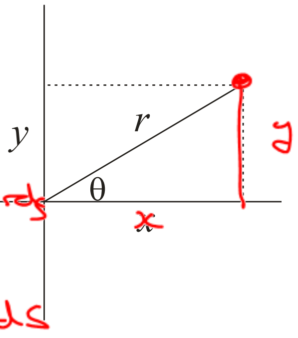
Recall : pos $\bar{u}(t) = [\underbrace{r\cos t}_{x}, \underbrace{r\sin t}_{y}]$

3. Polar coordinates.

Polar coordinates. For the point (x, y), we take r to be the distance of the point from the origin, and θ be the angle the line from the origin makes with the positive x-axis. Then



$$x = r\cos\theta$$
$$y = r\sin\theta$$

(r, θ) coords vs (x, y) coords

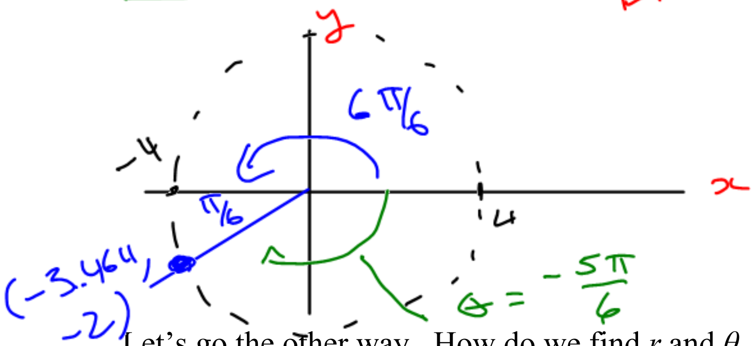
r = is called the *modulus* of (x, y). $= \sqrt{x^2+y^2}$

θ = is called the *argument* of (x, y).

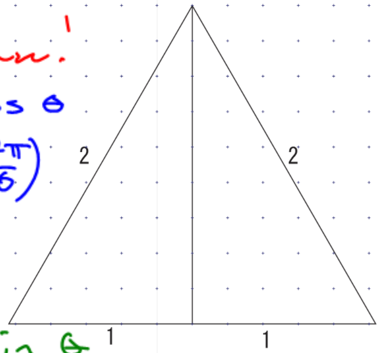
↳ $\tan\theta = \text{opp}/\text{adj} = y/x$

Example 3.1. Let (x, y) have modulus $r = 4$ and argument $\theta = 7\pi/6$. Find x and y.

Draw a diagram!



$x = r\cos\theta$
$= 4\cos\left(\frac{7\pi}{6}\right)$
$= -3.464$



$y = r\sin\theta$
$= 4\sin\left(\frac{7\pi}{6}\right) = -2$

Let's go the other way. How do we find r and θ in terms of x and y?

$r = \sqrt{x^2+y^2}$

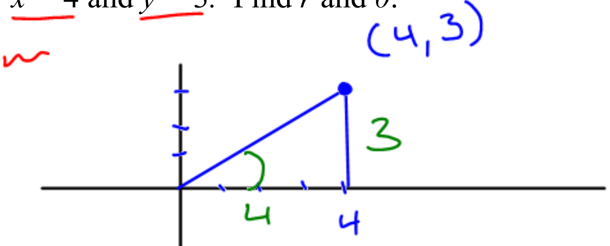
$\theta = \arctan\left(y/x\right)^*$

$* \rightarrow + \pi$, if • in Quad II, III or • if x negative

↳ only gives θ's $[-\pi/2, \pi/2]$

Example 3.2.
(a) Let $x = 4$ and $y = 3$. Find r and θ.

Diagram

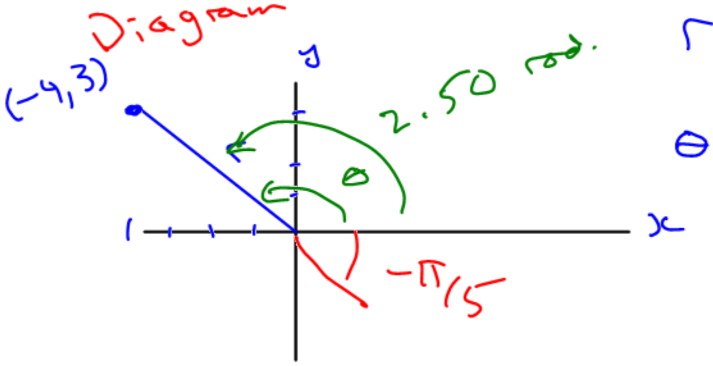


$r = \sqrt{3^2+4^2} = 5$

$\theta = \arctan\left(3/4\right) = 0.64$ rad ✓
$(\sim \frac{1}{5}\pi$ rad$)$

(b) Let $x = -4$ and $y = 3$. Find r and θ.

