## Week #1, Lecture 2: Fundamentals

- Notation, Sets and More

Don't forget to log in to `https://qlicker.queensu.ca` at the start of the lecture.
Qlicker enrolment code is **QM3CXK**

A pre-class question is up now.

We will start the class @ 4:30

 online textbook numbering

## Section 0 - Sets, Quantifiers, and Mappings

A **set** is a collection of objects. Examples might include:

finite

capital cities = { Ottawa, Paris, London ... }

prime numbers = { 1, 2, 3, 5, 7, 11, 13, ... }

infinite

integers b/w 1 and 10 = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }.

finite

negative whole numbers = { -1, -2, -3 ... }

infinite

Classify each of these sets as finite or infinite.

## Common Mathematical Sets

Some of our favourite sets are:

- $\mathbb{R}$: Real numbers e.g. 3, 4.725, $\pi$, $e$ . .
- $\mathbb{C}$: Complex numbers e.g. $3+0i$, $4.725+0i$, $2.5+6i$
- $\mathbb{N}$: "natural numbers" $= \{0, 1, 2, 3, 4 \ldots\}$
  whole / non-negative
  0 is in $\mathbb{N}$.
- $\mathbb{Z}$: integers $= \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$
- $\mathbb{R}^2$: "are two" – set of tuples, with element being a real #.
  eg. $(2, 5)$, $(\pi, e)$, . . .

- $C^\infty$: set of all infinitely diff'ble functions (on ~~some~~ domain)

  eg. $x$, $x^2$, $x^{10}$, $0$, $1$, $e^x$, and continuous on all of $\mathbb{R}$

  $\sin(x)$, $\cos(x)$, ~~tan~~$(x)$

  any poly'l.

- $P_n$: $n \geq 0$, $n \in \mathbb{N}$

  poly of degree $n$ or lower

  eg. $P_3$ contains eg. $x^3$, $3x^3-2x-1$, $x^2$, $0$, $1$, $10$, $.7$,

  $P_0$ e.g. $0$, $1$, $2.7$, $\pi$, ...

- $\emptyset$: empty set, $\{\}$ null set

  different from $\{0\}$.

Thinking of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^2$, $C^\infty$, $P_n$, and $\emptyset$.

Check-In: how many of these sets were known to you before today?

A. All of them

B. All of number-based ones

C. Just the reals and complex.

D. Never really catalogued these before...

Any other sets you can think of? You can type them into the Zoom chat.

## Set Notation

Examples: duplicates are ignored or don't set.

- Simple Lists

$$\underbrace{\{1,2,5,7\}}_{\text{curly brackets}} = \{1,1,2,2,2,5,7\} = \{7,5,2,1\}$$

- Set builder notation:
  { variable : test(s) element has to be pass to be in the set. }

"such that"

$$\{x : x \in \mathbb{N},\ x \leq 10\} = \{0,1,2,\dots 9,10\}$$

or "and"

$$\{x \mid$$

- (Informal) The implied . . ....

$$\{4,5,6,7,8,\underline{\dots}\} \qquad \{1,4,9,16,\dots\}$$

leave to context

Comments: order listing in a set does **not** matter.

Challenges: define the following sets using set notation.

(a) All the integers between -2 and 5 inclusive.

$\{-2,-1,0,1,2,3,4,5\}$ or $\{x : x \in \mathbb{Z}, -2 \leq x \leq 5\}$

(b) All the real numbers between -2 and 5 inclusive. cannot list these

$\{x : x \in \mathbb{R}, -2 \leq x \leq 5\}$ or $x \in [-2, 5]$ only for real #s

interval notation

$[\,,\,]$ = end point included

(c) The set of all squared natural numbers.

$x = 0, 1, 2$

$\{s : s = x^2, x \in \mathbb{N}\} = \{0, 1, 4, 9 ..\}$

$s = 0, s = 1^2, s = 2^2$

$(\,,\,)$ = end point not included

(d) The set of all $\underbrace{\text{infinitely differentiable functions}}_{C^\infty}$ which satisfy $\frac{d}{dx}f(x) = 1$.

$\{f : f \in C^\infty, d/dx\, f(x) = 1\}$, $\{f : f = x + a, a \in \mathbb{R}\}$

## Cartesian Product Sets

The $\times$ symbol between two sets represents the **Cartesian product** of the sets.
The Cartesian product set (or sometimes just "product set") is a new set whose elements are all the ordered pairs $(a, b)$ with $a \in A$ and $b \in B$.

Example: If $A = \{0, 1, 2\}$ and $B = \{5, 7\}$, define the following Cartesian products:

(a) $A \times B = \{ (0,5), (0,5), (1,5), (1,7), (2,5), (2,7) \}$

$|A| = 2$

$|B| = 3$

# elem

(b) $A \times A = \{ (0,0), (0,1), (0,2)$

$(1,0)\ (1,1)\ (1,2)$

$(2,0)\ (2,1)\ (2,2 \}$

$= A^2$

$|A| = 3$

# el

(c) $B \times B \times B = \{(5,5,5), (5,5,7) \ldots (7,7,7)\}$

$\overset{\shortparallel}{B^3}$

How many elements will each product set have? – Qlicker

$C = \{0, 1, 2\}$

$D = \{1, 2, 3, 4, 5\}$

# el in $C \times D$

$= |C| \cdot |D|$

$|C \times D| = 3 \cdot 5 = 15.$

Describe how order is and is not important in the product sets.

