $[a - d, a + d]$.

finite polynomial

missing/error

first omitted term

$$d = 5 - 3 = 2$$



Example 5.1. Use Taylor's inequality to decide how many terms of the series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

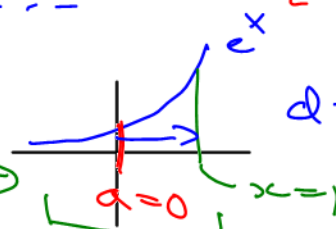
we need to take to approximate e to within 10^{-6} .

know

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

is this alternating @ $x=1$? No...

For error analysis, we must use Taylor's inequality



$$d = 1 - 0 = 1$$

$[-1, 1]$ is $[a-d, a+d]$ interval

error in $e^1 = e$ for sum up to x^n

$$\frac{M}{(n+1)!} d^{n+1} = \frac{e}{(n+1)!} < 10^{-6}$$

$M = (n+1)^{th}$ deriv of e^x maximized over $[-1, 1]$

$M = e^x$, maximized over $[-1, 1]$

$M = @ x=1, e^1$

Trial and error $\frac{e}{8!} = 67 \times 10^{-5} \notin 10^{-6}$

but $\frac{e}{9!} = 7.5 \times 10^{-6} \notin 10^{-6}$

try $\frac{e}{10!} = 7.5 \times 10^{-7} \leq 10^{-6}$

\Rightarrow exclude $\frac{x^{10}}{10!}$

loop $1 + x + \frac{x^2}{2} + \dots + \frac{x^9}{9!}$

so $e \approx 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \dots + \frac{1}{9!}$ error $\pm 7.5 \times 10^{-7}$

Example 5.2. Approximate the function $f(x) = x^{1/3}$ by a Taylor polynomial of degree 2 at $x = 8$. How accurate is this approximation on the interval $[7, 9]$? [When $x < 8$, you'll need to use Taylor's Inequality.]

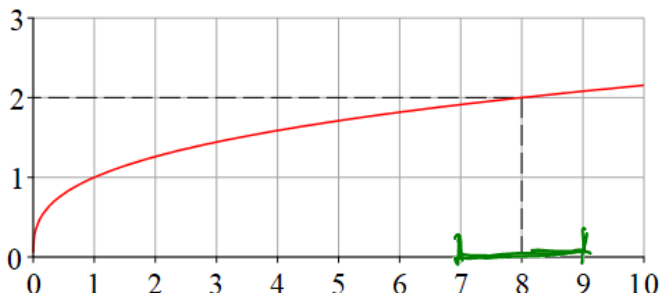
$a=8$

Solution.

sub in $a/x = 8$

$f(x)$

n	$f^{(n)}(x)$	$x = 8$	c_n
0	$x^{1/3}$	2	2
1	$(1/3)x^{-2/3}$	$(1/3)8^{-2/3}$	Div by 1!
2	$(-2/9)x^{-5/3}$	$(-2/9)8^{-5/3}$	Div by 2!
3	$(10/27)x^{-8/3}$	$(10/27)8^{-8/3}$	Div by 3!



We use a table to calculate the coefficients. I have not simplified the expressions as I want to be able to focus on the form of the terms of the series. First note that in the first column (the derivatives of f) the signs alternate from $n=1$ on. Here are the Taylor polynomials of degree 2 and 3:

$$T_2(x) = 2 + \frac{(1/3)8^{-2/3}}{1!}(x - 8) - \frac{(2/9)8^{-5/3}}{2!}(x - 8)^2$$

