

## Week #2: Vector Spaces

- The addition and scalar multiplication operators
- Axioms of a vector space
- Identifying vector spaces

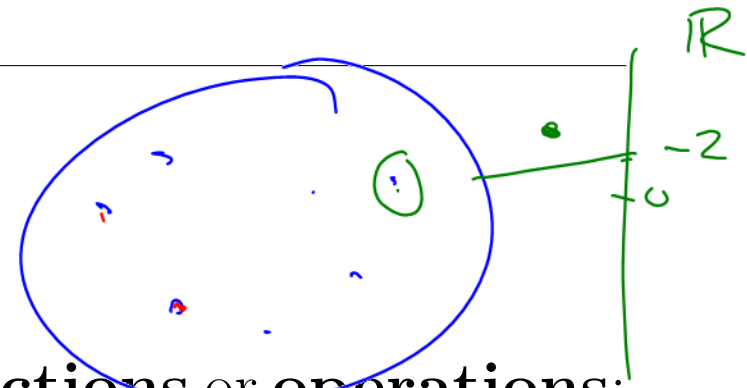
Don't forget to log in to <https://qlicker.queensu.ca> at the start of the lecture.  
Qlicker enrolment code is **QM3CXK**

A Qlicker question is open now.

(Fri) - We will also have EngSoc announcements.

## Section 2 - Vector Spaces

A vector spaces: a set combined with two specific functions.



Definition: A **real vector space** **V** is a set V along with **two functions or operations**:

*function name*

1. An addition map/operator called "+".

$$+ : (V \times V) \rightarrow V \quad \text{single element}$$

$$(\underline{v}, \underline{w}) \rightarrow \underline{v + w}$$

*2 elements of V*

2. A scalar multiplication map/operator called "·".

$$\cdot : (\mathbb{R} \times V) \rightarrow V$$

$$(a, w) \rightarrow a \cdot w$$

"real vector space"

if those functions satisfy specific properties.

Note: The 'real' in 'real vector space' comes from the scalar multiplication part. If we let people multiply by e.g. complex numbers,  $\underline{\mathbb{C}} \times \underline{V}$ , then we would be building a '*complex* vector space'.

We will stick mostly to *real* vector spaces thankfully!

Example: the way we usually use it, the set of real numbers  $\mathbb{R}$  fits the definition of a 'real vector space'.

$$+ : (\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{R}$$

$$(a, b) \rightarrow a + b$$

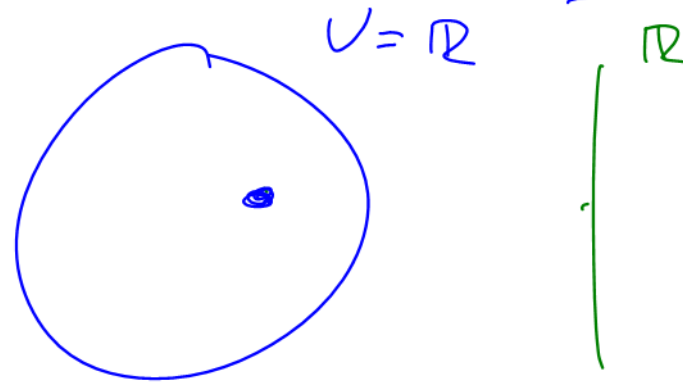
eg.  $+ (3, 4) \rightarrow 3 + 4 = 7$

Scalar multiplies

$$\odot : (\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{R}$$

$$(a, b) \rightarrow ab$$

$\odot (2, 7) \rightarrow 2 \cdot 7 = 14$



Example: Also, the set of functions  $C^\infty$ , infinitely differentiable functions, is a real vector space.

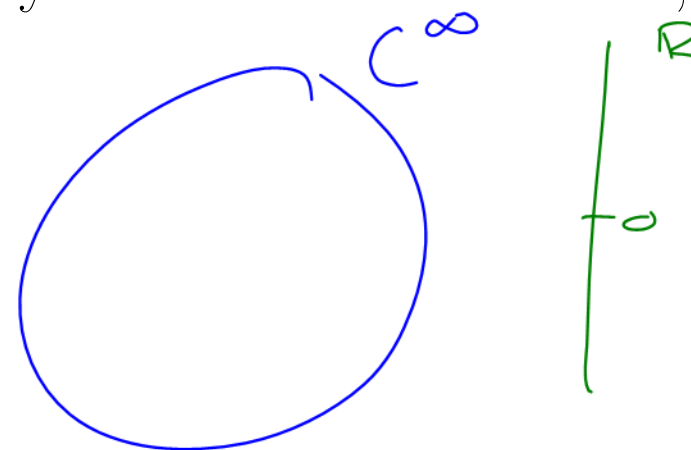
input set

$$\oplus: (C^\infty \times C^\infty) \rightarrow C^\infty$$

$$(f(x), g(x)) \rightarrow [f(x) + g(x)]$$

regular  $\mathbb{R}$  addition

point-by-point sum



eg.  $(\underbrace{\sin}_f + \underbrace{\cos}_g)(x) = \sin(x) + \cos(x)$

$$\cdot: (\mathbb{R} \times C^\infty) \rightarrow C^\infty$$

$$(a, f(x)) \rightarrow \underline{af(x)}$$

$\uparrow$   $\in \mathbb{R}$        $\uparrow$   $\in C^\infty$

point-by-point scalar multiplier

$$(3 \cdot \sin)(x) = 3 \cdot \sin(x)$$

$\uparrow$  regular  $\mathbb{R}$  multiplier

Now we hold up for a second though: we haven't actually **defined** 'addition' and 'scalar multiplication by a real'. In two previous examples we just did what we usually do, but without knowing how that might work in other contexts.