$(5,5,7) \neq (7,5,5)$

placement/order in tuples <u>does</u> matter

Some of our favourite product sets:

- $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ xy plane

- $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$ xyz 3D space

- $\underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ times}} = \mathbb{R}^n$ "are en"

  n-dimensional space

- $\underbrace{\mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C}}_{n \text{ times}} = \mathbb{C}^n$ n-dimensional complex space

  $\{ (\underbrace{1+0i, 2-3i, \ldots}_{\text{n-element tuple.}}) $

Example: Create a product set that could represent all possible 8-digit binary numbers.

$$S = \{(0,1,1,0,1,1,0,1), \ldots\}$$

each element in $B = \{0,1\}$

$$= B^8$$

# Membership, Set Equality, and Subsets

Symbol denoting that a single object belongs to/is an element of a set:

$\in$ "is an element of"

Symbol denoting that a single object **is not** an element of a set:

$\notin$ "is not an element of"

In the following examples, determine which symbol is appropriate between the object and the set:

| | | |
|---|---|---|
| $15.7 \in \mathbb{R}$ | $-3+2i \notin \mathbb{R}$ | $(x^2+1) \in C^\infty$ = continuous, inf'ly diff'ble function |
| $15.7 \notin \mathbb{N}$ | $-3+2i \in \mathbb{C}$ | $(x^2+1) \in P_2$ – Qlicker |
| $15.7 \notin \mathbb{Z}$ | $-3+2i \notin \mathbb{Z}$ | $(x^2+1) \notin P_1$ – Qlicker |

↳ poly'ls of degree 1 or lower i.e. linear.

$(x^2+1)' = x^2+1$

**Equality between sets. Definition:** Let $A$ and $B$ be two sets. $A$ is said to be equal to $B$ if they have exactly the same elements. In other words, for two sets, $A = B$ only if the following two conditions are met:

1. Every element in $A$ is also an element of $B$, and
2. Every element in $B$ is also an element of $A$.

ev $x$

Example: Determine if the two sets $A = \{5, 2, 3, 4, 6\}$ and
$B = \{x \in \mathbb{Z} : x > 1 \text{ and } x \leq 6\}$ are equal.

or $\{x : x \in \mathbb{Z}, x > 1, x \leq 6\}$

1. Is every element of $A$ also in $B$?

$5 \in A \to$ is 5 in $B$? Is $5 \in \mathbb{Z}$, $5 > 1$, $5 \leq 6$ so $5 \in B$

$2 \in A \to$ is 2 in $B$? $2 \in \mathbb{Z}$, $2 > 1$, $2 \leq 6$ yes so $2 \in B$

3
4 ✓
6

So all elements of $A$ <u>are</u> in $B$.

Example: Determine if the two sets $A = \{5, 2, 3, 4, 6\}$ and $B = \{x \in \mathbb{Z} : x > 1 \text{ and } x \leq 6\}$ are equal.

$B = \{\cancel{\ldots}, \cancel{-2}, \cancel{-1}, \cancel{0}, \cancel{1}, 2, 3, 4, 5, 6, \cancel{7}, \cancel{8}, \cancel{\ldots}\}$

$= \{2, 3, 4, 5, 6\}$

all elements are also in A.

so 1. all el'ts of A are also in B

and 2. all el'ts of B are also in A

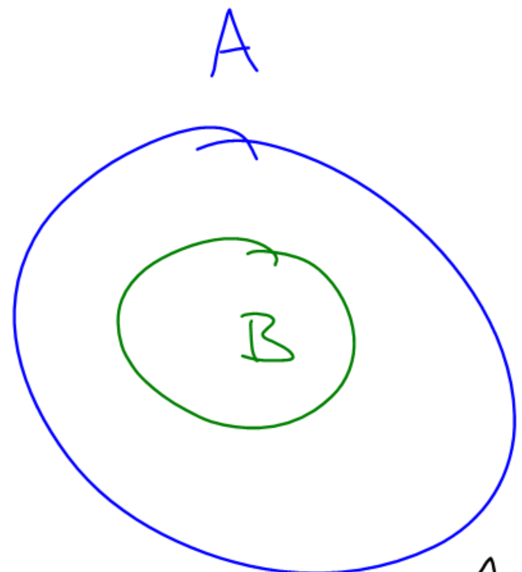
so $A = B$.

Example: Determine if two sets $A = \{1, 4, 9, \ldots\}$ and $B = \{s : s = x^2, x \in \mathbb{N}\}$ are equal.

