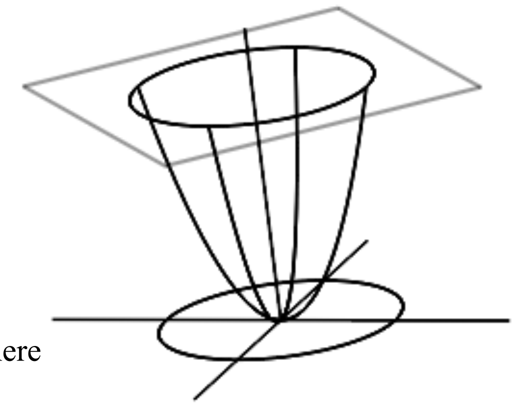
5. Triple integrals

Here we look at problems in which we have a function $f(x, y, z)$ of three variables and we want to integrate it over a solid region E in R^3. This will be written as the triple integral:

$$\iiint_E f(x,y,z)\,dV.$$

For example, if the temperature at a point in the region E is $T(x, y, z)$ then the average temperature over the region will be:



$$\overline{T} = \frac{1}{V}\iiint_E T(x,y,z)\,dV$$

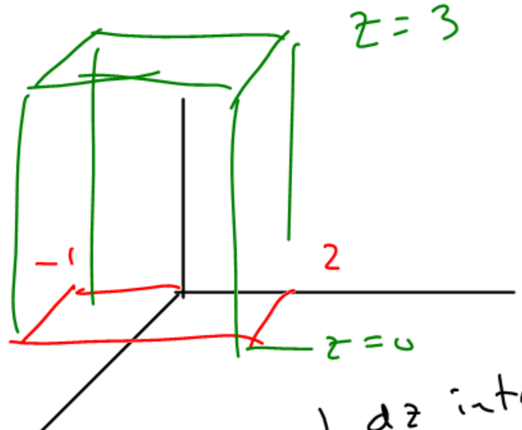
where

$$V = \iiint_E dV \text{ is the volume of } E.$$

In the examples we will work with it will always be the case that there is an axis (say z) for which (if we think of it as pointing "up") the region E will be described as all points lying below the top surface $z = H(x, y)$ and above the bottom surface $z = L(x, y)$. Then the triple integral can be written:

$$\iiint_E f(x,y,z)\,dV = \iint_R \int_{L(x,y)}^{H(x,y)} f(x,y,z)\,dz\,dA.$$

Once the dz-integral is calculated, the integral over R can be done in the usual way, either with Cartesian or polar coordinates. If we use polar coordinates in the x-y place the 3-D coordinate system is called *cylindrical coordinates.*



Example 5.1. Evaluate the triple integral $\iiint_B xyz^2\,dV$ where B is the rectangular box:

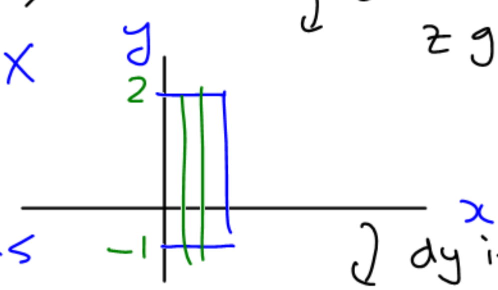
$$B = \{(x, y, z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

[Answer: 27/4.]

Fixed / copied

$$\int_{x=0}^{x=1}\int_{y=-1}^{2}\int_{z=0}^{z=3} xyz^2\,dz\,dy\,dx$$

↓ dz integ'l / z gone

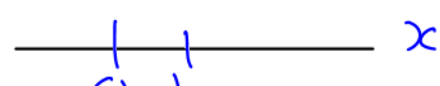


↑ diagrams

↓ dy int'l y gone

↓ z int'l

↓ anti deri



$$= \int_0^1 \int_{-1}^2 xy \frac{z^3}{3}\Big|_0^3 \, dy \, dx$$

z sub in

y intg'e

$$= \int_0^1 \int_{-1}^2 xy \, \cancel{\frac{27}{3}}\, 9 \, dy \, dx = \int_0^1 9x \frac{y^2}{2}\Big|_{y=-1}^{y=2} dx$$