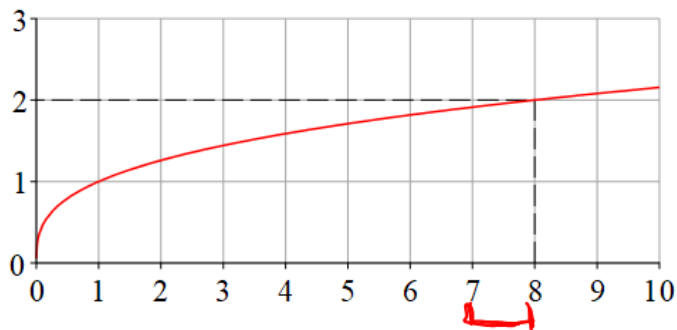
$$T_3(x) = 2 + \frac{(1/3)8^{-2/3}}{1!}(x - 8) - \frac{(2/9)8^{-5/3}}{2!}(x - 8)^2 + \frac{(10/27)8^{-8/3}}{3!}(x - 8)^3$$

Is the Taylor series of this function alternating? The answer is yes as long as all the terms $(x - 8)^n$ are positive. Otherwise it is not. Thus, we have an alternating series for $x > 8$ but not for $x < 8$. Thus, our error analysis for $x > 8$ can use the alternating series estimate, but for $x < 8$ we will use the Taylor estimate.

Case 1 ($x > 8$). The series is alternating. Here our error estimate will apply to the interval $[8, 9]$

Case 2 ($x < 8$). The series is not alternating.

n	$f^{(n)}(x)$	$x = 8$	c_n
0	$x^{1/3}$	2	2
1	$(1/3)x^{-2/3}$	$(1/3)8^{-2/3}$	Div by 1!
2	$(-2/9)x^{-5/3}$	$(-2/9)8^{-5/3}$	Div by 2!
3	$(10/27)x^{-8/3}$	$(10/27)8^{-8/3}$	Div by 3!



Here our error estimate will apply to the interval $[7, 8]$. As a reminder, here is the degree-3 Taylor polynomial:

$$T_3(x) = 2 + \frac{(1/3)8^{-2/3}}{1!}(x-8) - \frac{(2/9)8^{-5/3}}{2!}(x-8)^2 + \frac{(10/27)8^{-8/3}}{3!}(x-8)^3$$

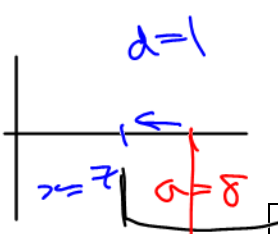
IF $x \in [7, 8]$, then $(x-8)$ is ~~pos~~ neg.
 signs of T_3 for $x \in [7, 8]$

$$= (+) + (+)(-) + (-)(+) + (+)(-) \dots$$

$$= (+) + (-) + (-) + (-) \dots$$

not alternating
 \Rightarrow cannot use alternating error formula
 \Rightarrow we use Taylor's inequality
 \rightarrow need $d, M, f^{(n+1)}$

error $< \frac{M}{(n+1)!} d^{n+1}$
 for up to n th power
 given in question.



	$x=7$	$x=9$
$x^{1/3}$	1.91293	2.08008
$T_2(x)$	1.91319	2.07986
Difference	0.00026	0.00022
Error Est	0.00034	0.00024

$M = \max$ of $f^{(3)}$ on $[7, 9]$
 $= \max$ of $\frac{10}{27} x^{-8/3}$ on $[7, 9]$

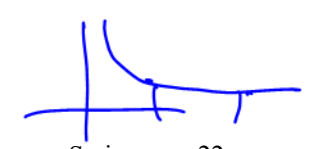
$$= \frac{10}{27} 7^{-8/3}$$

$$= \frac{10}{27} 7^{-8/3} \cdot (1) = 0.00034$$

$3 = (2+1)$
 $3! = (2+1)!$

$n=2$ so error of $T_2(x)$ on $[7, 8]$

smaller $x \Rightarrow$ bigger values
 $\frac{1}{x^{8/3}}$



6. The great Leonhard Euler (1707 – 1783)

Example 6.2 Does the relationship between the following three series, lead to any interesting formulae?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \dots$$



Example 6.2 Investigate Euler's formula for the sum of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$