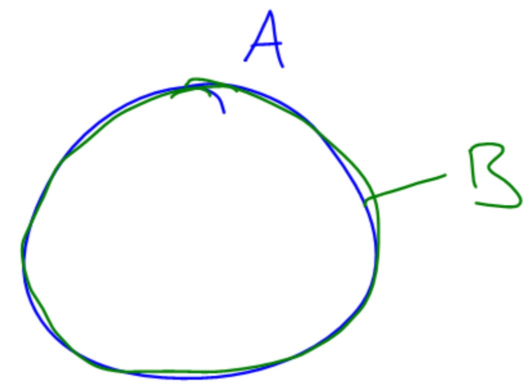
**Subsets. Definition:** Let $A$ and $B$ be two sets. $A$ is said to be a **subset** of $B$ if every element of $A$ is also an element of $B$; we write this as $A \subset B$.

subset

Example: Determine the subset relationships between $A = \{1, 4, 9, \ldots\}$ and $B = \{x^2 : x \in \mathbb{N}\}$.



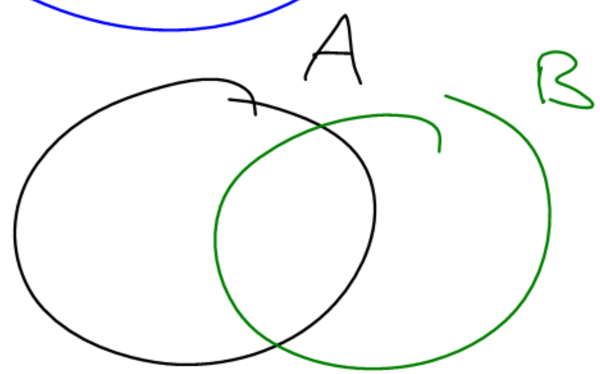

$B \subset A$




$B \subset A$

and

$A \subset B$




$B \not\subset A$

and

$A \subset B$

Example: Determine the subset relationships between the following sets:

integers $= \{ \ldots -3, -2, -1, 0, 1, 2 \ldots \}$

$\mathbb{Z} \subset \mathbb{R}$ $\qquad$ $\mathbb{N} \subset \mathbb{Z}$ ✓

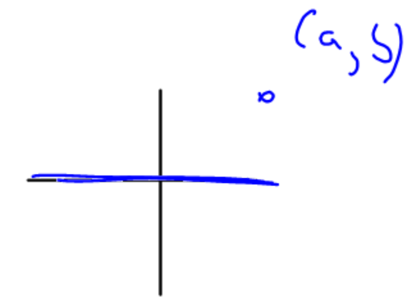
$\mathbb{Z} \not\subset \mathbb{N}$

natural numbers $= \{0, 1, 2, 3, 4 \ldots\}$

$\mathbb{Z} \subset \mathbb{Z}$ $\qquad$ $\subseteq$ $\quad$ $=$

$\mathbb{R} \not\subset \mathbb{C}$

$\mathbb{R} \not\subset \mathbb{R}^2 \to (a, b)$




$\mathbb{C} \not\subset \mathbb{R}^2$ $\qquad$ $\mathbb{C}$ and $(a, b)$ incompatible

Comment: We can state that two sets $A$ and $B$ are **equal** *if and only if* both $A \subset B$ and $B \subset A$.

$$A = B \iff A \subset B \text{ and } B \subset A$$

Example: Use subsets to show that $A = \{x \in \mathbb{Z} : x \geq 0\}$ and $B = \mathbb{N}$ are equal.

Those writing the deferred 171 exam:

Good Luck!

## Week #1, Lecture 3 : Fundamentals

- Set operators
- Functions
- Linear Systems of Equations

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Opening Qlicker question is up now.

## Section 0 - Set Operators, Functions and Mappings

Last lecture we saw sets, and we saw ways to compare them through:

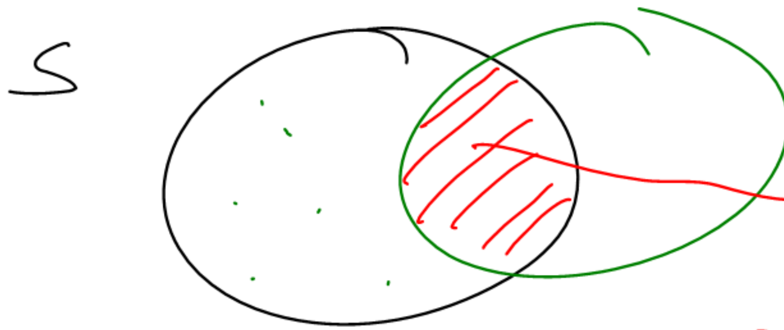
- equality: $A = B$, and
- subsets: $A \subset B$.

We will also need ways to **combine** existing sets.

∩ Sound "∩ shape"



**Set Intersection. Definition:** Let $S$ and $T$ be sets. We denote $S \cap T$ as the set of all elements that are in **both** $S$ **and** in $T$. We call $S \cap T$ the **intersection** of the sets $S$ and $T$.