y sub in

tidy

$$= \frac{9}{2} \int_0^1 x \left[\underbrace{2^2 - (-1)^2}_{=3}\right] dx = \frac{27}{2} \int_0^1 x \, dx$$

$$= \frac{27}{2} \frac{x^2}{2}\Big|_0^1$$

$$= \frac{27}{4}(1) = 6.75$$

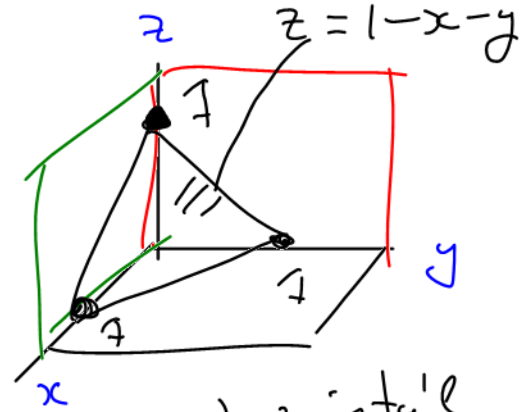
4 sided die / triangle base pyramid

Example 5.2. (a) Evaluate the triple integral $\iiint_E z\,dV$ where E is the tetrahedron bounded by the four planes:

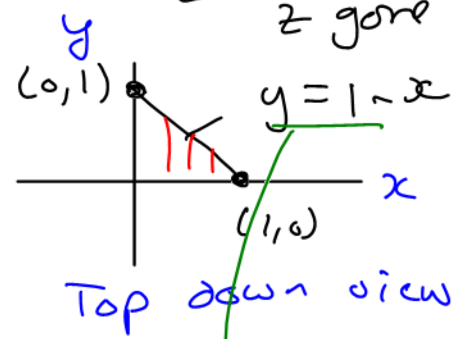
$$x = 0,\ y = 0,\ z = 0 \ \text{ and } \ x+y+z = 1 .$$

[Answer: 1/24.]

$$z = 1 - x - y$$



z intg'e, z gone



$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} z \, dz \, dy \, dx$$

for any x, y

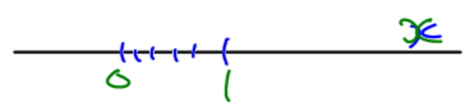
z intg'e

$$= \int_0^1\int_0^{1-x} \frac{z^2}{2}\Big|_{z=0}^{z=1-x-y} dy\,dx$$

z sub in

intersection $z = 1-x-y$ and $z = 0$

$$= \int_0^1\int_0^{1-x} \frac{1}{2}(1-x-y)^2 \, dy\,dx$$



(b) Does your answer allow you to easily find the centroid of E?

y intg'e

$$= \int_0^1 \frac{1}{2}\,\frac{(1-x-y)^3}{3}(-1)\Big|_{y=0}^{y=1-x} dx$$

tidy

$$= -\frac{1}{6}\int_0^1 (1-x-y)^3\Big|_{y=0}^{y=1-x} dx$$

sub in y's

$$= -\frac{1}{6}\int_0^1 \underbrace{\left(1-x-\cancel{(1-x)}\right)^3}_{0} - (1-x-0)^3 \, dx$$

tidy

$$= \frac{1}{6}\int_0^1 (1-x)^3 \, dx$$

$$= \frac{1}{6} \left. \frac{(1-x)^4}{4}(-1) \right|_{x=0}^{x=1} = -\frac{1}{24} (1-x)^4 \Big|_0^1$$

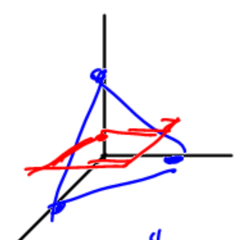
$$= -\frac{1}{24}\left[(1-1)^4 - (1-0)^4\right]$$

$$= \frac{1}{24}.$$

(b) Does your answer allow you to easily find the centroid of E?

