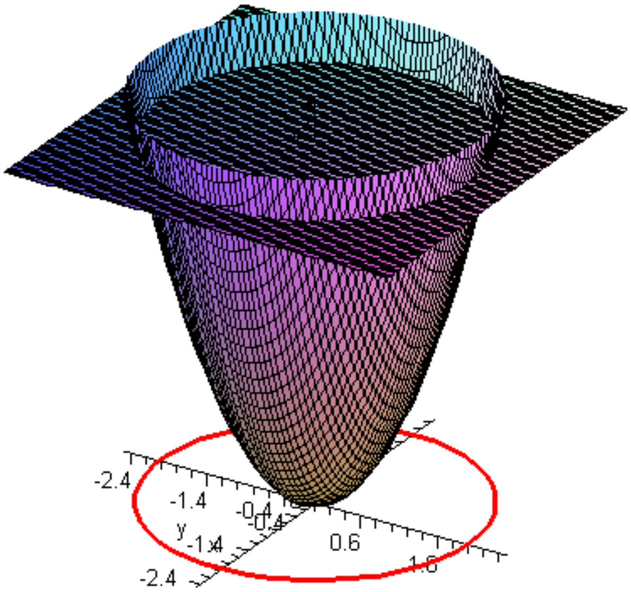
# Chapter 3

# Multiple Integration



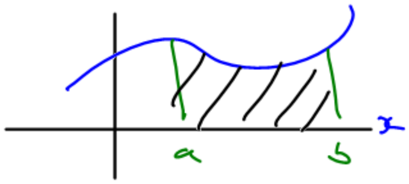
4 weeks

1. **Double integrals: basic concepts.**
2. **Double integrals over general regions.**
3. **Polar coordinates.**
4. **Centre of mass.**
5. **Triple integrals**

surface $z = f(x,y)$

## 1. Double integrals: basic concepts.

In 1D $f(x)$



Area $= \int_a^b f(x)\,dx$

If $f(x, y) \geq 0$, the volume $V$ of the solid lying below the surface $z = f(x, y)$ and above the region $R$ in the $x$-$y$ plane is called the double integral of $f$ over $R$ and is written

double integral

$$V = \iint_R f(x,y)\,dA.$$

"small unit of area"

[Note: If $f(x, y)$ is allowed to be negative, the integral gives us a "signed" volume where volumes below the $x$-$y$ plane are counted negative.]

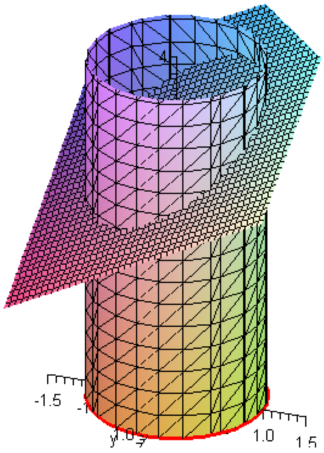
*Example 1.1.*
The volume over the disk $D = \{x^2 + y^2 \leq 1\}$ lying below the plane

$$z = \frac{y-x}{2} + 3$$

and above the $x$-$y$ plane is written

$$V = \iint_D \left(\frac{y-x}{2} + 3\right) dA.$$

z height or $f(x,y)$



The volume $V$ of the solid lying below the surface

$$z = f(x, y)$$

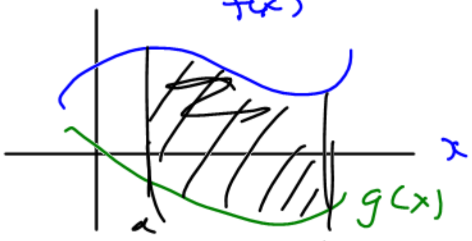
above the surface

$$z = g(x, y)$$

and "over" the base region $R$ in the $x$-$y$ plane is written

$$V = \iint_R f(x,y) - g(x,y)\,dA.$$

(top z) − (bottom z)



$z=4$ top surface

*Example 1.2.* The volume of the solid above the parabolic bowl

$$z = x^2 + y^2$$

and below the plane $z = 4$ is written

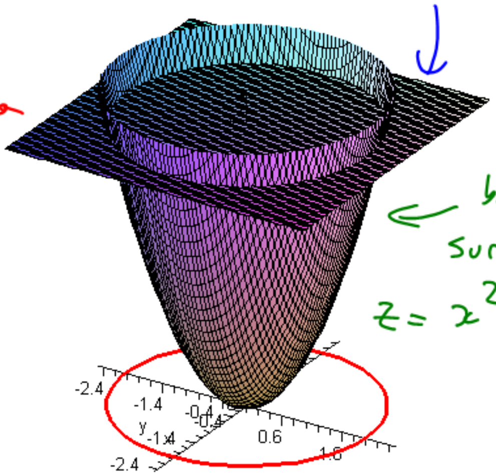
integrand little area units

$$V = \iint_D 4 - (x^2 + y^2)\,dA.$$

domain

where $D$ is the disk:

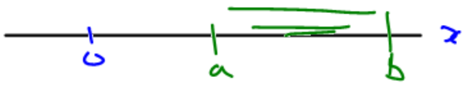
$$D = \{x^2 + y^2 \leq 4\}.$$



bottom surface $z = x^2+y^2$

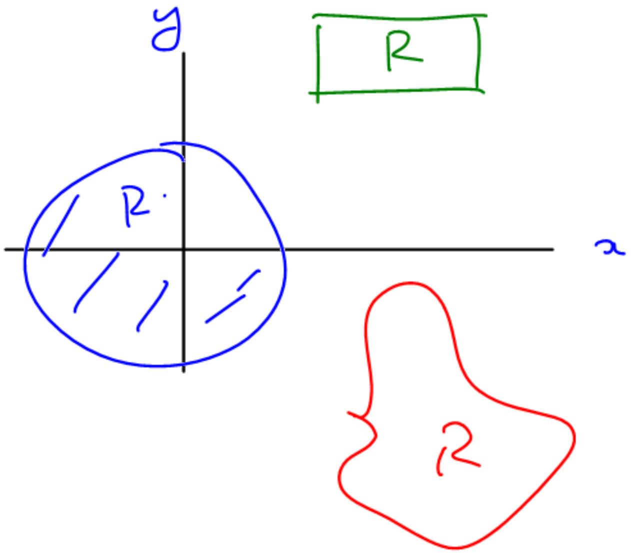
# Domains / Intervals

in 1D



in 2D

Top down view



Full

Here is a wonderful general principle for calculating such volumes. You've already seen special case of this in the method of calculating volumes of revolution. For example, you already know how to calculate the volume in Example 1.2 above by slicing the solid horizontally into disks and integrating the area of the slices.

vol of a slice

= A(x)· thickness

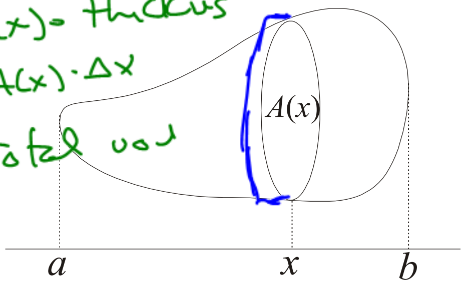
= A(x)·Δx

dx / Δx thick

Total vol

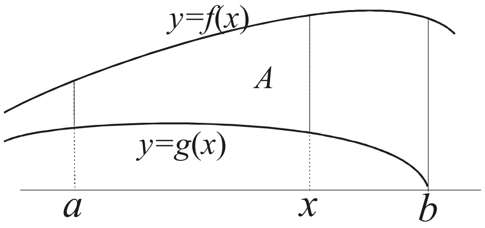
***The sliced potato principle.***

Suppose we have a general solid in 3-space (think of it as a potato). Suppose that if we slice it perpendicular to some suitable axis, say *x*, we have a way of calculating the cross-section area *A*(*x*) at *x*. Then the volume of the solid is the integral of these areas *A*(*x*) over the interval containing the solid.



$$V = \int_a^b A(x)dx$$

Note that this is a natural generalization of the standard formula for calculating the area under a graph with an integral or (perhaps more generally) of calculating the area between two curves.



$$A = \int_a^b f(x) - g(x)dx$$

*Example 1.3.* Use the sliced potato principle to calculate the volume of the solid in Ex. 1.2:

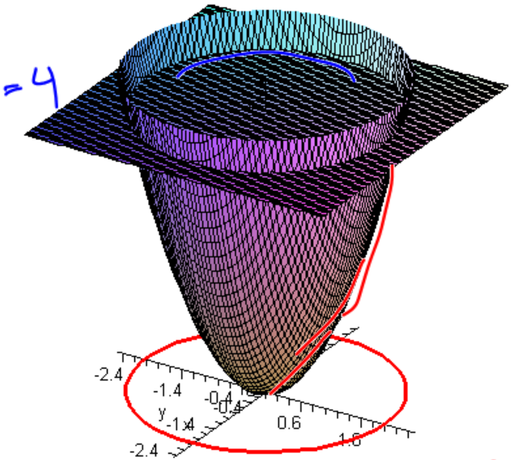
$$V = \iint_D \boxed{4} - (x^2 + y^2)dA.$$

z=4

where *D* is the disk of radius 2:

$$D = \{x^2 + y^2 \le 4\}.$$

[Answer: $8\pi$]



Take horizontal slices (circles)

↓

need radius to get area

Total Volume

$$= \int_{z=0}^{z=4} \text{Area}(z)\, \underset{\text{Thickness}}{dz}$$

$\overset{\downarrow}{2D} \cdot \overset{\downarrow}{1D} = \text{vol}$

$$= \int_{z=0}^{z=4} (\pi z)\, dz$$

This is a 1D integral!