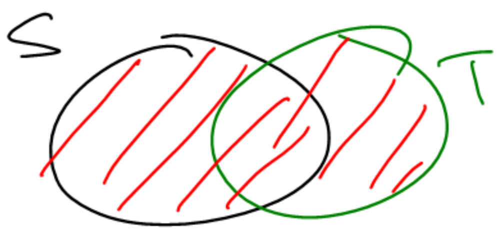



new set $S \cap T$ = all elements

$= \{x : x \in S \text{ and } x \in T\}$

**Set Union. Definition:** Let $S$ and $T$ be sets. We denote $S \cup T$ as the set of all elements that are in **either** $S$ **or** in $T$. We call $S \cup T$ the **union** of the sets $S$ and $T$.

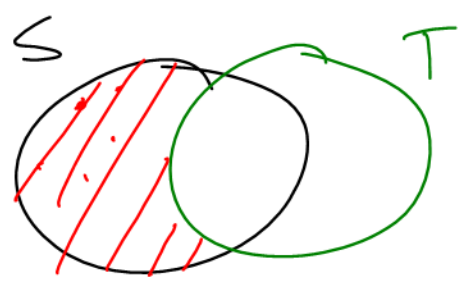
"∪ shape"



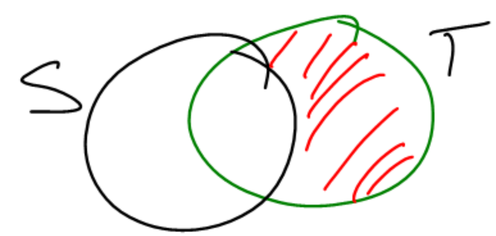

$S \cup T = \{x : x \in S \text{ or } x \in T\}$

**Set Difference. Definition:** Let $S$ and $T$ be sets. We denote $S \setminus T$ as the set of all elements that **are** in $S$ **but not** in $T$. We call $S \setminus T$ the **set difference** of the sets $S$ and $T$.

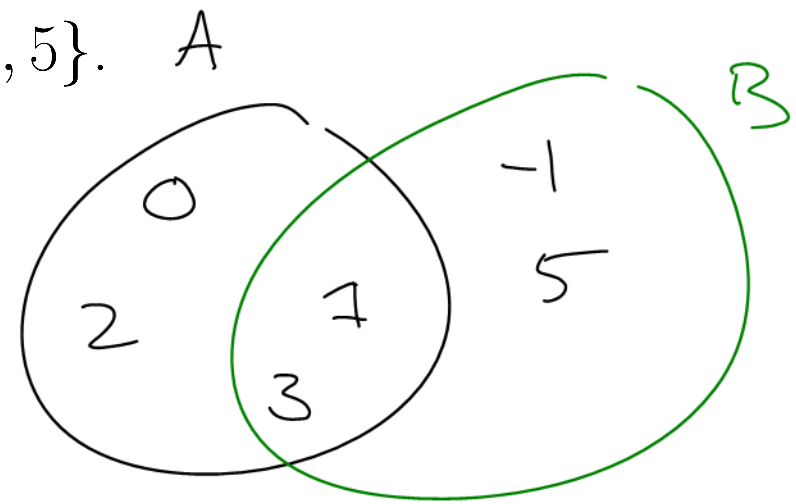
Set "$S - T$"




$T \setminus S$



Example: Let $A = \{0, 1, 2, 3\}$, and $B = \{-1, 1, 3, 5\}$.




Compute:

(a) $A \cup B$ $= \{0, 1, 2, 3, -1, 5\}$

(b) $A \cap B$ $= \{1, 3\}$

(c) $A \setminus B$ Qlicker

$= \{0, 2\}$

$\{\ldots -27, -8, -1, \cancel{0}, \cancel{1}, \cancel{8}, \cancel{27}, \ldots\}$

Example: Let $A = \{c : c = x^3, x \in \mathbb{Z}\}$, and $B = \mathbb{N}$.

natural $= \{0, 1, 2, \ldots\}$

$\{\ldots -3, -2, -1, 0, 1, \ldots\}$

Compute:

(a) $A \cup B$

$= \{\ldots -27, -8, -1, 0, 1, 2, 3, 4 \ldots\}$

(b) $A \cap B = \{0, 1, 8, 27, \ldots\}$

(c) $A \setminus B = \{\ldots -27, -8, -1.\}$

$A \cap B = B \cap A$

$A \cup B = B \cup A$

Yes!

Question: Does the order of the sets listed matter for intersection, union and set difference?

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$A \cap B$ or $B \cap A$ $\qquad$ $A \setminus B$ or $B \setminus A$

commutativity. -> change order w/o changing value.

## Quantifiers

- $\forall$ - "For all ...", or "For any..."

  Example: Rewrite the formula $A \subset B$ using $\forall$:

  any element $x$ in $A$ is also in $B$.