Diagram



$r = \sqrt{(-4)^2+(3^2)} = 5$

$\theta = \arctan\left(3/{-4}\right)^* = -0.64$ rad $+ \pi$
$+ \pi$ or $\approx -\frac{\pi}{5}$

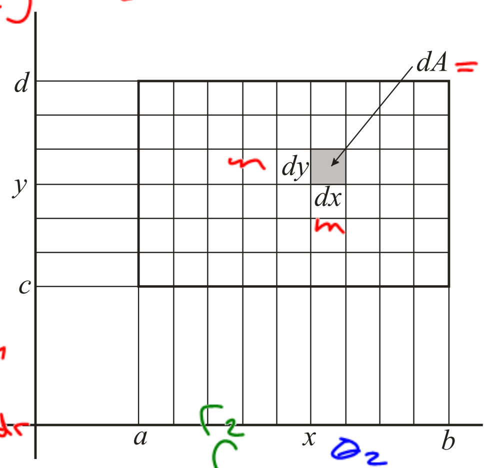
$= 2.50$ rad

$\text{vol} = \sum (\text{heights})(\text{areas})$

$z = f(x,y)$ dA

Example 3.3. Integrating in polar coordinates. ✓

When calculating a double integral $\iint_R f(x,y)\,dA$ it will be sometimes much better to use polar coordinates to work with the base region R and in this case, the question arises as to what the differential area dA should be. When working with Cartesian coordinates, dA was the area of a differential rectangle with sides dx and dy and thus, when we moved to an iterated integral, dA turned into the area $dxdy$ of this rectangle. For the rectangular region at the right:

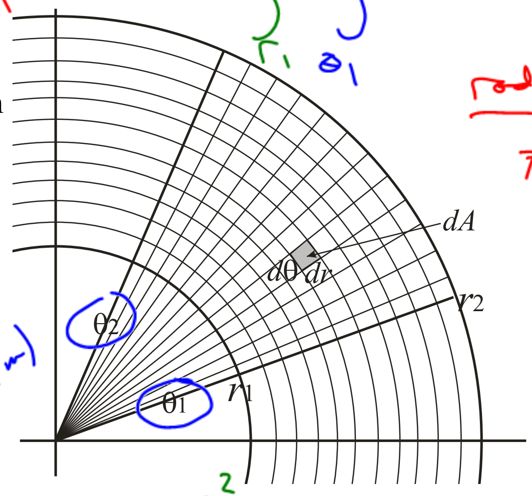


$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,\underline{dxdy} = \cancel{\iint d\theta dr}$$

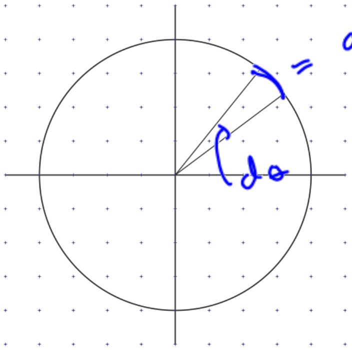
$$\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} d\theta dr$$

$$\frac{\text{rad} \times \text{m}}{\neq \text{m}^2}$$

Suppose, however, that R is a *polar* rectangle, defined by the inequalities $\theta_1 \le \theta \le \theta_2$ and $r_1 \le r \le r_2$. Then, if $f(x, y)$ could be readily expressed in polar coordinates, we might want to use those coordinates to do the integration. Thinking about Riemann sums, we'd decompose the region into differential polar rectangles with angular width $d\theta$ and distance from the origin increased by dr. But what is the corresponding differential area dA in terms of $d\theta$ and dr? Is it the product $d\theta dr$?

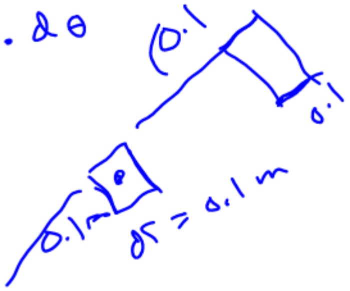


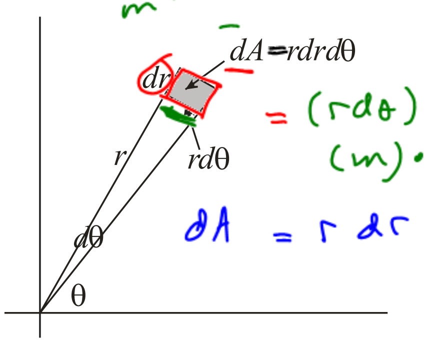
Convince yourself that when integrating over a region in the x-y plane using polar coordinates, the area dA of a differential polar rectangle must be converted to $rdrd\theta$.