# Axioms of a real vector space

With those two operations in place,  $V$  is a real vector space if the following axioms hold.

0. Both the  $+$  and  $\cdot$  are well-defined functions. *on  $V \times V, \mathbb{R} \times V \rightarrow V$*   
 *$+$ ,  $\cdot$  are defined for all inputs*
1. The operation  $+$  is *associative* and *commutative*.  
*outputs are all in  $V$*
2. There exists in  $V$  an element called the **zero vector**,  $\mathbf{0}$ , such that for any  $v \in V$  we have  
*or zero element*

$$v + \mathbf{0} = v, \text{ and } \mathbf{0} + v = v$$

3. Each  $v \in V$  has at least one negative/opposite/inverse called  $-v$ , such that

$$v + (-v) = \mathbf{0}, \text{ and } (-v) + v = \mathbf{0}. \quad \text{potential}$$

4. The scalar multiplication is *associative*.

*vector space  $(V, +, \cdot)$*

5. The scalar multiplication and addition combinations are distributive:

$$\begin{array}{c} a(v + w) = av + aw, \text{ and } (a + b)v = av + bv \\ \begin{array}{ccc} \in \mathbb{R} & \in V & \in V \end{array} \end{array}$$

6. For any  $v \in V$ , we must have  $1 \cdot v = v$ .

*unit multiplier*

Check-in: how familiar are you with the terms *associative*, *commutative* and *distributive*?

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## Reminder:

- An operation is *associative* if you can change groupings of the inputs in a triple operation with no change in the final value. E.g.  $(3+2)+5 = 3+(2+5)$ , or  $(3 \cdot 2) \cdot 5 = 3 \cdot (2 \cdot 5)$

- An operation is *commutative* if you can swap the order of the inputs with no change in the final value. E.g.  $2 + 7 = 7 + 2$ , or  $2 \cdot 7 = 7 \cdot 2$ .



matrices

$$AB \neq BA$$

mult is not commutative.

- Two operations are *distributive* if you can expand them without changing the value. Edited E.g.  $3 \cdot (2 + 7) = 3 \cdot 7 + 3 \cdot 2$ , or  $(2 + 7) \cdot 3 = 2 \cdot 3 + 7 \cdot 3$ .

$\uparrow$  addition-inside  
 scalar mult.  
 outside

Notation: when we deal with vector spaces, we will write either:

• V, or where + and · are left to context  
 addition scalar mult

• (V, +, ·)

list these explicitly  
 if there is a risk  
 of ambiguity.

$\mathbb{R} \rightarrow$  assume  
 '+' means regular '+'

$\mathbb{R}^n$ ,

$C^\infty$  + scalar

Wait! If these + and · are just functions, why don't we call them  $f(u, v)$  or  $g(a, v)$  like we did earlier?

easier to read  $3+4=7$

vs  $f(3, 4) = 7$

or  $4 \cdot (3+7) = 40$

vs  $g(4, f(3, 7)) = 40$

$+ : (V \times V) \rightarrow V$   
 $\equiv (v, w) \rightarrow v + w$   
 alternatives  
 $f : (V \times V) \rightarrow V$   
 $\equiv (v, w) \rightarrow v + w$

Example: Our most-used vector space:  $\mathbb{R}^n$ . *n-dimensional vector of real #'s*  
 In the notation  $(\mathbf{V}, +, \cdot)$ , what are each of the vector space ingredients?

set  $V = \mathbb{R}^n$

where each  $x \in V = \mathbb{R}^n$  is an *n-tuple* of real numbers,

eg.  $x \in (a_1, a_2, \dots, a_n)$  where  $a_1, a_2, \dots, a_n \in \mathbb{R}$

$+$ :  $\underbrace{\mathbb{R}^n \times \mathbb{R}^n}_{\text{input set}} \rightarrow \underbrace{\mathbb{R}^n}_{\text{output = single vector}}$   
 = pair of vectors

$[(a_1, \dots, a_n), (b_1, \dots, b_n)] \rightarrow (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$   
 simple  $\mathbb{R}$  addition.

$\cdot$ :  $\underbrace{\mathbb{R}}_{\text{scalar}} \cdot \mathbb{R}^n \rightarrow \mathbb{R}^n$   
 regular / simple  $\mathbb{R}$  multiplication.  
 $(k, (a_1, \dots, a_n)) \rightarrow (ka_1, ka_2, \dots, ka_n)$

$$= \{0, 1, 2, 3, \dots\}$$

Example: The natural numbers  $(\mathbb{N}, +, \cdot)$  **are not** a vector space if we use regular addition and real scalar multiplication.

regular addition      scalar mult w/ real #s

Prove that, using the vector space axiom/property that:

“Each  $v \in V$  has a negative/opposite/inverse called  $-v$ , such that

$$v + (-v) = \mathbf{0}, \text{ and } (-v) + v = \mathbf{0}.”$$

need to find

$$(-v)$$

single element of  $\mathbb{N}$

regular integer/real addition

that satisfies. eg.