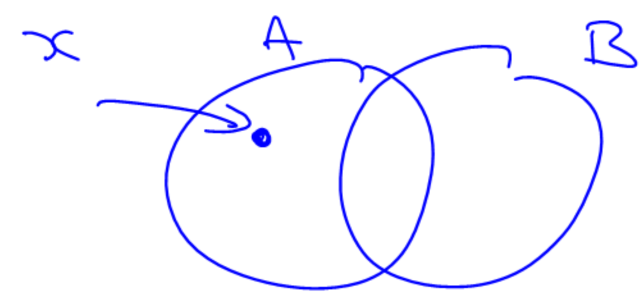
If $A \subset B$, then $\forall\ x \in A \Rightarrow x \in B$

implies that

$A \setminus B \neq \emptyset$

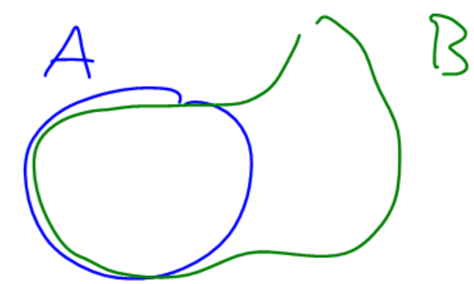
- $\exists$ - "There exists ..."

  Example: Rewrite the statement $A \setminus B \neq \emptyset$ using $\exists$:




$A \setminus B \neq \emptyset$

then $\exists\ x \in A$, but $x \notin B$



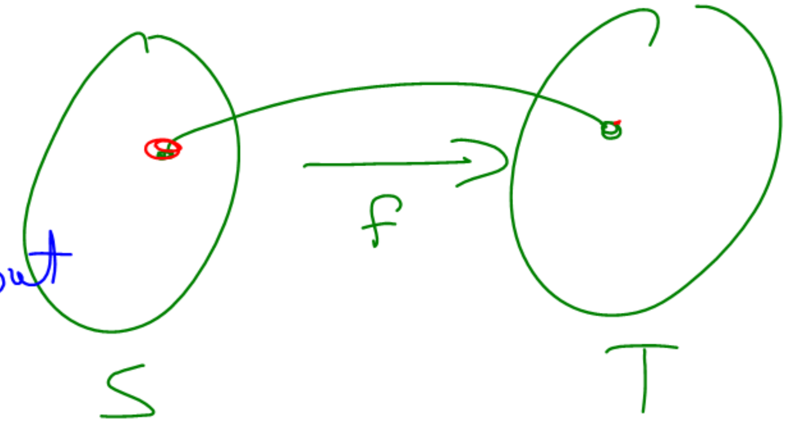

$A \setminus B = \emptyset$

**Mappings and Functions. Definition:** Let $S$ and $T$ be two sets. A **mapping** or **function** from $S$ to $T$ is a rule which assigns to each element of $S$ **one and only one** element of $T$.

new

like calculus.

every element in input set has a defined output



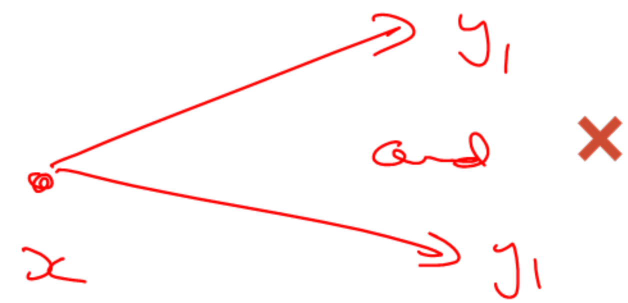

Notation:

$$f : S \to T$$

$$x \to f(x)$$

$S$ is called the "input set" or "domain set"

$T$ is called the "output set" or "target set"




Qlicker

Example: Write out some functions we know from calculus in this new style.

(a) $y = \sin(x)$

$f: \mathbb{R} \to \mathbb{R}$

$x \to \sin(x)$

(b) $y = e^{3x}$

$f: \mathbb{R} \to \mathbb{R}$

$x \to e^{3x}$

(c) $y = \frac{1}{x}$

not a function on $\mathbb{R} \to \mathbb{R}$

$\cancel{f: \mathbb{R} \to \mathbb{R}}$

$\cancel{x \to \frac{1}{x}}$

we need every input $x$ to have a defined output

but $0 \to \frac{1}{0} \notin \mathbb{R}$

Note any issues with these functions.

Name other mappings from $\mathbb{R} \to \mathbb{R}$ that you have seen that would **not** be functions under this definition.