= arc length
= $r \cdot d\theta$

(0.1 rad)(0.1 m)





m^2

$= (r\,d\theta)\,dr$

$(\text{m})\cdot(\text{m})$

$dA = r\,dr\,d\theta$

Calculating a double integral using polar coordinates: $\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \left(\int_{r=g(\theta)}^{r=h(\theta)} f(r,\theta)\,\underset{\uparrow}{r}\,dr \right) d\theta$

r multiplier added when we use polar coords hard to anti-diff'l te in xy coords!

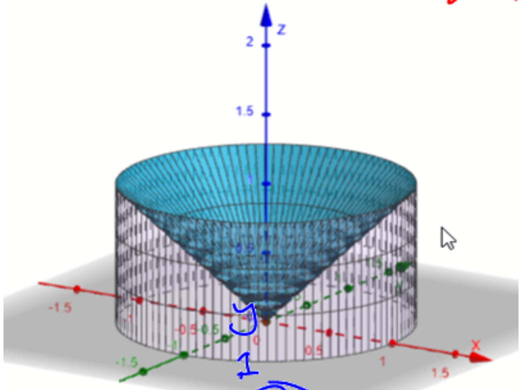
Example 3.4. Use polar coordinates to calculate $\iint_R \underbrace{\sqrt{x^2+y^2}}_{z=}\,dA$ ✓

where R is the disk $\{x^2+y^2 \le 1\}$.
[Answer: $2\pi/3$: can you deduce that from Example 1.4?]

memorize. $r = \sqrt{x^2+y^2}$

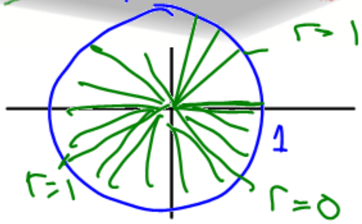
$r^2 = x^2+y^2$

Will be simple if we express everything in polar coords.



$$\iint_R \sqrt{x^2+y^2}\ \boxed{dA}$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} \underbrace{r}_{\text{z height}}\cdot\underbrace{(r\,dr\,d\theta)}_{\text{replaces } dA}$$



or $\int_{\theta=-\pi}^{\theta=+\pi}$

or $\int_{\theta=-2\pi}^{\theta=0}$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r^2\,dr\,d\theta$$

sub in $r=1, r=0$

int'g r

$$= \int_{\theta=0}^{\theta=2\pi} \frac{r^3}{3}\Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi}\left(\frac{1}{3}\right)d\theta$$

θ intg'd

$$= \frac{1}{3}\theta\Big|_0^{2\pi} = \frac{1}{3}(2\pi) = \frac{2\pi}{3}.$$

Example 3.5. Let E be the solid bounded by the plane $z = 0$ and the paraboloid $\underline{z} = \underline{1 - x^2 - y^2}$. Write an iterated integral for the volume of E. [Answer: $\int_0^{2\pi}\int_0^1 r - r^3\,dr\,d\theta$.]

$$z = 1 - x^2 - y^2$$
$$= 1 - (x^2+y^2)$$
$$\boxed{z = 1 - r^2}$$

$$\iint_{R} z \cdot dA \quad \text{concept}$$

↳ $z = 1 - x^2 - y^2$ given

R ↳ boundary

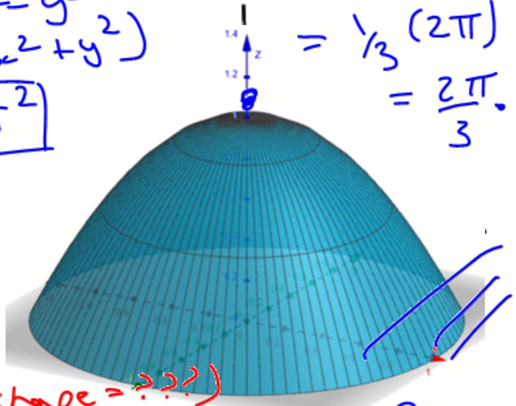
↳ intersection of ($z = 1 - x^2 - y^2$ and $z = 0$) set equal

$1 - x^2 - y^2 = 0$

$1 = x^2 + y^2$

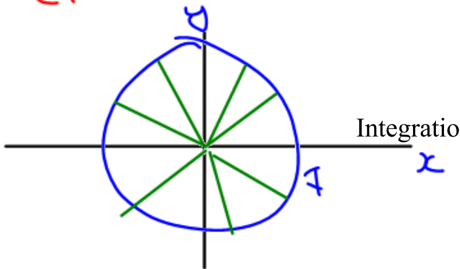
(shape = ???)

circle w/ radius 1.