= center of mass if density is constant/uniform.

$$\bar{z} = \frac{\iiint_E z \, dV}{\iiint_E dV} \quad \begin{matrix} \leftarrow \text{part (a)} = 1/24 \\ = \text{total vol} = \frac{1}{6} \end{matrix}$$



"cone"

$\bar{z} = 1/4$

from symmetry $\bar{x} = \bar{y} = 1/4$ too

$V = \frac{1}{3}(\text{base area})(h)$

$= \frac{1}{3}\left(\frac{1}{2}\right)\cdot 1 = \frac{1}{6}$ cubic units

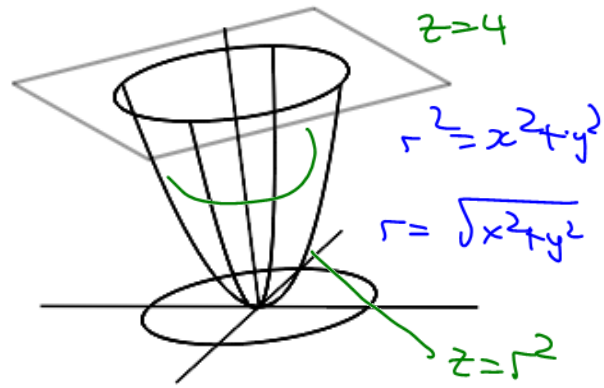
$(1/4, 1/4, 1/4)$

is the tetrahedron centroid or center of mass

Example 5.3. Evaluate the triple integral $\iiint_E \sqrt{x^2+y^2}\,dV$ where E is the region bounded by the plane $z = 4$ and the paraboloid $z = x^2 + y^2$.
[Answer: $128\pi/15$.]

screams polar coords!

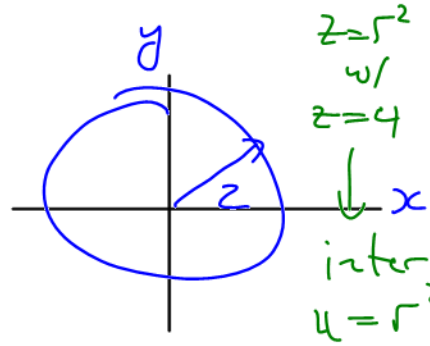
$z = r^2$



$$I = \iint \int r \, dz \, dA$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{r=2} \int_{z=r^2}^{z=4} r \, dz \, (r \cdot dr \cdot d\theta)$$

r const for dz int



$z=r^2$ w/ $z=4$ ↓ inter $4=r^2$, $2=r$

$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 1 \, dz \, (r^2) \, dr \, d\theta$$

z intg'l

$$= \int_0^{2\pi} \int_0^2 z \Big|_{z=r^2}^{z=4} (r^2) \, dr \, d\theta$$

sub in z's

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) \cdot r^2 \, dr \, d\theta$$

& tidy

$$= \int_0^{2\pi} \int_0^2 4r^2 - r^4 \, dr \, d\theta$$

r intg'l

$$= \int_0^{2\pi} \frac{4r^3}{3} - \frac{r^5}{5} \Big|_0^2 \, d\theta$$

r sub in

corrections!

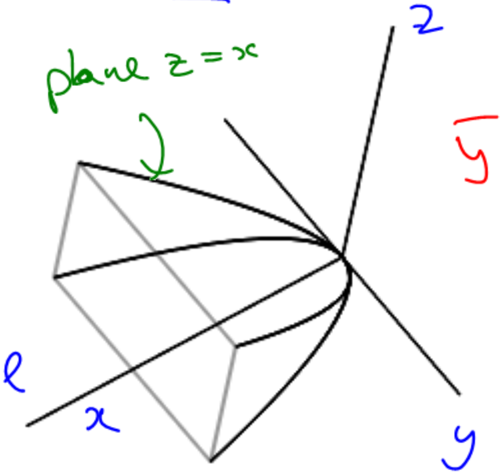
$$= \int_0^{2\pi} \frac{4}{3} \cdot 8 - \frac{32}{5} \, d\theta$$

θ intg'l

$$= \int_0^{2\pi} \frac{64}{15} \cancel{\frac{48}{5}} \, d\theta \rightarrow = \frac{64}{15} \cancel{\frac{48}{5}} \cdot \theta \Big|_0^{2\pi} = \frac{64}{15}(2\pi) = \frac{128}{15}\pi$$

centroid = center of mass if density is constant / uniform

Example 5.4 Find the centroid of the region bounded by the parabolic cylinder $x = y^2$, $\rightarrow y = \pm\sqrt{x}$
and the planes $z = x$, $z = 0$, and $x = 1$.
[Answer: $M = 4/5$, $(\bar{x}, \bar{y}, \bar{z}) = (5/7, 0\ 5/14)$.]