$\tan(x)$

$\ln(x)$

$\sqrt{x}$

$y = \frac{1}{x}$

$f: S \to \mathbb{R}$

$x \to 1/x$

$S = \{x \mid x \in \mathbb{R}, \text{ but } x \neq 0\}$

Example: Consider the three mappings

What is the range = part of target set that IS reached (mapped to be at least one input

$\{0, 1, 2, 3, ..\}$

$$f : \mathbb{N} \to \mathbb{N}$$
$$x \to x^2$$

$\{0, 1, 4, 9, 16, ...\}$

$\{.. -2, -1, 0, 1, 2, ..\}$

$$g : \mathbb{Z} \to \mathbb{Z}$$
$$x \to x + 1$$

→ surjective

$\{.. -1, 0, 1, 2, 3, ...\}$

$\{0, 1, 2 ..$

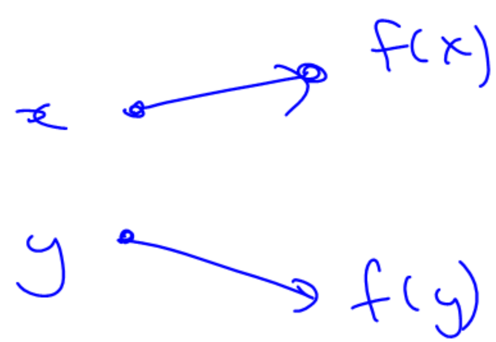
$$h : \mathbb{N} \to \mathbb{Z}$$
$$x \to x + 1$$

$\{1, 2, 3, 4, ...\}$.

Note they differ in how they cover the target set, and whether have unique or non-unique outputs. It is often useful to classify functions by these properties.

**Injective. Definition:** Let $f : S \to T$. $f$ is said to be **injective** if $\forall x, y \in S$: $x \neq y \Rightarrow f(x) \neq f(y)$.

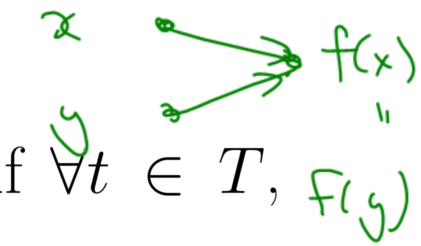
Paraphrase:




$\Updownarrow$

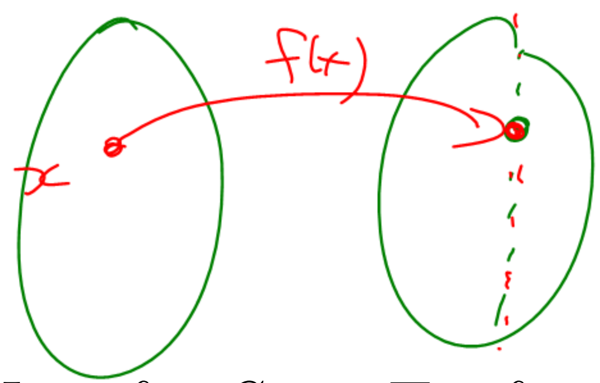
"one to one"

not inj. if




**Surjective. Definition:** Let $f : S \to T$. $f$ is said to be **surjective** if $\forall t \in T$, $\exists x \in S : f(x) = t$.

Paraphrase:




onto / fully covers output.

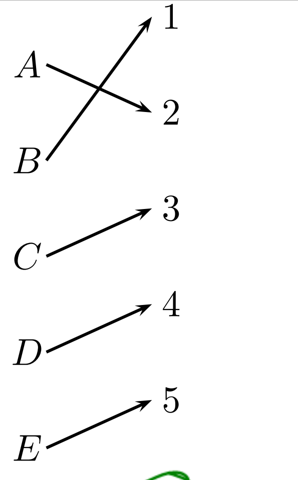
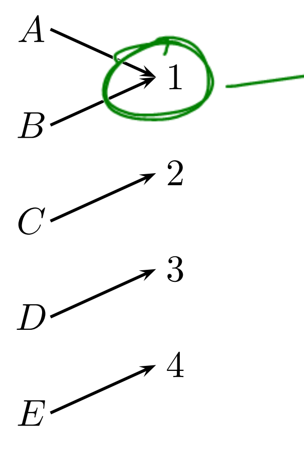
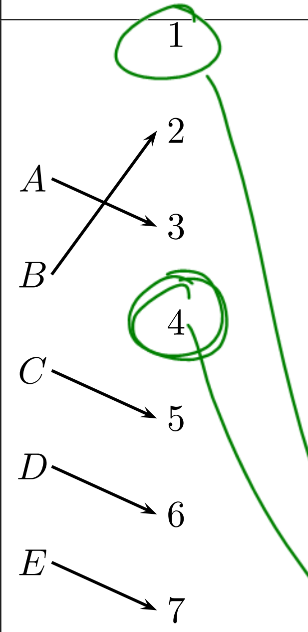
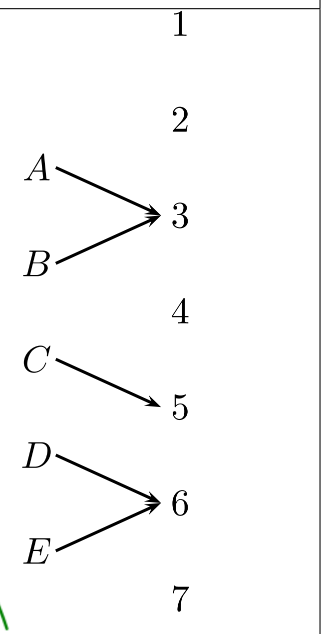
**Bijective. Definition:** Let $f : S \to T$. $f$ is said to be **bijective** if $\forall t \in T$, there exists one and only one element $x \in S : t = f(x)$.