↑ conceptual

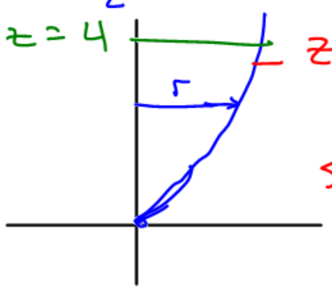
↓ computational antideriv

$$= \pi \frac{z^2}{2} \Big|_0^4$$

$$= \left(\pi \frac{16}{2}\right) - (\pi \cdot 0) = \boxed{8\pi}$$

cubic units.

formula



$z = x^2 + y^2$

set y=0

r = x coord on

$z = x^2 \cancel{+ y^2}$ =0

↓

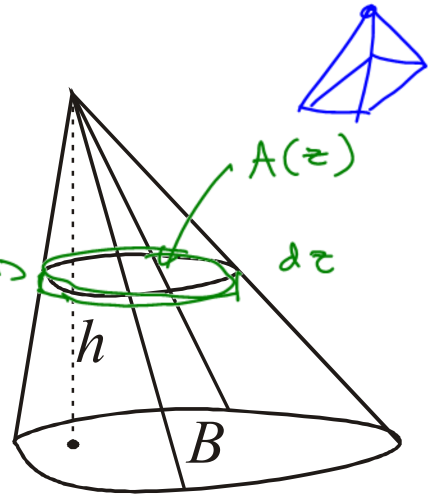
$\boxed{r} = x = +\sqrt{z}$

$$\text{Area}(z) = \pi r^2$$

$$= \pi(\sqrt{z})^2 = \pi z$$

*Example 1.4.* Use the sliced potato principle to calculate the volume of a cone with height $h$ and base *of any shape* with area $B$.
[Hint: It's technically easier if you take $z$=0 to be the vertex of the cone.]
*Answer*: $Bh/3$.

Circular base cone



Recall $V = \frac{1}{3}\underbrace{\pi r^2}_{\text{Area of base}} h$

General B case:

Volume of a slice

$= \underset{\text{area}}{A(z)} \cdot \underset{\text{thickness}}{dz}$

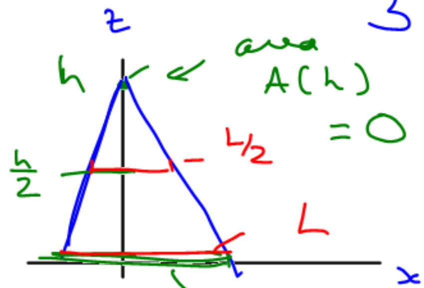
Hypothesis $V = \frac{1}{3}\overset{\text{height}}{\overset{\downarrow}{}}\underset{\text{Base area}}{\underset{\uparrow}{B}} \cdot h$

Q: what is $A(z)$?

$A(z) = B \cdot \left(\frac{h-z}{h}\right)^2$

fraction of tot height, measured from the peak

Side view

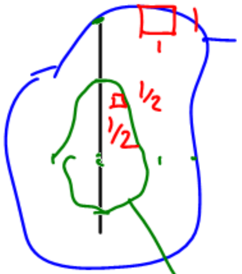


area $A(h) = 0$ (point/vertex)

lengths in red; using similar triangles

Area at $z=0$ is B.

Now look down from above



B at $z=0$

area at $z = h/2$

area $= \boxed{\frac{B}{4}}$

$$\text{vol} = \int_{z=0}^{z=h} \underbrace{B\left(\frac{h-z}{h}\right)^2}_{\text{Area of slice}} \, \underset{\text{thickness}}{dz}$$

↑ conceptual

↓ calculation

B, h constants

$$= \int_0^h \frac{B}{h^2}(h-z)^2 \, dz$$

anti deriv

$$= \frac{B}{h^2} \frac{(h-z)^3}{3}(-1) \Bigg|_{z=0}^{z=h}$$

check: $\frac{d}{dz}$

$$= B/h^2 \left[\left(-\frac{1}{3}(\cancel{h-h})^3\right) - \left(-\frac{1}{3}(h-0)^3\right)\right]$$

$0$

$$= \frac{B}{h^2} \cdot \frac{h^3}{3}$$

$$= \frac{Bh}{3}.$$ ✓

Volume small area unit

||

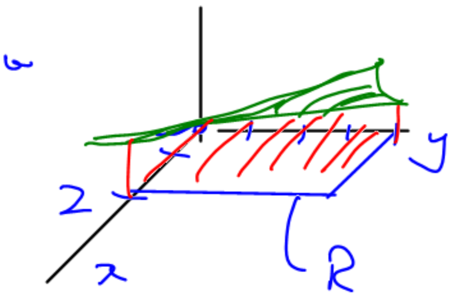
xy region

*Example 1.5.* Use the sliced potato principle to evaluate the double integral $\iint_R (3x^2 + 4xy)\,dA$ where $R$ is the rectangle $\{0 \le x \le 2,\ 0 \le y \le 4\}$. Do this in two ways:

$z = f(x,y)$ surface

(a) Slice perpendicular to the $y$-axis
(b) Slice perpendicular to the $x$-axis.
[*Answer*: 96.]