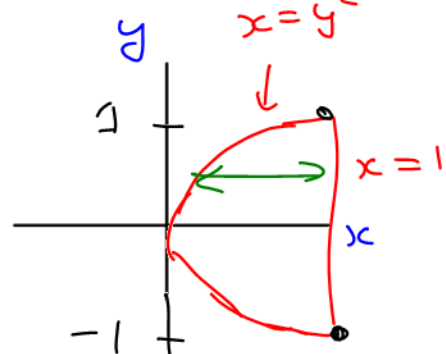
$\bar{y} = 0$

$$\text{Vol} = \int_{y=-1}^{y=1} \int_{x=y^2}^{x=1} \int_{z=0}^{z=x} dz\, dx\, dy$$

↓ do z intg'l

$$= \int_{y=-1}^{y=1} \int_{x=y^2}^{1} z \Big|_{z=0}^{z=x} dx\, dy$$

↓ z gone



↓ sub in z's

$$= \int_{y=-1}^{y=1} \int_{x=y^2}^{x=1} x \, dx\, dy$$

↓ do x intg'l

$$= \int_{-1}^{1} \frac{x^2}{2} \Big|_{x=y^2}^{y=1} dy$$

↓ sub in y's

$$= \int_{-1}^{1} \frac{1}{2} - \frac{(y^2)^2}{2} \, dy = \int_{-1}^{1} \frac{1}{2}(1 - y^4)\, dy$$

$$= \frac{1}{2}\left(y - \frac{y^5}{5}\right)\Big|_{-1}^{1} = \frac{1}{2}\left[\left(1 - \frac{1}{5}\right) - \left(-1 - \frac{(-1)^5}{5}\right)\right]$$

$$= \frac{1}{2}\left(\frac{4}{5} + \frac{4}{5}\right) = \frac{4}{5}$$

$$\boxed{\text{Vol} = 4/5 \ m^3}$$

To find $\bar{x} = \dfrac{\iiint_E x\cdot\cancel{\rho}\cdot dV}{\iiint_E \cancel{\rho}\, dV}$

w/ constant density ρ

$$= \frac{\iiint_E x\, dV}{\iiint_E dV}$$

let $I_2 = \iiint_E x\, dV$ — only need this

$\iiint_E dV$ = total volume $= \frac{4}{5}$

$$I_2 = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} (x)\, dz\, dx\, dy = \int_{-1}^{1}\int_{y^2}^{1} x\cdot z\Big|_{z=0}^{z=x} dx\, dy$$

$$= \int_{-1}^{1}\int_{y^2}^{1} x^2\, dx\, dy = \int_{-1}^{1} \frac{x^3}{3}\Big|_{x=y^2}^{x=1} dy$$

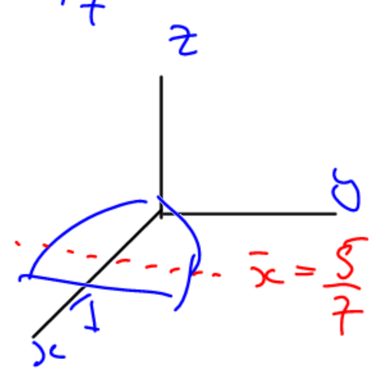
$$= \int_{-1}^{1} \frac{1}{3}\left(1^3 - (y^2)^3\right) dy \xrightarrow{\text{tidy}} = \int_{-1}^{1} \frac{1}{3}\left(1 - y^6\right) dy$$

$$= \frac{1}{3}\left(y - \frac{y^7}{7}\right)\Big|_{y=-1}^{y=1} = \frac{1}{3}\left[\left(1 - \frac{1}{7}\right) - \left(-1 - \frac{(-1)^7}{7}\right)\right]$$