$$3 + (-v) = \mathbf{0}$$



$$-v = -3 \text{ but}$$

$$-3 \notin \mathbb{N} \quad \times$$

$\Rightarrow$  for  $\mathbb{N}$ ,  $\exists v \in \mathbb{N}$

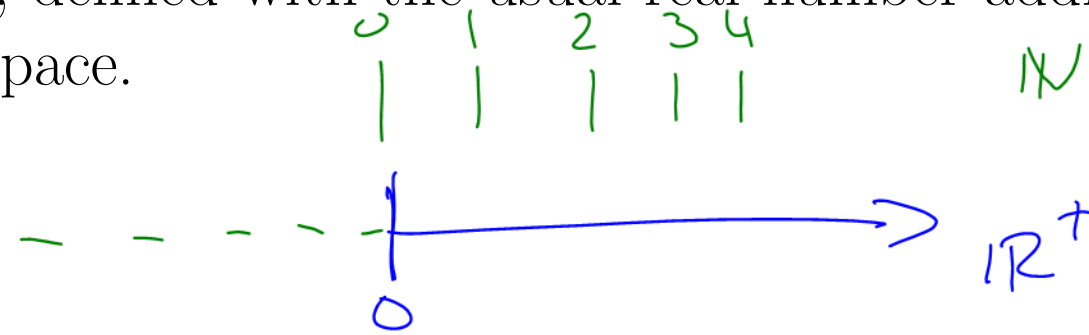
for which there does not

exist an additive inverse,

eg.  $3 \in \mathbb{N}$

$\Rightarrow (\mathbb{N}, +, \cdot)$  is not a real vector space

Example: Use similar logic to show that the set of non-negative reals,  $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ , defined with the usual real number addition and scalar multiplication **is not** a vector space.



no additive inverse for eg  $3.792 \in \mathbb{R}^+$

b/c  $3.792 + (-v) = 0$   
 $\hookrightarrow -3.792$   
 has no solution where

$(-v) \notin \mathbb{R}^+$  or  $(-v) \in \mathbb{R}^+$

also scalar mult w/ reals is not a well defined function.  
 $\mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$   
 eg. pick  $-2 \in \mathbb{R}$  multiplier  $\rightarrow -2 \cdot v, v \in \mathbb{R}^+$  gives a value outside  $\mathbb{R}^+$

$$\{ \dots -3, -2, -1, 0, 1, 2, 3 \dots \}$$

Example: The integers  $(\mathbb{Z}, +, \cdot)$  **are also not** a vector space if we use regular addition and real scalar multiplication.

Show why not, this time looking at the scalar multiplication,  $\cdot$ , defined as a function:

$$\cdot : (\mathbb{R} \times \mathbb{Z}) \rightarrow \mathbb{Z}$$

$$(\underline{a}, \underline{n}) \rightarrow \underline{a} \cdot \underline{n}$$

eg.  $-3.79 \in \mathbb{R}$

and  $5 \in \mathbb{Z}$

but  $-3.79 \cdot 5 = -18.95 \notin \mathbb{Z}$

so scalar mult is not a well defined function

$$\text{as } \bullet : \mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{Z}$$

$\Rightarrow \mathbb{Z}$  are not a real vector space.

Example: The set of infinitely differentiable functions  $(C^\infty(\mathbb{R}), +, \cdot)$  **is** a vector space, if we define:

- Addition as:

point-by-point addition

$$+ : (C^\infty \times C^\infty) \rightarrow C^\infty$$

$$f + g \rightarrow \underbrace{f(x)} + \underbrace{g(x)}$$

eval both functions @ a point

add the real values of each function

↑ on reals  
 set of continuous functions that are infinitely diff'ble  
 $\sin(x), e^x, x^2$

$$h(x) = \sin(x) + x^2$$

- Real scalar multiplication as:

same point-by-point

scalar multiplication

$$f(x) = \sin(x)$$

vs

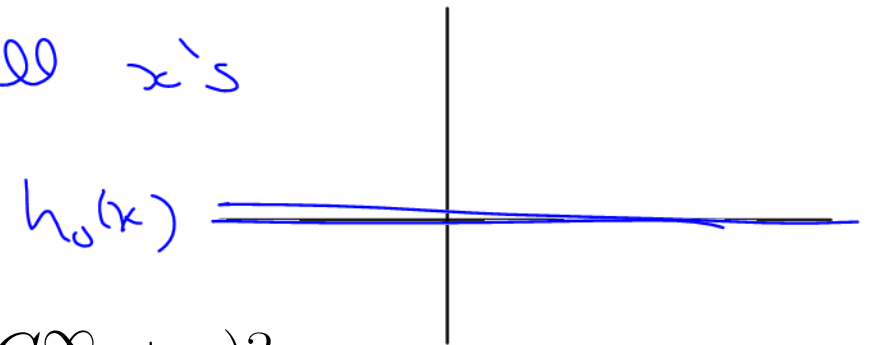
$$g(x) = 3 \sin(x)$$

What is the **zero element** in  $(C^\infty, +, \cdot)$ ?

Need  $f(x) + \bar{0} = f(x)$  (for all  $x$ 's)  
 or

$0$  element in  $C^\infty$  is the always-zero function

call it  $h_0(x) = 0$  for all  $x$ 's



How can we build the opposite/negative of any element in  $(C^\infty, +, \cdot)$ ?

Need  $f + (-f) = \bar{0}$   
 or

then  $-f$  can be built by

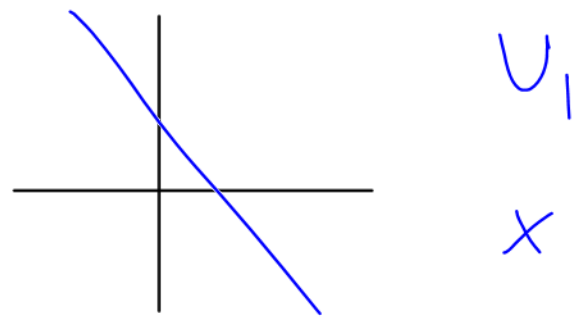
computing  $\underline{(-1) \cdot f}$

scalar multiplier by  $-1$ ,

Summary: Vectors Spaces are sets on which we can define an appropriate  $+$  and  $\cdot$  operation.