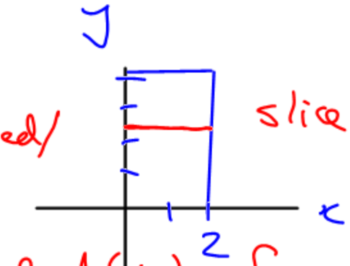
3D view



(a) Total vol $= \int_{y=0}^{y=4} A(y)\,dy$

↑ formula?

Top down 2D view



fix $y = y_0$  $A(y_0) = \int_{x=0}^{x=2} (\text{height})\,dx$

constant

$= \underbrace{\int_{x=0}^{x=2} 3x^2 + 4x\cdot y_0 \,dx}_{=\text{ area of one slice}}$

y fixed/ $y = y_0$

↳ find $A(y_0)$ of that slice.

anti deriv $= \cancel{3}\frac{x^3}{\cancel{3}} + \overset{2}{\cancel{4}}\frac{x^2}{\cancel{2}}\cdot y_0 \Big|_{x=0}^{x=2}$

sub in x's $= (2^3 + 2\cdot 2^2\cdot y_0) - (0+0)$

tidy $A(y_0) = 8 + 8y_0$

$\updownarrow$

$A(y) = 8 + 8y \rightarrow$ Total vol $= \int_{y=0}^{y=4} A(y)\,dy$

sub in $A(y)$ $= \int_0^4 8 + 8y\,dy$

int $= 8y + \frac{8y^2}{2}\Big|_0^4$

$= (32 + 4\cdot 16) - (0)$

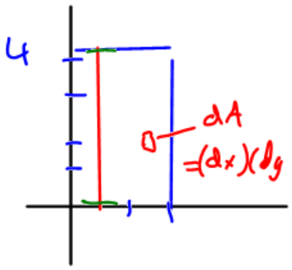
$= 96.$

Observe that this approach has calculated the volume essentially as an *iterated integral*. The two calculations are:

(a) $\int_0^4 \left[\int_0^2 (3x^2 + 2xy)\,dx\right] dy$ where the inner integral treats $y$ as constant

(b) $\int_0^2 \left[\int_0^4 (3x^2 + 2xy)\,dy\right] dx$ where the inner integral treats $x$ as constant

(b) $V = \iint_R 3x^2 + 4xy \;\; dA$



$= \int_{x=0}^{x=2} \overset{}{\underset{\downarrow}{A(x)}} \; dx$

height

$= \int_{x=0}^{x=2} \left( \int_{y=0}^{y=4} (3x^2 + 4xy)\, dy \right) dx$

↓ as we evaluate $\int dy$, treat $x$ as a constant!

work from inside out

$= \int_{x=0}^{x=2} \left( \underset{\text{const for } \int dy}{\underline{3x^2}} \cdot y + 4x \cdot \frac{y^2}{2} \Bigg|_{y=0}^{y=4} \right) dx$

sub in y's

$= \int_{x=0}^{x=2} \left( \left( 3x^2 \cdot 4 + 4x \cdot \frac{4^2}{2} \right) - (0 + 0) \right) dx$

tidy

$= \int_0^2 12x^2 + 32x \;\; dx$

nice! only x's left.

integrate x's

$= 12\frac{x^3}{3} + 32\frac{x^2}{2} \Bigg|_0^2$

$= (4 \cdot 2^3 + 16 \cdot 2^2) - (0 + 0)$

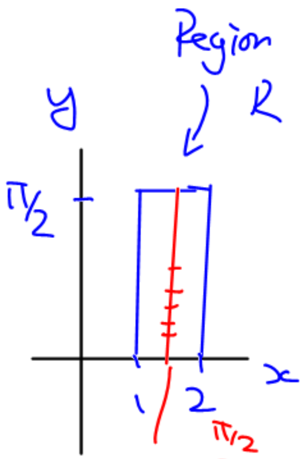
$= 32 + 64 = 96$

same as (a)! yay!

*Example 1.6.*

(a) Use an iterated integral to evaluate the double integral $\iint_R \overbrace{y\sin(xy)}^{\text{height } z=f(x,y)}dA$ where $R$ is the rectangle $\{1\le x\le 2,\ 0\le y\le \pi/2\}$.

(b) Write the integral that you get by slicing perpendicular to the $x$-axis.
(c) Do the calculation by slicing perpendicular to the $y$-axis. [Order sometimes matters.]
[Answer: 1.]