$= 6/7 - (-6/7)$

$$= \frac{1}{3}\left(\frac{6}{7} + \frac{6}{7}\right) = 4/7$$

$$\bar{x} = \frac{I_2}{\text{Total } V} = \frac{4/7}{4/5} = \frac{5}{7}$$



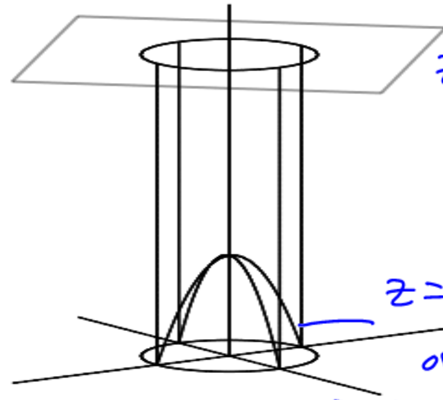
$\bar{x} = \frac{5}{7}$, $\bar{y} = 0$ from symmetry of the region

$$\bar{z} = \frac{\iiint z \, dV}{\text{Total vol}}$$

Example 5.5 A solid E lies inside the cylinder $x^2 + y^2 = 1$, and between the plane $z = 4$, and the paraboloid

$$z = 1 - x^2 - y^2.$$

The density at any point is equal to its distance from the axis of the cylinder. Find the mass of E.
[Answer: $12\pi/5$.]



Polar coords:
Recall: $r^2 = x^2 + y^2$

$r = \sqrt{x^2+y^2}$

$dA = r \cdot dr \cdot d\theta$

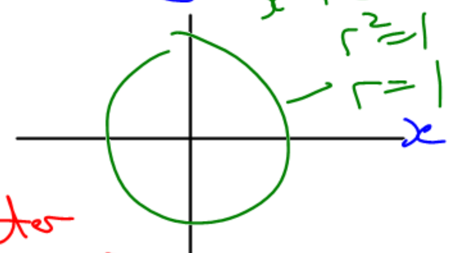
$$M = \iiint_E \rho \cdot dV$$

$\frac{kg}{m^3} \cdot m^3 = kg$ ✓

Do z's range

$$= \iint_{\text{circle w/ } r=1} \int_{z=1-r^2}^{z=4} \rho \, dz \, dA$$

$\rho = r =$ dist from center z axis



do x,y / r,θ ranges

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=1-r^2}^{4} r \, dz \, (r \cdot dr \cdot d\theta)$$

tidy

$$= \int_0^{2\pi} \int_0^1 \int_{z=1-r^2}^{4} r^2 \, dz \, dr \, d\theta$$

do z integ'l, r const

$$= \int_0^{2\pi} \int_0^1 r^2 z \Big|_{z=1-r^2}^{z=4} \, dr \, d\theta$$

sub in z's

$$= \int_0^{2\pi} \int_0^1 r^2 \underbrace{(4-(1-r^2))}_{3+r^2} \, dr \, d\theta$$

tidy

$$= \int_0^{2\pi} \int_0^1 3r^2 + r^4 \, dr \, d\theta$$

do r integ'l

$$= \int_0^{2\pi} \frac{3r^3}{3} + \frac{r^5}{5} \Big|_{r=0}^{r=1} \, d\theta$$

$$= \int_0^{2\pi} \left(1 + \frac{1}{5}\right) d\theta = \int_0^{2\pi} \frac{\cancel{4}6}{5} \, d\theta = \frac{\cancel{4}6}{5}\theta \Big|_0^{2\pi}$$

$$= \frac{12\pi}{5}$$

$$M = \frac{12\pi}{5} \text{ kg}$$

Example 5.6 Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\right)dy\,dx$.

[Answer: $16\pi/5$.]