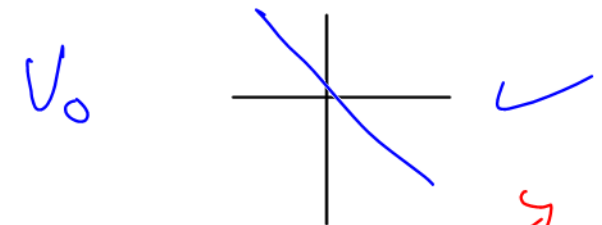
### Not Vector Spaces

$\mathbb{N}$  .....  
 $\mathbb{Z}$  - scalar multiple by reals  $\rightarrow a \cdot x \notin \mathbb{Z}$   
 $\{x \in \mathbb{R} : x \geq 0\}$   
 - missing  $(-x)$   
 for some values



### Vector Spaces

$\mathbb{R}$   
 $\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$   
 $C^\infty$  functions  
 $P_1, P_2, \dots, P_n$   
 poly's of degree  $n$



$\omega_L$   
 $\omega / \text{no } "+", "\cdot"$



xy plane

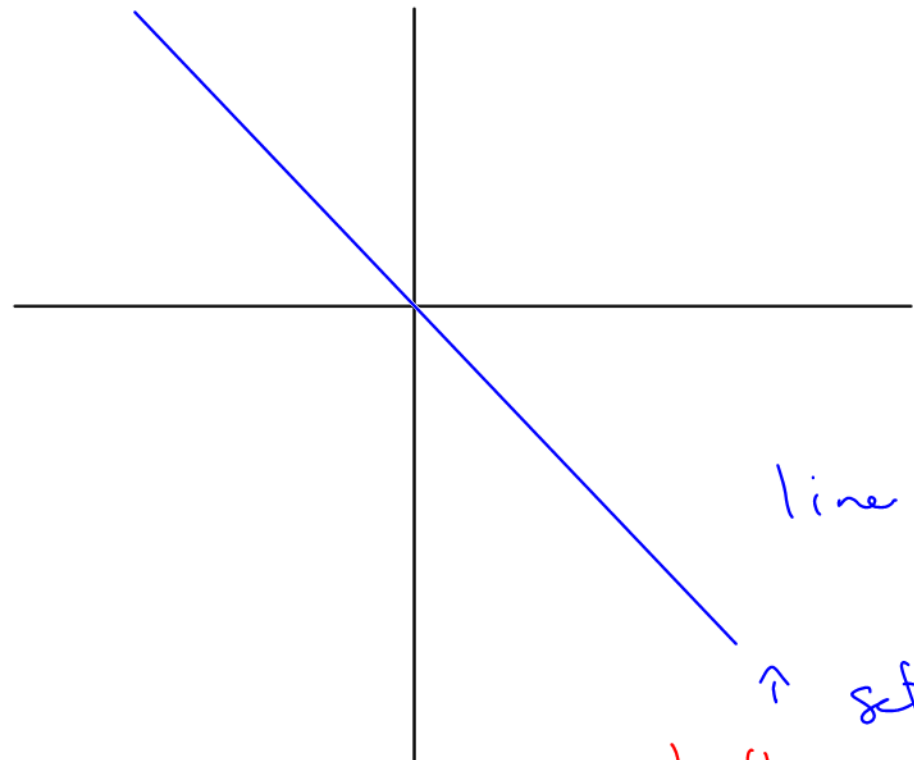
Example: Consider the subset  $V_0 \subset \mathbb{R}^2$  which satisfies  $V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$ , with the usual  $\mathbb{R}^2$  addition and scalar multiplication.

such that

What is the shape of this set, geometrically, in  $\mathbb{R}^2$ ?

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all points satisfying  $x + y = 0$   
 or  $y = -x$



upper left

line  $y = -x$

↑ set of points =  $V_0$

bottom right

$$\underline{V_0} = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}.$$

Is  $(\underline{V_0}, +, \cdot)$  a vector space? To show this we would need to show that **all** the axioms for a vector space are fulfilled.

(We will just show some of these...)

0. Both the  $+$  and  $\cdot$  are well-defined functions.

$\rightarrow$  if we add two elements of  $V_0$ , do we get another element of  $V_0$ ? Yes!

$$\text{Let } (a, b), (c, d) \in V_0 \quad \text{then } (a, b) + (c, d) = (a+c, b+d)$$

(usual vector +)

Is  $(\underbrace{a+c}_x, \underbrace{b+d}_y)$  in  $V_0$ ?

1. The operation  $+$  is *associative* and *commutative*.

$$\text{so } (a+c, b+d) \in V_0$$

Is  $(a+c) + (b+d) = 0$ ?

Yes b/c know  $a+b=0$   $(a,b) \in V_0$   
and  $c+d=0$ ,  $(c,d) \in V_0$

$$\text{so } (a+c) + (b+d) = 0$$

$$\leftarrow \text{so } (a+b) + (c+d) = 0$$

$$V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}.$$

2. There exists in  $V$  an element called the **zero vector**  $\mathbf{0}$ , such that for any  $v \in V$  we have

$$v + \mathbf{0} = v, \text{ and } \mathbf{0} + v = v$$

Try  $(0,0)$  vector as  $\bar{0}$

Let  $(a,b) \in V_0$ , then

$$(a,b) + (0,0) = (a+0, b+0) = (a,b)$$

adding w/ iReals  
 using + for vectors  
 original  $(a,b)$

3. Each  $v \in V$  has a negative/opposite/inverse called  $-v$ , such that

$$v + (-v) = \mathbf{0}, \text{ and } (-v) + v = \mathbf{0}.$$

so  $(0,0)$  is the  $\bar{0}$  for  $V_0$

$$V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}.$$

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4. The scalar multiplication is *associative*.