Paraphrase:

combo of inject + surjective

guarantees invertibility.

# Diagrams

| | Injective | Not Injective |
|---|---|---|
| Surjective |  |  |
| Not surjective |  |  |

two inputs → same output

lonely uncovered output values.

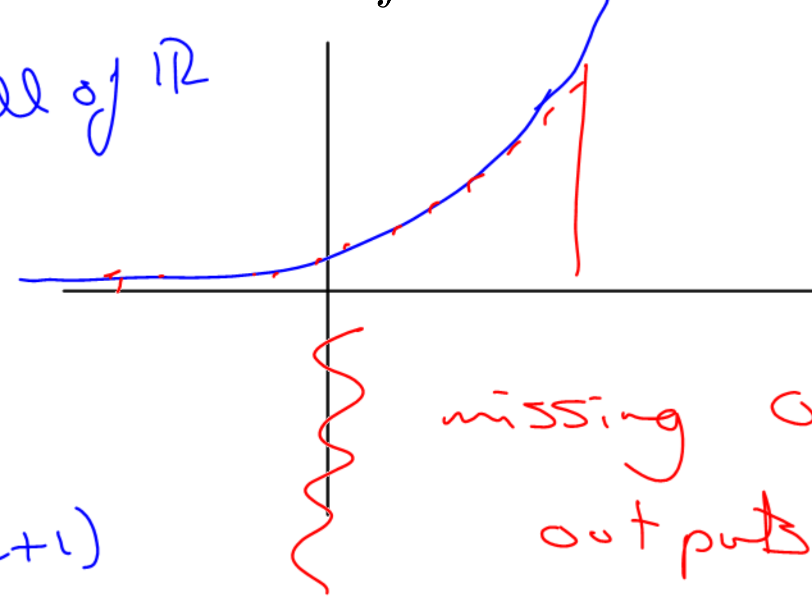
Example: Identify functions from calculus, with the sets $f : \mathbb{R} \to \mathbb{R}$ that would be:

- Injective, but not surjective. output not all of $\mathbb{R}$

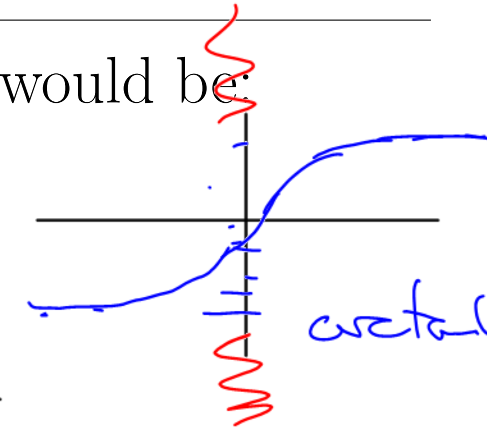
one-to-one

one output = 1 single input

$y = e^x$



missing 0, neg real outputs.



$\arctan(x)$

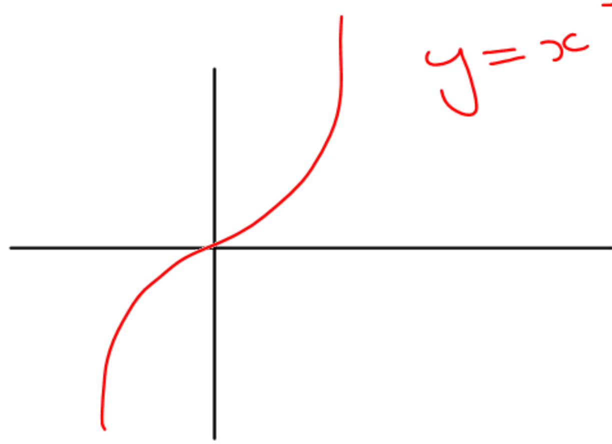
- Surjective, but not injective.

outputs cover all of $\mathbb{R}$

$y = x(x-1)(x+1)$

get same $y$ w/ two different $x$'s

- Bijective.