Polar coords

$r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2}$

$dA = r\cdot dr\cdot d\theta$

$x = r\cos\theta$

$y = r\sin\theta$

↓ Translate into set of boundaries

$$= \int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=\sqrt{x^2+y^2}}^{z=2}(x^2+y^2)\,dz\,dy\,dx$$

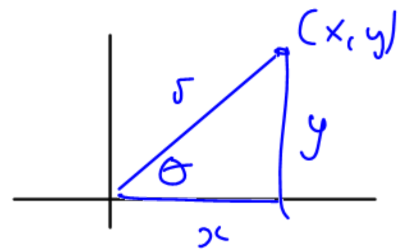
$z=\sqrt{x^2+y^2}$ ↓ $z = r$

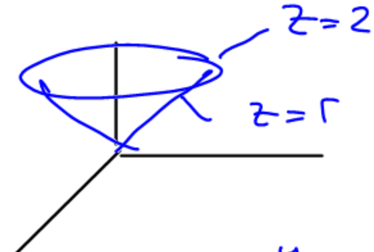
$y = -\sqrt{4-x^2}$ ↓

$y^2 = 4-x^2$

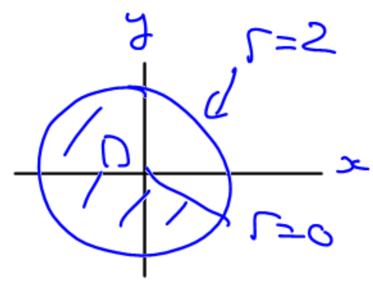
$x^2+y^2 = 4$

circle / $r=2$





$$= \iint_{D}\int_{z=r}^{z=2} r^2\,dz\,dA$$



$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\int_{z=r}^{z=2} r^2\,dz\,(r\cdot dr\cdot d\theta)$$

$$= \int_0^{2\pi}\int_0^{2}\int_r^{2} r^3\,dz\,dr\,d\theta$$

↓ do z integ'l
r^3 const

$$= \int_0^{2\pi}\int_0^{2} r^3\cdot z\Big|_{z=r}^{z=2}\,dr\,d\theta$$

↓ sub in z's

$$= \int_0^{2\pi} \int_0^2 r^3 (2 - r) \, dr \, d\theta$$ ↲ tidy / expand

$$= \int_0^{2\pi} \int_0^2 2r^3 - r^4 \, dr \, d\theta$$ ↲ do r integ'l

$$= \int_0^{2\pi} \left. \frac{2r^4}{4} - \frac{r^5}{5} \right|_{r=0}^{r=2} d\theta$$ ↲ sub in r's

$$= \int_0^{2\pi} \frac{1}{2} 2^4 - \frac{2^5}{5} \, d\theta$$ ↲ tidy

$$= \int_0^{2\pi} 8 - \frac{32}{5} \, d\theta = \left. \frac{8}{5} \theta \right|_{\theta=0}^{\theta=2\pi} = \frac{16\pi}{5}$$

(8/5)

do θ integral

$$\approx 10.05$$

Example 5.7 For the integral $\boxed{\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} dzdydx}$ change the order of integration to *dydxdz*.

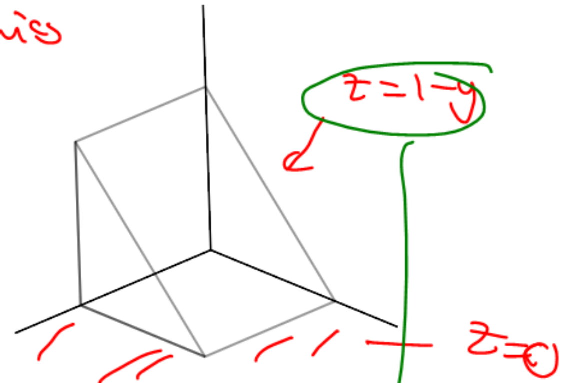
explicitly add variables to limits/boundaries

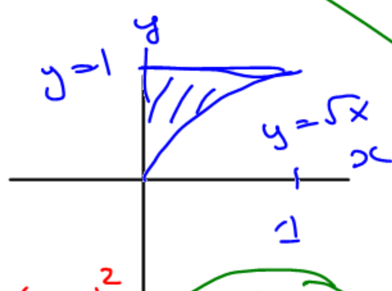
$$= \int_{x=0}^{x=1} \int_{y=\sqrt{x}}^{y=1} \int_{z=0}^{z=1-y} dz\, dy\, dx$$