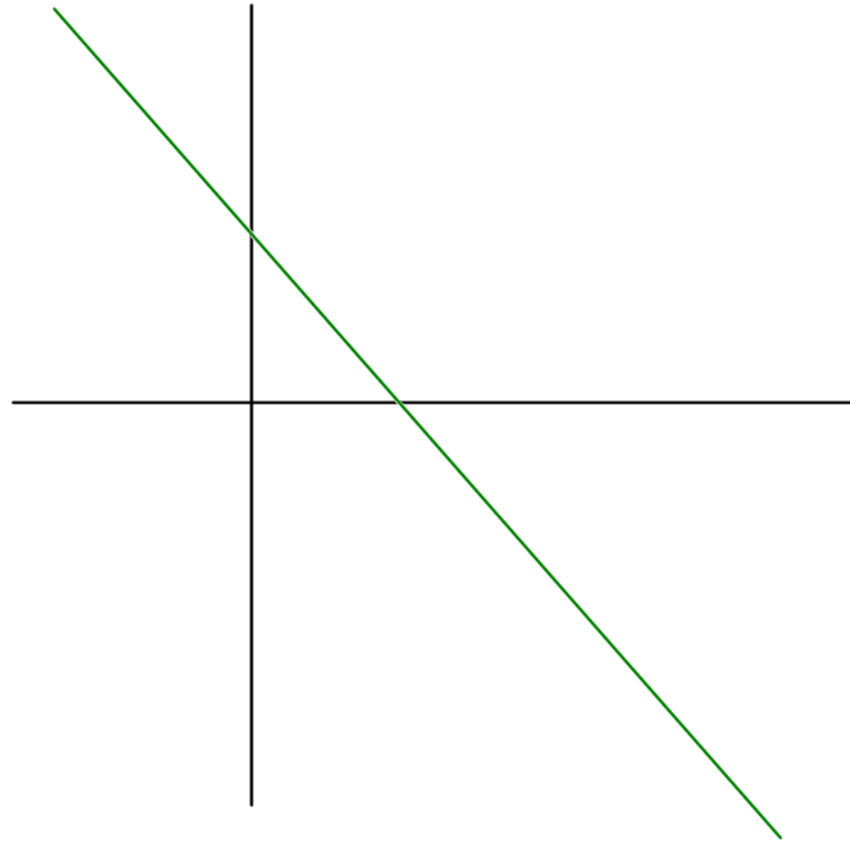
5. The scalar multiplication and addition combinations can be expanded:

$$a(v + w) = av + aw, \text{ and } (a + b)v = av + bv$$

$V_0$  is a vector space.

Example: Consider the similar subset  $V_1 \subset \mathbb{R}^2$  which satisfies  $V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = \mathbf{1}\}$ , with the usual  $\mathbb{R}^2$  addition and scalar multiplication. What is the shape of this set, geometrically, in  $\mathbb{R}^2$ ?

$$x + y = 1$$
$$\Downarrow$$
$$y = 1 - x$$



line  
 $y = 1 - x$   
all points on line are in  
 $V_1$

$$V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = 1\}.$$

Does this space satisfy the criterion that:

“0. Both the  $+$  and  $\cdot$  are well-defined functions.”

No!  $\Rightarrow V_1$  is not a vector space

$$\text{Let } (a, b) \in V_1 \rightarrow a + b = 1$$

We want to show that  $\cdot$  is not a function here,  
so we look any single counter-example.

e.g. let  $k = 5 \in \mathbb{R}$

then  $k \cdot (a, b)$  must be in  $V_1$  if  $V_1$  is a vector space

but  $5(a, b)$

using regular scalar mult on vectors  $\hookrightarrow = (5a, 5b)$ , but  $5a + 5b = 5(a+b) \text{ b/c } (a, b) \in V_1$

so  $5(a, b) \notin V_1$

$$= 5 \neq 1$$

$$V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = \mathbf{1}\}.$$

Does  $V_1$  violate any other vector space axioms?

$\rightarrow \bar{0}$  is not in  $V_1$

(Not necessary b/c  
any single missing  
axiom is enough to  
disqualify a set)

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Intuitions: what seems to help make a vector space different from non-vector-spaces?

"weird" vector space

Example: Let  $W_2$  be a subset of  $\mathbb{R}^2$ , with  
 $W_2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ .

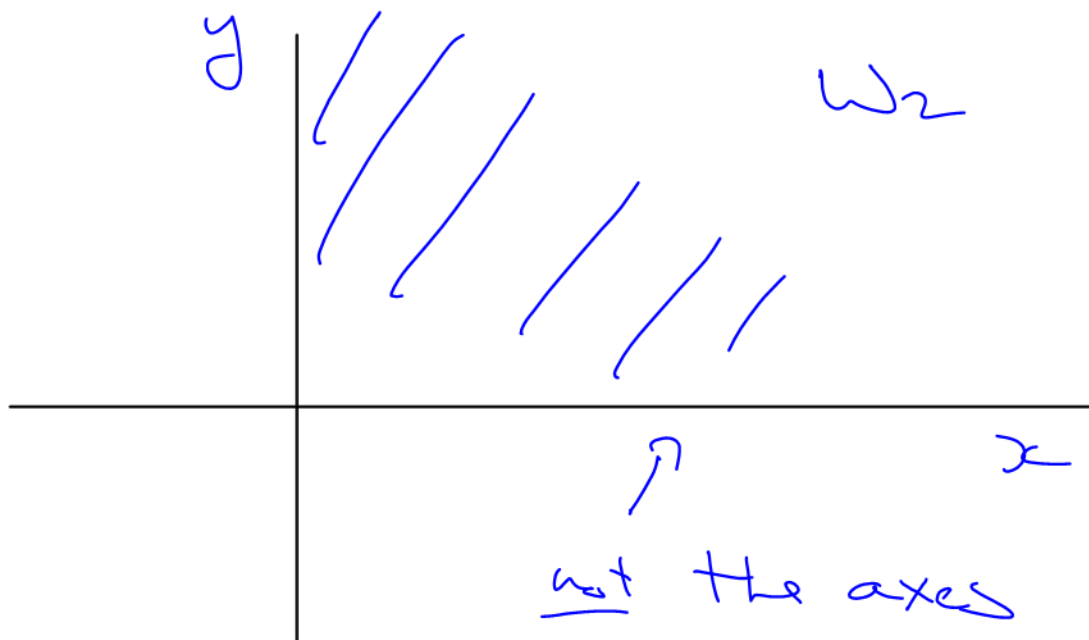
$$\sum W_2 = (W_2, \overset{\text{weird}}{+}, \overset{\text{weird}}{\cdot})$$