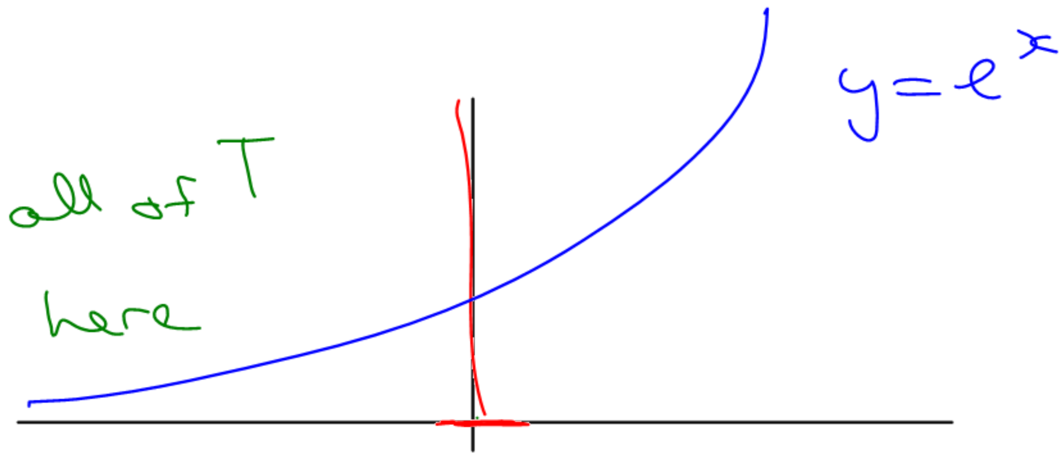


$y = x^{3 \text{ or } 5}, x^5, x^7$ – output covers all $\mathbb{R}$

Could the classification of those functions be changed if we redefined either the domain set or the target set for the functions?

let $g: \mathbb{R} \to \underline{\underline{T}}$ where $T = \{x : x \in \mathbb{R}, x > 0\}$

$x \to e^x$

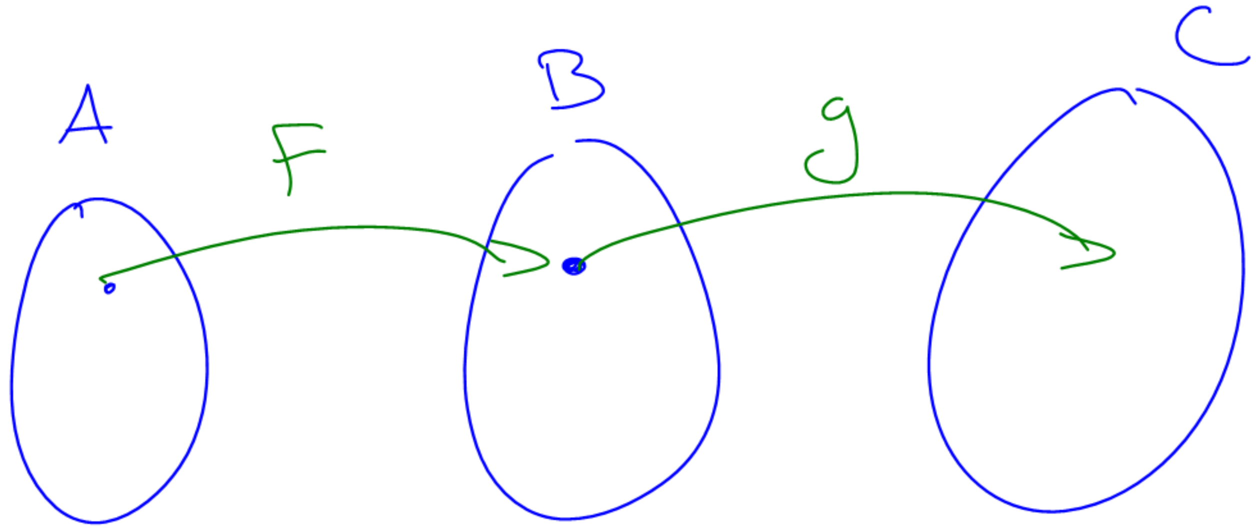



output of $e^x$ $\underline{\text{is}}$ all of $T$

so $g$ $\underline{\text{is}}$ surjective here

## Composition

Suppose that $f : A \to B$, and $g : B \to C$.
The we can combine or **compose** these maps to get a map directly from $A \to C$:




Notation is $g \circ f$ reads as either:

- $g$ follows $f$, or
- $g$ composed with $f$.

# Section 1 - Systems of Linear Equations

Number of solutions to equations

When we solve equation(s), we get a **set** of solutions.

Examples:

(a) Find the set of real solutions to $x^2 = 9$.

(b) Find the set of (possibly complex) solutions to $y^4 = 1$.

(c) Find the set of **only real** solutions to $u^2 = -1$.

Those equations are simple but still relatively challenging to solve because they are **non-linear** equations.

When we look at systems of **purely linear** equations though, we find some simple patterns in the possible solutions.

Example A: Find the set of solutions to the set of linear equations

$$3x + y = 4$$
$$2x + y = 1$$

Example B: Find the set of solutions to:

$$\begin{aligned} 3x + y &= 1 \\ 6x + 2y &= 4 \end{aligned}$$

# Example C: Find the set of solutions to:

$$3x + y = 2$$
$$6x + 2y = 4$$

Categorize the types of solutions we found:

- System A:

- System B:

- System C:

It will take some work over the next few weeks, but we will prove that these are in fact the **only** possible outcomes when we solve systems of linear equations.

Next week: Section 2, Vector Spaces, or “Sets with Math”.