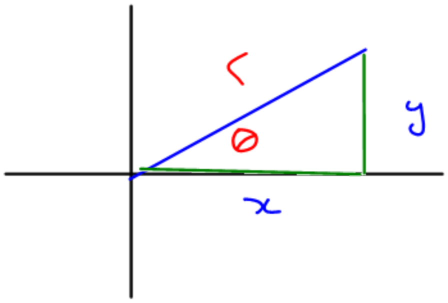


z=0

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (1-r^2)(r\,dr\,d\theta)$$



Polar coords



$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

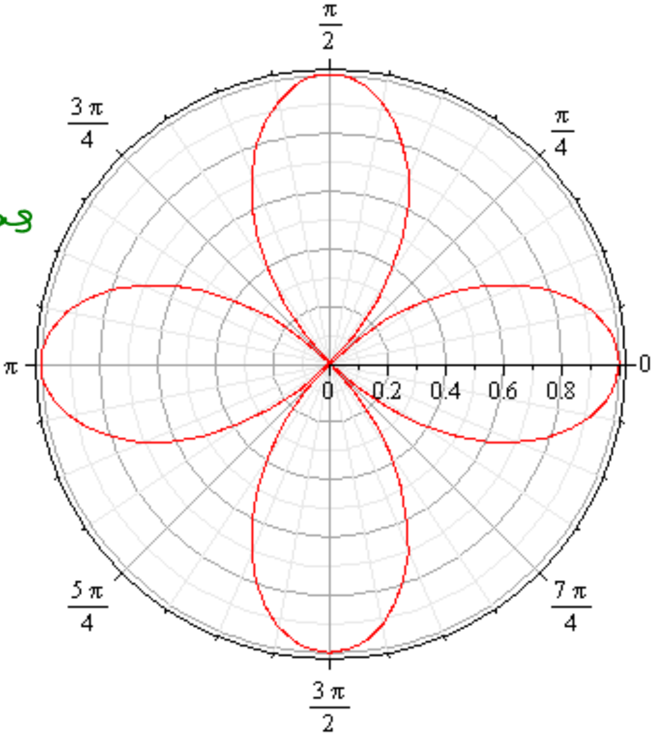
$y = f(x)$ $\qquad$ $r = f(\theta)$

Example 3.6

Write an iterated integral in polar coordinates for the area enclosed by one loop of the four-leaved rose $\boxed{r = \cos 2\theta}$.

[Tech-help: $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin(2u)$.]

[Answer: $\pi/8$.]

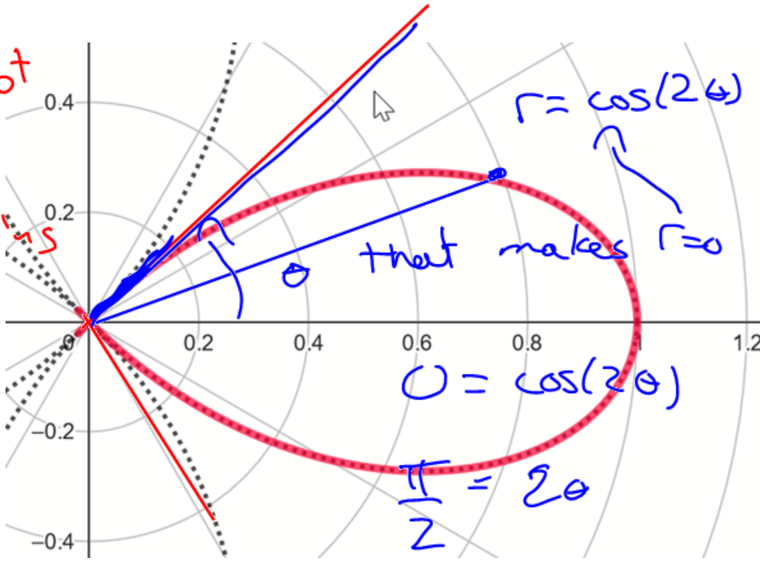


earlier $\quad \mathrm{Vol} = \iint\limits_{\substack{R \\ xy\ \text{plane}}} \underbrace{z}_{\text{heights}} \cdot \underbrace{dA}_{\text{areas}}$