Is  $W_2$  a real vector space, if we use the usual addition and scalar multiplication for  $\mathbb{R}^2$  vectors? Why or why not?

↓  
 No.  
 b/c  $\bar{0}$  is not in  $W_2$

so  $(a, b) + \bar{0} = (a, b)$   
 ↑

need this to  $(0, 0)$   
 but  $(0, 0) \notin W_2$



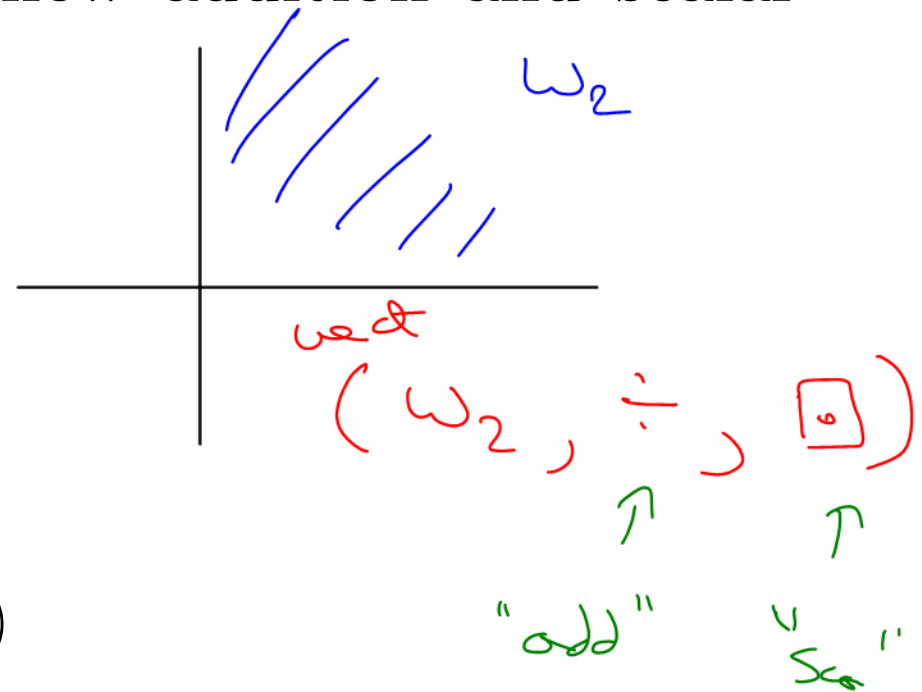
Now let's try expanding our ideas of what "+" and "." can mean. Imagine the same set  $W_2$ , but making a different vector space by defining **new addition and scalar multiplication** rules.

First, addition:

$$+ : (W_2 \times W_2) \rightarrow W_2$$

$$(x, y) + (a, b) \rightarrow (xa, yb)$$

our "+" is element-by-element product.



$\boxed{+}$

Example: Try 'adding' the following in  $(W_2, +, \cdot)$ :

(a)  $(1, 2) + (3, 4)$

$$= (1 \cdot 3, 2 \cdot 4)$$

$$= (3, 8)$$

(b)  $(4, 5) + (2, 2)$

$$= (8, 10)$$

commutativity ✓

(d)  $(2, 2) + (4, 5)$

$$= (8, 10)$$

(e)  $(0, 5) + (7, 2)$  X

$$=$$

off limits  
why?  
 $(0, 5) \notin W_2$

Will this "addition" be a well-defined function on  $W_2$ ? **Yes!**

$$W_2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

Next, scalar multiplication:

$$\cdot : (\mathbb{R} \times W_2) \rightarrow W_2$$

$$\alpha \cdot (x, y) \rightarrow (x^\alpha, y^\alpha)$$

any real      any  $\in W_2$

Example: Try “scalar multiplying” the following in  $(W_2, +, \cdot)$ :

(a)  $2 \cdot (1, 2)$

$$= (1^2, 2^2) = (1, 4)$$

(d)  $0 \cdot (2, 3)$

$$= (2^0, 3^0) = (1, 1)$$

(b)  $3 \cdot (0.5, 4)$

$$= (0.5^3, 4^3) = (0.125, 64)$$

(e)  $-\pi \cdot (2, 3)$

$$= (2^{-3.1415\dots}, 3^{-\pi}) \quad \checkmark$$

Will this “scalar multiplication” be a well-defined function?

Yes!

$$W_2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

$$+ : (x, y) + (a, b) \rightarrow (xa, yb)$$

$$\cdot : \alpha \cdot (x, y) \rightarrow (x^\alpha, y^\alpha)$$

Example: What is the **0** element in  $W_2$ ? Recall that this requires **0** +  $(x, y)$  =  $(x, y)$ .

so let  $\bar{0} = (a, b)$   $a, b$  unknown.

then we need  $(a, b) + (x, y) = (x, y)$

but  $(a, b) + (x, y) = (ax, by)$  regular IR mult.

by our "+" operator

so  $a=1, b=1$

so in  $(W_2, +, \cdot)$ , the  $\bar{0} = (1, 1)$

$$W_2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

$$+ : (x, y) + (a, b) \rightarrow (xa, yb)$$

$$\cdot : \alpha \cdot (x, y) \rightarrow (x^\alpha, y^\alpha)$$

Example: For any  $(x, y) \in W_2$ , how do you build the negative/opposite value such that  $\underline{\underline{-}}(x, y) + (x, y) = \mathbf{0}$ ?