$A(x) = \int_{y=0}^{\pi/2} dy$

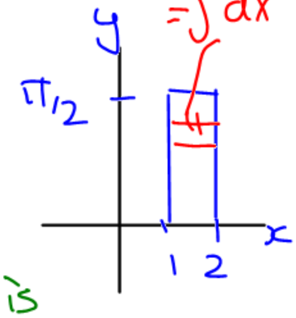
(a) $\iint_R y\sin(xy)\,dA$ $z = f(x,y)$ heights.

$$= \underbrace{\int_{x=1}^{x=2}\int_{y=0}^{y=\pi/2}}_{\text{defines our } xy \text{ region } R} y\sin(xy)\,dy\,dx$$

compute integrals from the inside first

$\int y\sin(ky)\,dy$ do by parts

hard



$A(y) = \int dx$

or (b) slice ⊥ to y axis instead

$$\text{total} = \int_{y=0}^{y=\pi/2}\int_{x=1}^{x=2} y\sin(\underset{=}{x}y)\,dx\,dy$$

x is the variable, y is a constant

like $\int c\sin(kx)\,dx$ ↓ ↑ $d/dx$ $-\frac{1}{k}\cos(kx)$

(do dx integral

$$= \int_{y=0}^{y=\pi/2} \overbrace{y\cdot\left(-\frac{1}{y}\cos(xy)\right)}^{\text{check } \frac{\partial}{\partial x}}\Bigg|_{x=1}^{x=2} dy$$

tidy $= \int_0^{\pi/2} -\cos(xy)\Big|_{x=1}^{x=2} dy$

sub in x's $= \int_0^{\pi/2} -\cos(2y) - (-\cos(1\cdot y))\,dy$

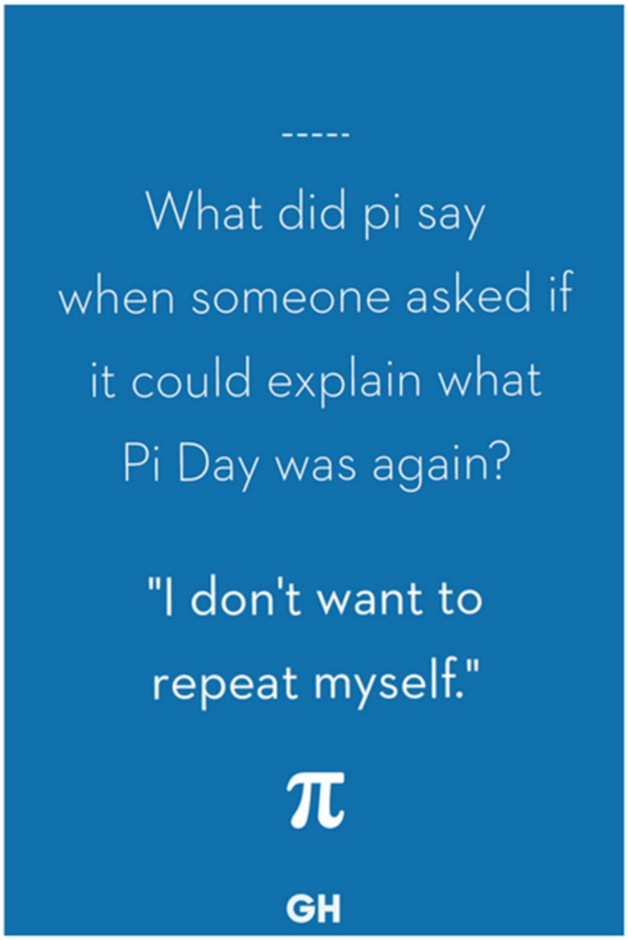
tidy $$= \int_0^{\pi/2} \cos(y) - \cos(2y)\, dy$$

anti diff $$= \sin(y) - \frac{1}{2}\sin(2y) \Big|_{y=0}^{y=\pi/2}$$

sub in $$= \left[\sin\left(\tfrac{\pi}{2}\right) - \frac{1}{2}\underset{0}{\cancel{\sin\left(\overset{\pi}{\cancel{2\cdot\tfrac{\pi}{2}}}\right)}}\right] - \left[\underset{0}{\cancel{\sin(0)}} - \frac{1}{2}\underset{0}{\cancel{\sin(0)}}\right]$$

$$= 1.$$

# Happy Pi Day!



What did pi say
when someone asked if
it could explain what
Pi Day was again?

"I don't want to
repeat myself."