$\Rightarrow \quad \underset{\text{of } R}{\text{Area}} = \iint\limits_{\substack{R \\ xy\ \text{plane}}} 1 \cdot \underline{dA}$

$$= \int_{\theta=-\pi/4}^{\theta=\pi/4} \int_{r=0}^{r=\cos(2\theta)} (r\,dr\,d\theta)$$

concept ↑ do calculus



$r = \cos(2\theta)$

$r=0$

θ that makes $r=0$

$0 = \cos(2\theta)$

$\frac{\pi}{2} = 2\theta$

$\frac{\pi}{4} = \theta$ limit

$$= \int_{\theta=-\pi/4}^{\pi/4} \left. r^2/2 \right|_{r=0}^{r=\cos(2\theta)} d\theta$$

sub in r's

$$= \int_{\theta\ -\pi/4}^{\pi/4} \frac{1}{2}\left[\cos^2(2\theta) - 0\right] \underline{d\theta}$$

$$= \int_{u=-\pi/2}^{u=+\pi/2} \frac{1}{2}\cos^2(u)\left(\frac{1}{2}du\right)$$

To make $2\theta \to u$,

subs let $u = 2\theta$

so $\frac{du}{d\theta} = 2$ $\quad$ here ↙

$\frac{1}{2}du = d\theta$

Bounds : $\theta = -\pi/4 \to u = 2\cdot\frac{-\pi}{4}$

$\boxed{u = \frac{-\pi}{2}}$

$\overbrace{\int \cos^2 u\,du} = \frac{1}{2}u + \frac{1}{4}\sin(2u)$]

tidy

$\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin(2u)$.]

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2 u \, du$$

$$= \frac{1}{4}\left[\frac{1}{2}u + \frac{1}{4}\sin(2u)\right]_{u=-\pi/2}^{u=\pi/2}$$

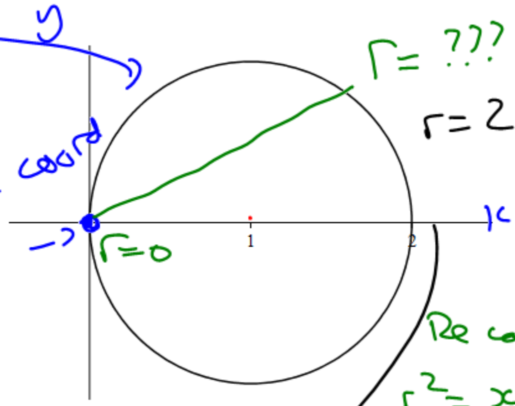
$$= \frac{1}{4}\left[\left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\underset{0}{\cancel{\sin\left(2\frac{\pi}{2}\right)}}\right) - \left(\frac{1}{2}\left(\frac{-\pi}{2}\right) + \frac{1}{4}\underset{0}{\cancel{\sin\left(2\cdot\frac{-\pi}{2}\right)}}\right)\right]$$

$$= \frac{\pi}{16} + \frac{\pi}{16} = \frac{\pi}{8} \simeq 0.4 \text{ sq units.}$$

Example 3.7 Use polar coordinates to find the volume of the solid bounded by
the *x-y* plane,
the cylinder $x^2 + y^2 = 2x$, $\rightarrow (x-1)^2+y^2=1$
the paraboloid $z = x^2 + y^2$.

[Tech: $\int \cos^4 u\,du = \frac{1}{4}(\cos^3 u)\sin u + \frac{3}{8}\cos u \sin u + \frac{3}{8}u$.]

[Answer: $3\pi/2$.]



still polar coord center →

$$\text{Vol} = \iint_R z \cdot dA$$

heights · areas

$$= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{r=2\cos\theta} \underbrace{r^2 (r}_{= r^3}\, dr\, d\theta)$$

Do r integral

Recall $r^2 = x^2+y^2$

① $x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$

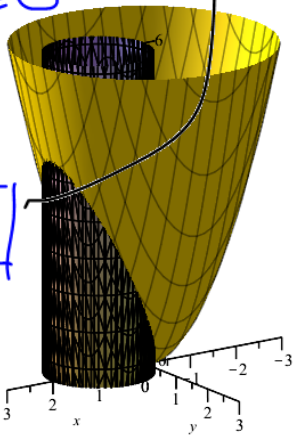
Boundary $x^2+y^2=2x$

or $\underbrace{x^2+y^2}_{r^2} - 2\underbrace{x}_{\theta} = 0$ } write w/ only r, θ's

$$r^2 - 2(r\cos\theta) = 0$$

factor $r(r - 2\cos\theta) = 0$

$r = 0$ or $\boxed{r = 2\cos\theta}$
point.