Let  $(c, d) = \underline{\underline{[-(x, y)]}}$

not  $-1 \cdot (x, y)$  for now,  
it's just some vector satisfying

$\underline{\underline{-(x, y)}}$

$$(c, d) + (x, y) = \bar{0} = (1, 1)$$

but  $(c, d) + (x, y) = (cx, dy) = (1, 1)$

$$\Rightarrow \textcircled{c}x = 1 \Rightarrow c = \frac{1}{x}$$

$$\dots \qquad \qquad \qquad d = \frac{1}{y}$$

regular real mult  $\underline{\underline{[-(x, y)]}} + (x, y) = \bar{0}$   
so for  $(x, y) \in W_2$

$$\underline{\underline{-(x, y)}} = \left( \frac{1}{x}, \frac{1}{y} \right)$$

Discussion: If most of our examples are just about  $\mathbb{R}^n$  regular vectors anyway, what is the point of this abstract definition of a vector space?

We can use intuition about the **geometry** of  $\mathbb{R}^2$  and  $\mathbb{R}^3$  vectors to inform us about the **algebra** of other harder-to-visualize vector spaces.

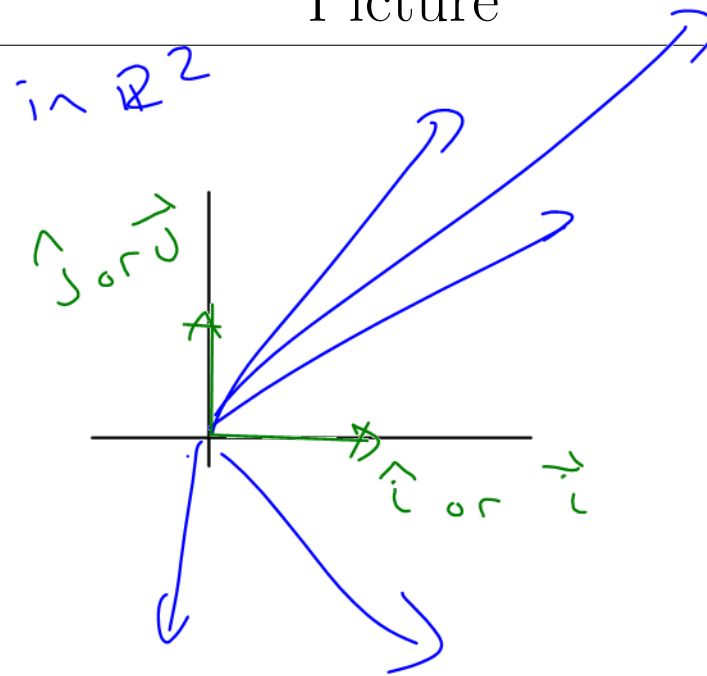
Concept	Picture	Algebra
Vector addition $\underline{\vec{u} + \vec{v}}$ $=$ $\vec{v} + \vec{u}$	<p style="text-align: center;">commutativity</p>	$\underline{w = u + v}$ $= v + u$ $h = f + g = g + f$

Concept

All vectors are linear combinations of a core set of "building block" vectors.

Building blocks for  $\mathbb{R}^2$  are just  $\{\hat{i}, \hat{j}\}$

Picture



any vector  $\vec{v} \in \mathbb{R}^2$

can be written as

$$\vec{v} = a \hat{i} + b \hat{j}$$

for some  $a, b \in \mathbb{R}$

Algebra

$\mathbb{R}^2$  = lin comb'n of some "small" set of building block vectors

$\mathbb{I}_n$   $\mathbb{P}_2$ , poly's degree 2  $\{1, x, x^2\}$  or less.

Other similar features we will see in Section 3.

## Deductions from the Vector Space Axioms

We call a set  $(V, +, \cdot)$  if and only if it satisfies the vector space axioms.

However, the axioms lead to a set of secondary properties that are also then guaranteed to be true for any vector space. We will describe and prove several of these, as a nice collection of proof examples.

**Theorem 1.** In any vector space  $V$ , there is only one zero vector  $\vec{0}$ .

(Recall: the axiom was just that there exists a zero vector, but maybe there is more than one equivalent vector.)

Proof:

take any  $\vec{u} \in V$

$$\text{then } \vec{u} + \vec{0} = \vec{u}$$

no change  
after adding  $\vec{0}$

Assume we have two zero vectors. Goal: show that they must be equal.

Give names  $\vec{0}_A, \vec{0}_B$

both satisfy the  $\vec{0}$  conditions/axioms for a vector space.