π

GH

Source: Good Housekeeping (yeah, really!)

non-rectangular

## 2. Double integrals over general regions.

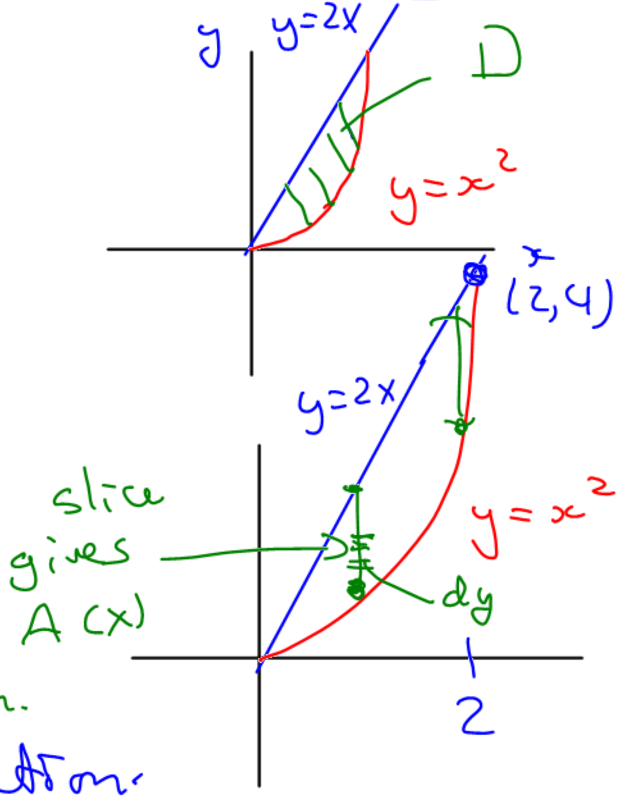
z height ↓

*Example 2.1* Find the volume of the solid that lies under the plane $z = x + 2y$ and above the region $D$ in the $x$-$y$ plane bounded by the line $y = 2x$ and the parabola $y=x^2$.
[Answer: 28/5]

$$V = \int_{x=0}^{x=2} A(x)\, dx$$

z height ↓

$$= \int_{x=0}^{x=2} \left( \int_{y=x^2}^{y=2x} (x+2y)\, dy \right) dx$$



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$\int dy$, treat $y$ as the variable ⇒ $x$ is constant

~~calculation.~~

calculation.

$$= \int_{x=0}^{x=2} xy + \frac{\cancel{2}y^2}{\cancel{2}} \Bigg|_{y=x^2}^{y=2x} dx$$

**The big idea.** The sliced potato principle allows us to convert a double integral:

$$\iint_R f(x,y)\,dA$$

into an iterated integral:

$$\int_a^b \left( \int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

where in the inner $y$-integral, $x$ is held constant.

sub in $y=2x$, $y=x^2$

$$= \int_{x=0}^{x=2} \left( x(2x) + (2x)^2 \right) - \left( x\cdot x^2 + (x^2)^2 \right) dx$$ no y's

tidy

$$= \int_0^2 \underset{6x^2}{\cancel{2x^2 + 4x^2}} - x^3 \overset{-}{\cancel{+}} x^4 \, dx$$

$$= 2\cancel{6}\frac{x^3}{\cancel{3}} - \frac{x^4}{4} - \frac{x^5}{5} \Big|_0^2$$

$$= \left( 2\cdot 2^3 - \frac{2^4}{4} - \frac{2^5}{5} \right) - (0 - 0 - 0)$$

$$= 16 - 4 - \frac{32}{5} = \ldots -$$ #

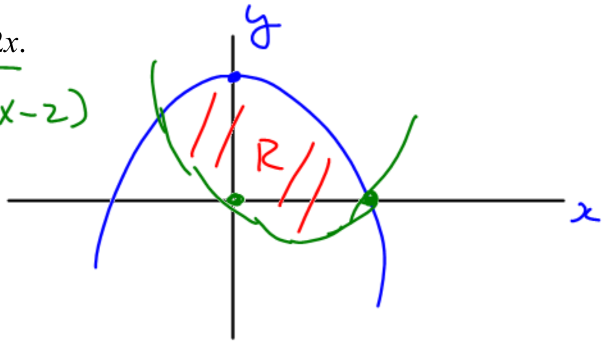
$z = f(x,y)$

*Example 2.2* Evaluate the integral $\iint_R x\,dA$ where $R$ is the region bounded by the parabolas

$$y = 4 - x^2 \text{ and } y = x^2 - 2x.$$

$= x(x-2)$

[Answer: 9/2.]