$$= \int_{\theta=-\pi/2}^{\pi/2} \frac{r^4}{4}\Big|_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}\left((2\cos\theta)^4 - 0\right) d\theta$$

$$\frac{2^4}{4} = \frac{16}{4} = 4$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta$$

ick...

$$\int \cos^4 u\,du = \frac{1}{4}(\cos^3 u)\sin u + \frac{3}{8}\cos u \sin u + \frac{3}{8}u\,.]$$

$$= \overset{\downarrow}{4}\left(\frac{1}{4}\cos^3\theta\,\sin\theta + \frac{3}{8}\cos\theta\,\sin\theta + \frac{3}{8}\theta\right)\Bigg|_{\theta=-\pi/2}^{\theta=\pi/2}$$

$$= 4\left[\left(0+0+\frac{3}{8}\left(\frac{\pi}{2}\right)\right) - \left(0+0+\frac{3}{8}\left(-\frac{\pi}{2}\right)\right)\right]$$

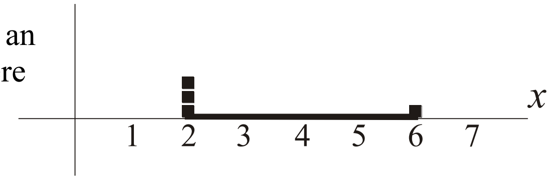
recall $\cos \pi/2 = 0$

$\cos(-\pi/2) = 0$

$$= 4\cdot\frac{3}{8}\pi = \frac{3\pi}{2}$$ cubic units

4. Centre of mass.

Imagine we have a seesaw running between x=2 and x=6, and an adult at x=2 weighing 3 times as much as a child at x=6. Where should the fulcrum be so that the adult will balance the child? The answer is that it should be 3 times as far from the child as from the adult. And that places it at $\bar{x}$ =3.



If the adult is 3 times as big as the child, the child should be 3 times as far from the "center."

Another way to say this is that the products of the mass and the distance from the center must be same for adult and child:

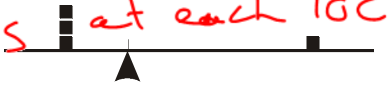
$$m_1(\bar{x}-x_1)=m_2(x_2-\bar{x})$$

where $\bar{x}$, the position of the balance point, is called the *center of mass*. Solving:

x center of mass

$$\boxed{\bar{x}} = \frac{m_1x_1+m_2x_2}{m_1+m_2}$$

$$\frac{\sum (\underline{x\text{ loc}})\cdot(\text{mass at that loc})}{\sum \text{mass at each loc}}$$



This equation displays the center of mass $\bar{x}$ as a weighted average of x_1 and x_2 where the m_i are used as weights:

$\Downarrow$

$$(x)\cdot\rho(x,y)\cdot dA$$

planar density

$$\underbrace{m\cdot\left(g/m^2\right)\cdot\left(m^2\right)}_{\text{mass}}$$

For our example: $\bar{x}=\dfrac{3\cdot 2+1\cdot 6}{3+1}=3$.

More generally, suppose the mass is continuously distributed over the x-axis with density $\rho(x)$. Then the center of mass formula becomes:

$$\bar{x}=\frac{\int_a^b \overset{\text{loc}}{x}\,\overset{\text{mass}}{\overbrace{\rho(x)dx}}}{\underset{\text{gains}\,\cdot\,\text{length}}{\int_a^b \rho(x)dx}}=\frac{1}{M}\int_a^b x\rho(x)dx$$

where the total mass M is the integral of the density.

Total mass g/m · m = g

Finally suppose the mass is distributed over a planar region R. Then the centroid will have two components $(\bar{x},\bar{y})$. The first will balance the masses in the x-direction and the second will balance the masses in the y-direction. The formulae are:

$$\bar{x}=\frac{\iint_R x\rho(x,y)dA}{\boxed{\iint_R \rho(x,y)dA}}=\frac{1}{M}\iint_R x\rho(x,y)dA$$

$$\bar{y}=\frac{\iint_R y\rho(x,y)dA}{\boxed{\iint_R \rho(x,y)dA}}=\frac{1}{M}\iint_R y\rho(x,y)dA.$$

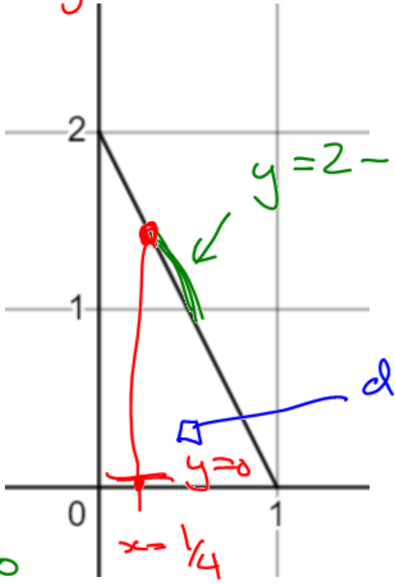
=

= total mass

g/m²

Example 4.1 Let *E* be a triangular lamina with vertices (0, 0), (1, 0) and (0, 2) and density function is $\rho(x, y) = x$. more dense to the right