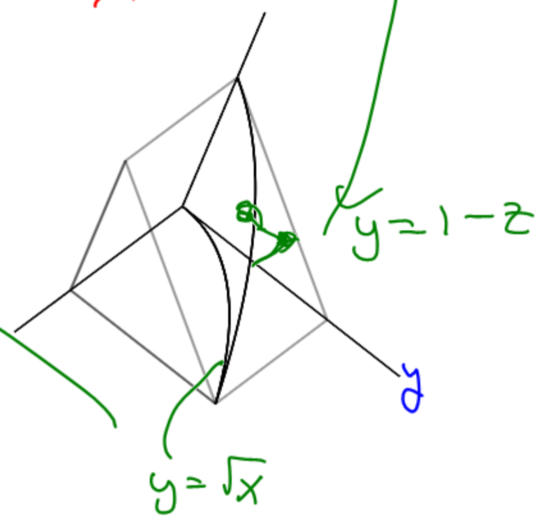
plane

$x=\text{const.}$ $y=g(x)$ $z=F(x,y)$

diagram →

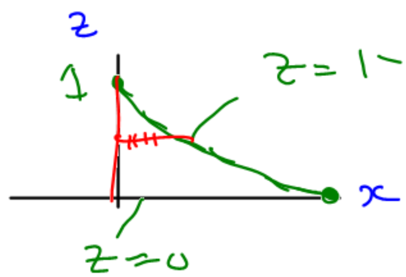






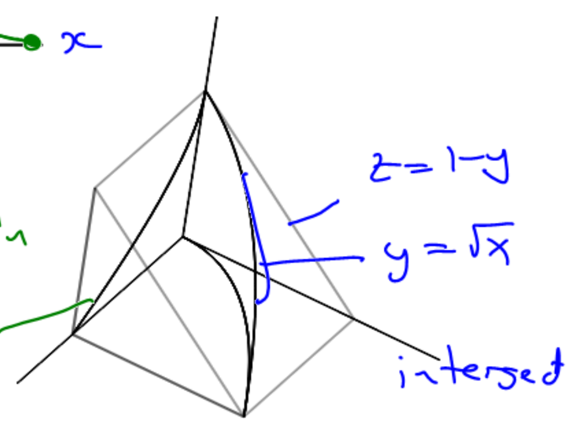
$$= \int_{z=0}^{z=1} \int_{x=0}^{x=(1-z)^2} \int_{y=\sqrt{x}}^{y=1-z} dy\, dx\, dz$$

$z=\text{const}$ $x=g(z)$ $y=f(x,z)$



$z = 1-\sqrt{x} \iff \sqrt{x} = 1-z$

$x = (1-z)^2$



xz proj of inters'n of $z=1-y$, $y=\sqrt{x}$

set y's equal → $z = 1-\sqrt{x}$

1) write all limits as formulas

$$\int_0^y dx \qquad x = y \;\Rightarrow\; x = 0$$

2) Sketch 3D region using these boundaries

3) Build new limits based on the 'd' order in the problem

$$\int_{y=\text{const}} \int_{x=g(y)} \int_{z=f(x,y)} dz\, dx\, dy$$

Week 11: Done!

No more new material in APSC 172.

Week 12:

Mon	Tue	Wed	Thu	Fri	Sat	Sun	Mon2
Apr 1	Apr 2	Apr 3	Apr 4	Apr 5	Apr 6	Apr 7	Apr 8
Series Review/ Problems	Differential Review/ Problems		Integral Review/ Problems				No Class. Eclipse Day!

April 2024 (Canada)

Sun	Mon	Tue	Wed	Thu	Fri	Sat
31 Easter Sunday	1 Easter Monday (NT, NU, QC) Week 12	2	3	4	5	6 Tartan Day
7	8 Make-Up Day. Solar Eclipse!	9 Vimy Ridge Day	10	11 APSC 112/114 Exam - 2 PM	12	13 APSC 172 Exam - 2 PM
14	15 APSC 132 Exam - 2 PM	16	17 APSC 174 Exam - 2 PM	18	19 APSC 142 Exam - 2 PM	20 APSC 182 Exam - 2 PM
21	22	23	24 APSC 103 Exam - 2 PM	25 Last Day of Exams	26	27
28	29	30	1	2	3	4