Find intersection b/w two parabolas

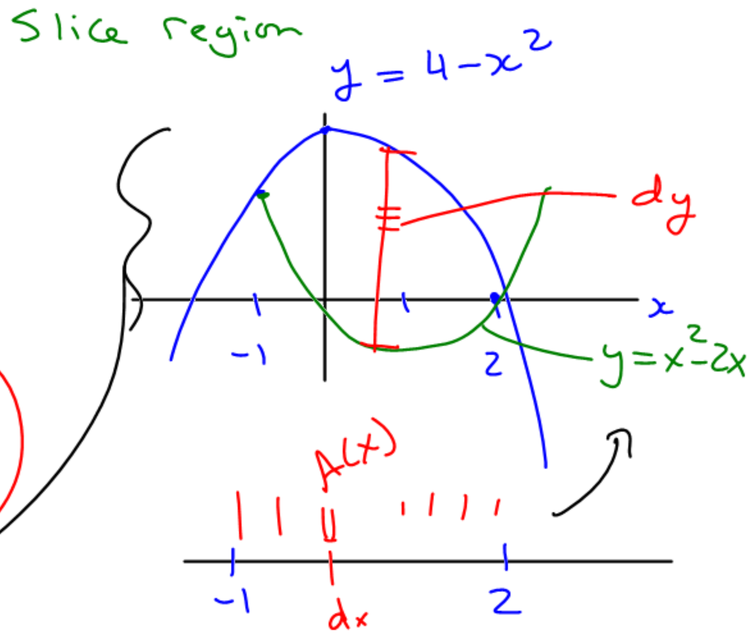
set $y = 4 - x^2 = x^2 - 2x = y$

$0 = 2x^2 - 2x \overset{-}{\cancel{+}} 4$

or $0 = x^2 - x \overset{-}{\cancel{+}} 2$

$0 = (x-2)(x+1)$

intersect @ $x = -1$, $x = 2$



$$V = \int_{x=-1}^{2} A(x)\,dx$$

$$= \int_{x=-1}^{2} \int_{y=x^2-2x}^{y=4-x^2} x \; dy \; dx$$

const

↑ concepts diagrams

---

↓ calculations

y integ $= \displaystyle\int_{x=-1}^{x=2} \left( x \cdot y \,\Big|_{y=x^2-2x}^{y=4-x^2} \right) dx$

sub in y's $= \displaystyle\int_{x=-1}^{x=2} \left(x \cdot (4-x^2)\right) - \left(x(x^2-2x)\right) dx$

only x's now!

tidy $= \displaystyle\int_{-1}^{2} 4x \underset{-2x^3}{\cancel{-x^3 - x^3}} + 2x^2 \, dx$

do
x
int'le

$$= \frac{4x^2}{2} - \frac{2x^4}{4} + \frac{2x^3}{3} \Big|_{-1}^{2}$$

sub in

$$= \left(2 \cdot 2^2 - \frac{1}{2} 2^4 + \frac{2}{3} \cdot 2^3\right) - \left(2 \underset{1}{\cancel{(-1)^2}} - \frac{1}{2} \underset{1}{\cancel{(-1)^4}} + \frac{2}{3} \underset{-1}{\cancel{(-1)^3}}\right)$$

$$= 8 - 8 + \frac{2}{3} \cdot 8 - 2 + \frac{1}{2} + \frac{2}{3}$$

$$= 9/2.$$

$\frac{1}{3} B \cdot h \rightarrow B = \frac{1}{2}, h = 2 \rightarrow$ 4 sided die, pyramid $v = \frac{1}{3} \cdot \frac{1}{2} \cdot 2 = \frac{1}{3}$ w/ triangle base

*Example 2.3 Find* the volume of the tetrahedron bounded by the planes

general ones

$$z = 2 - x - 2y, \quad x = 2y, \quad x = 0, \text{ and } z = 0.$$

xy plane

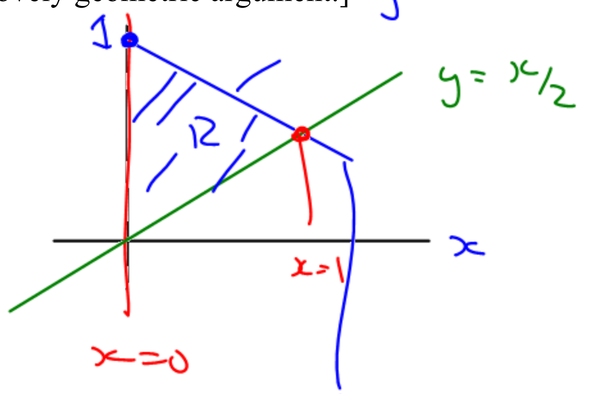
[Answer: 1/3: can you deduce that from Example 1.4? There's a lovely geometric argument.]