(a) Find the mass of *E*. [Answer: $M = 1/3$]



Mass on a small dA area

= dens · dA

$(g/m^2)\cdot(m^2) \to g$

$= (x)\, dA$

Total mass $= \iint_R x \cdot dA$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=2-2x} x\, dy\, dx = \int_0^1 xy\Big|_{y=0}^{y=2-2x} dx$$

(b) Calculate the coordinates of the center of mass. [Answer: $(\bar{x}, \bar{y}) = (1/2, 1/2)$.]

both $\bar{x}$, $\bar{y}$

$$= \int_0^1 \Big(x\,(\overset{y=2-2x}{2-2x}) - x\cdot \overset{y=0}{0}\Big)\, dx$$

$$= \int_0^1 2x - 2x^2\, dx = \frac{2x^2}{2} - \frac{2x^3}{3}\Big|_0^1$$

$$= \left(1 - \tfrac{2}{3}\right) - (0-0) = \tfrac{1}{3} \text{ grams.}$$

$M = \frac{1}{3}$ grams

To find $\bar{x}$, x center of mass,

$$\bar{x} = \frac{1}{M}\iint_R x \cdot (\overset{\text{density or } \rho}{x}\, dA)$$

finding x center of mass

same region so same limits of integration

$$= \frac{1}{M}\int_0^1\int_{y=0}^{y=2-2x} x^2\, dy\, dx$$

↓ y integral

$$= \frac{1}{M}\int_0^1 x^2 y\Big|_{y=0}^{y=2-2x} dx$$

↓ sub in y's

$$= \frac{1}{M}\int_0^1 x^2(2-2x)\, dx$$

tidy

$$= \frac{1}{M}\int_0^1 2x^2 - 2x^3 \, dx$$

↓ x integral

$$= \frac{1}{M}\left(\frac{2x^3}{3} - 2\frac{x^4}{4} \Big|_{x=0}^{x=1} \right)$$

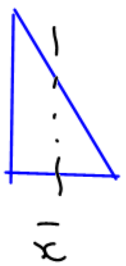
↓ sub in x's

from (a), $M = 1/3$

$$= \frac{1}{M}\left(\left(\frac{2}{3} - \frac{2}{4}\right) - 0\right) = \frac{1}{M}\left(\frac{1}{6}\right)$$

$$= \frac{1}{1/3}\cdot\frac{1}{6} = 3\cdot\frac{1}{6} = \frac{1}{2}$$

$$\bar{x} = \frac{1}{2}$$



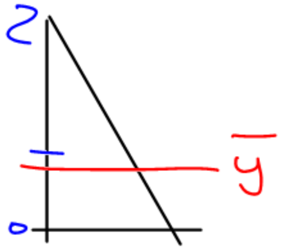
x balancing line.
@ $\bar{x} = 1/2$

Find $\bar{y}$ = y center of mass

$$\bar{y} = \frac{1}{M}\iint_R y\,(\underbrace{x\,dA}_{\text{mass of a small slab}})$$

density

y b/c we are finding y center of mass



$$= \frac{1}{M}\int_0^1\int_{y=0}^{y=2-2x} y\cdot x\cdot dy\,dx$$

same R → same limits of integration

↓ y integral

$$= \frac{1}{M}\int_0^1 \frac{y^2}{2}\cdot x \Big|_{y=0}^{y=2-2x} dx$$

↓ sub in y's

$$= \frac{1}{M}\int_0^1 \frac{1}{2}(2-2x)^2\cdot x \quad dx$$

↓ tidy

$$(2+2x)^2$$
$$= 2 + 8x + 4x^2$$

$$= \frac{1}{M}\int_0^1 \frac{1}{2}\left(4-8x+4x^2\right)x\,dx$$

↓ tidy again

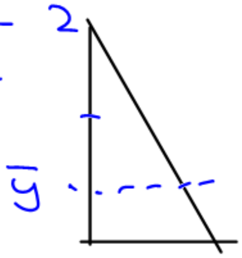
$$= \frac{1}{M}\int_0^1 2x-4x^2+2x^3\,dx$$

↓ x integral

$$= \frac{1}{M}\left(\frac{2x^2}{2}-\frac{4x^3}{3}+\frac{2x^4}{4}\right)\Bigg|_0^1$$

$$= \frac{1}{M}\left(1-\frac{4}{3}+\frac{1}{2}\right) = \frac{1}{M}\cdot\frac{1}{6} = \dots = \frac{1}{2}$$

so $\bar{y} = \frac{1}{2}$



both $\bar{x}, \bar{y}$

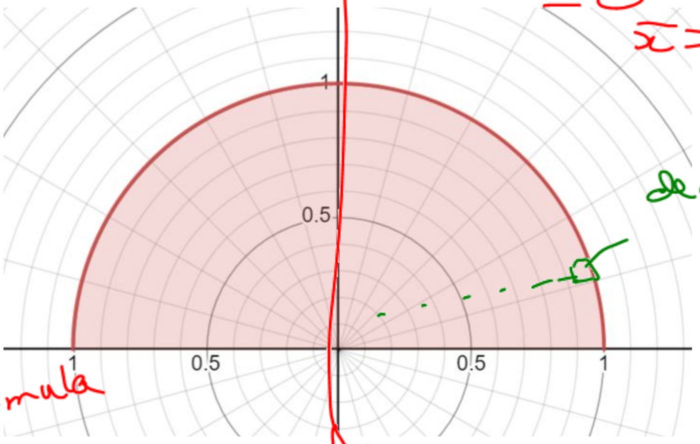
↳ $\bar{x}$ by the symmetry of dens + shape $= 0$

$\bar{x} = 0$

Example 4.2. Find the mass and center of mass of the semicircular lamina

$$\{x^2 + y^2 \le 1, y \ge 0\}$$

if the density equals the distance r from the origin.
[Answer: $M = \pi/3$, $(\bar{x}, \bar{y}) = (0, 3/2\pi)$].]



dens $= 1$

dens $= 0$

$\bar{x}$

Mass

$$= \iint_R \underbrace{(\text{dens})}_{(g/m^2)} \cdot \underbrace{dA}_{m^2} = g$$

polar dA formula

$$= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1} \underbrace{(r)}_{\text{dens}} \left(r \cdot dr \cdot d\theta \right)$$

tidy

$$= \int_0^\pi \int_{r=0}^1 r^2 \, dr \, d\theta$$

calculation

r integral

$$= \int_0^\pi \left. \frac{r^3}{3} \right|_{r=0}^{r=1} d\theta$$

sub in r's

$$= \int_0^\pi \frac{1}{3} \, d\theta$$

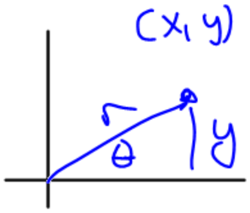
$$= \left. \frac{1}{3}\theta \right|_0^\pi = \frac{1}{3}(\pi - 0) = \pi/3$$

$$\boxed{M = \pi/3 \text{ grams}}$$

To find $\bar{y}$,

$$\bar{y} = \frac{1}{M}\iint_R \underset{\uparrow}{y}\,(\overbrace{\rho\cdot dA}^{\text{small mass units}})$$

add y multiplier b/c finding $\bar{y}$



into polar coords

$$= \frac{1}{M}\int_0^{\pi}\int_0^{1} y\cdot\rho\,(r\,dr\,d\theta)$$

& $\rho = r$ given

$$= \frac{1}{M}\int_0^{\pi}\int_0^{1}(r\cdot\sin\theta)\,r\cdot r\,dr\,d\theta$$

$y = r\sin\theta$ polar identity

tidy

$$= \frac{1}{M}\int_0^{\pi}\int_0^{1} r^3\sin\theta\,dr\,d\theta$$

↓ $\sin\theta$ const for r integral

& r integral

$$= \frac{1}{M}\int_0^{\pi}\frac{r^4}{4}\sin\theta\,\Big|_0^1\,d\theta$$

$$= \frac{1}{M}\int_0^{\pi}\frac{1}{4}\sin\theta\,d\theta$$

$$= \frac{1}{M}\,\frac{1}{4}\,(-\cos(\theta))\,\Big|_0^{\pi}$$

$M = \pi/3$ →

$$= \frac{1}{M}\,\frac{1}{4}\,(-(-1)-(-(1))) = \frac{1}{M}\cdot\frac{1}{4}\cdot 2$$

$$= \frac{1}{2}\,\frac{1}{M} = \frac{1}{2}\,\frac{1}{\pi/3}$$

$$= \frac{1}{2}\,\frac{3}{\pi} = \frac{3}{2\pi}$$

$$\doteq 0.48$$