$y = \frac{x}{2}$ $\quad y = 1 - x/2$

$$\text{Vol} = \iint_R z = f(x,y)\, dA$$

define → R

know $z = 2 - x - 2y$



• at intersection

$y = 1 - x/2$ and $y = x/2$ set equal

$$1 - \frac{x}{2} = \frac{x}{2}$$

$$1 = \frac{x}{2} + \frac{x}{2} = x$$

$$\boxed{x = 1}$$

intersection b/w $z = 2 - x - 2y$ and $z = 0$

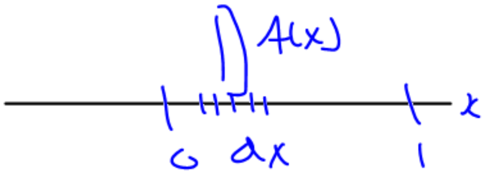
$$0 = 2 - x - 2y$$

$$2y = 2 - x$$

$$\boxed{y = 1 - \frac{x}{2}}$$

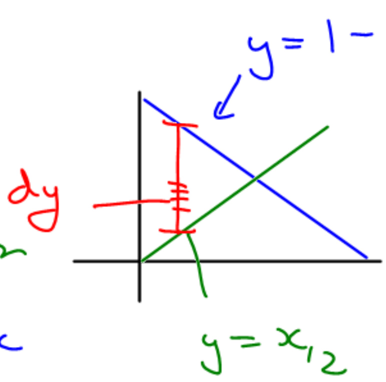
So Total vol

$$= \int_{x=0}^{x=1} A(x)\, dx$$



$$= \int_{x=0}^{x=1} \int_{y=x/2}^{y=1-x/2} 2 - x - 2y \; dy \; dx$$

const



calculation

y int'le

$$= \int_{x=0}^{x=1} \left( 2y - xy - \frac{\cancel{2}y^2}{\cancel{2}} \Bigg|_{y=x/2}^{y=1-x/2} \right) dx$$

sub in y's

$$= \int_{x=0}^{x=1} \left[ 2\left(1 - \frac{x}{2}\right) - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 \right.$$

$$- \left[2\left(\frac{x}{2}\right) - x\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^2\right] \, dx$$

only x's, no y's.

tidy

$$= \int_0^1 \left(2 - x - x + \frac{x^2}{2} - \left(1 - x + \cancel{\frac{x^2}{4}}\right) - x + \frac{x^2}{2} + \cancel{\frac{x^2}{4}}\right) dx$$

more tidy

$$= \int_0^1 1 + x^2 - 2x \quad dx$$

x integ

$$= x + \frac{x^3}{3} - \cancel{2}\frac{x^2}{\cancel{2}} \Big|_0^1$$

$$= \left(1 + 1/3 - 1\right) - \left(0 + 0 - 0\right)$$

$$= 1/3.$$

*Example 2.4* Evaluate the iterated integral $\int_0^1\int_x^1 \exp(y^2)\,dy\,dx$ by interchanging the order of integration (represent it first as a double integral).
[Answer: $(e-1)/2$]

defines the region R in $\iint_R e^{y^2}\,dA$

$\int_{x=0}^{x=1}\int_{y=x}^{y=1} e^{y^2}\,dy\,dx$

y intg'l

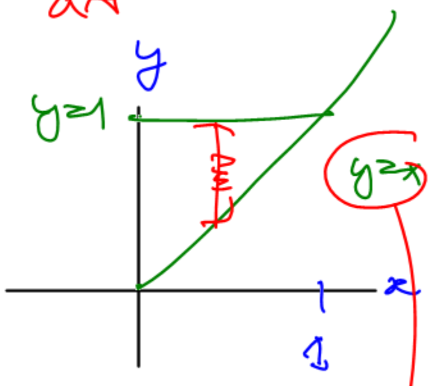
XX

no non-series anti deriv.

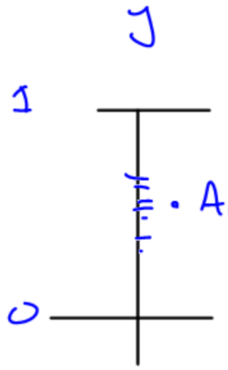
if we intg'te w/ dy

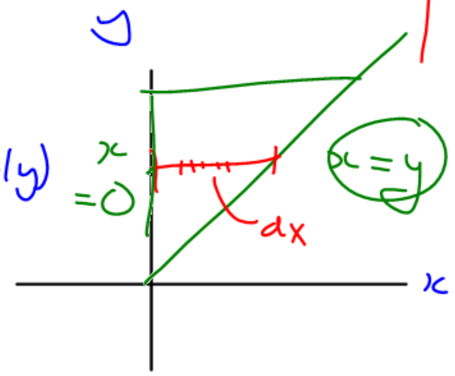
IF we $\int e^{y^2}\,dx$ ✓

const.



or





$= \int_{y=0}^{y=1} A(y)\,dy$

$= \int_{y=0}^{y=1}\int_{x=0}^{x=y} e^{y^2}\,dx\,dy$

---

do x intg'l $= \int_{y=0}^{1} e^{y^2}\cdot x \Big|_{x=0}^{x=y}\,dy$

calculation

sub in

$= \int_{y=0}^{y=1} e^{y^2}\, y\,dy$

← inside

outside

Substitution!

let $w = y^2$

so $\frac{dw}{dy} = 2y$ or $\frac{1}{2y}\, dw = dy$

$$= \int_{y=0}^{y=1} \cancel{y}\, e^{w} \left( \frac{1}{2\cancel{y}}\, dw \right) = \frac{1}{2} \int_{y=0}^{y=1} e^{w}\, dw$$

integrate

$$= \frac{1}{2} e^{w} \Big|_{y=0}^{y=1}$$

back to y's

$w = y^2$

$$= \frac{1}{2} e^{y^2} \Big|_{y=0}^{y=1}$$

$$= \frac{1}{2} e^{1} - \frac{1}{2} e^{0}$$

$$= \frac{1}{2} (e - 1).$$