Start w/ one of  $\vec{0}$ 's:  
adding  $\vec{0}_B$  doesn't change value.

$$\vec{0}_A = \vec{0}_A + \vec{0}_B = \vec{0}_B$$

$$\text{so } \vec{0}_A = \vec{0}_B$$

so both zero's

are actually the same element of  $V$ .

like  $\vec{u}$  in  $\vec{u} = \vec{u} + \vec{0}$  treat  $\vec{0}_B$  as  $\vec{0}$

**Theorem 2.** In any vector space  $V$ , each element  $v$  has only one (i.e. unique) additive inverse  $-v$ .

Proof:

Let  $v \in V$  and assume that  $v$  has two additive inverses  $[-v_A]$  and  $[-v_B] \Rightarrow$   
 $v + [-v_A] = \vec{0}$   
 and  
 $v + [-v_B] = \vec{0}$

Goal: use axioms to show  $[-v_A] = \dots = [-v_B]$

Start w/  $[-v_A] = [-v_A] + \vec{0} = [-v_A] + (v + [-v_B]) = ([-v_A] + v) + [-v_B]$

$\vec{a} = \vec{a} + \vec{0}$  axiom

axiom on additive inverses

axiom on additive inverses

axiom on additive inverses

axiom on additive inverses

axiom on additive inverses

so  $[-v_A] = [-v_B]$

so both additive inverses are the same.

$\vec{0} + [-v_B] = [-v_B]$

$$[-v_A] = [-v_A] + \bar{0}$$

$$= [-v_A] + (v + [-v_B])$$

$\bar{0}$  addition axio

$v_B$  is an additive inverse.

**Theorem 3** - Cancellation property. Given a vector space  $V$ , and three vectors  $u, v, w \in V$  that satisfy  $u + v = w + v$ , then we can conclude  $u = w$ .

Proof:

Start w/

$$u = u + \bar{0}$$

$\bar{0}$  add axiom

$$= u + (v + (-v))$$

$$\bar{0} = v + (-v)$$

$$= (u+v) + (-v)$$

associat.

so if

$$u+v = w+v$$

$$= (w+v) + (-v)$$

from given  $u+v = w+v$

then  
we can "cancel"  
the  $v$ 's!

$$= w + (v + (-v))$$

assoc

$$= w + \bar{0}$$

axiom add inv.

$$u = w$$

$$= w$$

$\bar{0}$  add axiom

**Theorem 4.** Let  $V$  be a vector space. Then

(a) For any vector  $v \in V$ ,  $0 \cdot v = \vec{0}$ .

get  $\vec{0}$  in any vector space

(b) For any scalar  $a \in \mathbb{R}$ ,  $a \cdot \vec{0} = \vec{0}$

by computing

scalar product  $0 \cdot v$

for any vector  $v$

Proofs:

(a) Start w/

$$0 \cdot v = (0 + 0) v$$

$$\textcircled{1} = 0v + 0v$$

$0 + 0 = 0$  in  $\mathbb{R}$

distribut.

$0 \rightarrow \mathbb{R}$  scalar  $0$

$\vec{0}, \vec{0}$  both  $0$  vector

Now consider

$$0 \cdot v = \vec{0} + 0 \cdot v \textcircled{2}$$

$\vec{0}$  add axiom

$$u + v = w + v$$

$$\cancel{0} + \cancel{0 \cdot v} = 0 \cdot v + \cancel{0 \cdot v}$$

cancellation prop

set  $\textcircled{1} = \textcircled{2}$

$$\vec{0} = 0 \cdot v \text{ for any } v \in V.$$

(b) For any scalar  $a \in \mathbb{R}$ ,  $a \cdot \bar{0} = \bar{0}$

i.e., If you multiply the  $\bar{0}$  by any scalar, it stays as the  $\bar{0}$ .

Try same strategy as (a): use  $u + \cancel{v} = w + \cancel{v}$  cancellation  
 $\forall \underline{a \cdot \bar{0} + v} = \bar{0} + v$

Let  $a \in \mathbb{R}$ :  $\swarrow$  use this as our  $v$ :  $\downarrow$

$$\begin{aligned} a \cdot \bar{0} + \bar{0} &= a \cdot \bar{0} \\ &\searrow \searrow \\ &= a(\bar{0} + \bar{0}) \end{aligned}$$

$$= a \cdot \bar{0} + a \cdot \bar{0}$$

Ah!

$$\frac{a \cdot \bar{0}}{v} + \frac{\bar{0}}{v} = \frac{a \cdot \bar{0}}{v} + \frac{a \cdot \bar{0}}{w}$$

By cancellation property, can cancel

first  $a \cdot \bar{0}$ 's:  $\cancel{a \cdot \bar{0}} + \bar{0} = \cancel{a \cdot \bar{0}} + a \cdot \bar{0} \Rightarrow \bar{0} = a \cdot \bar{0}$  for any real  $a$ .

Axiom

$$v + \bar{0} = v$$

same axiom, but inside the multi.

$$\bar{0} = \bar{0} + \bar{0}$$

distributivity

**Theorem 4 cont.** Let  $V$  be a vector space. Then

(c) For any vector  $v \in V$ ,  $-v = -1 \cdot v$   
 i.e. the scalar product  $-1 \cdot v$  is the additive inverse of  $v$ .

you can build the additive inverse of any

Proofs:

Axiom 6:  $1 \cdot v = v$

Start w/

$$(-1) \cdot v + v = (-1) \cdot v + (1) \cdot v$$

$$= (-1+1) \cdot v$$

$$= 0 \cdot v$$

$$= \vec{0}$$

$$(-1) \cdot v = [-v]$$

distributive

$-1 \in \mathbb{R}$  scalar prod.

vector  $v$  by computing  $(-1) \cdot v$

last page Thm 4 (a)

$$0 \cdot v = \vec{0}$$

$$\text{Then } (-1) \cdot v + v = \vec{0}$$

So  $(-1) \cdot v$  is the additive inverse

Werkstatt

$$a \cdot (u_1, u_2) = (a u_1 - 3a + 3, a u_2 + 2a - 2)$$

$$a = -1$$

$$- (x, y) = (-1) \cdot (x, y)$$

$$= (-1 x - 3(-1) + 3, (-1) y + 2(-1) - 2)$$

$$= (6 - x, -4 - y)$$

$$(x, y) + \cancel{[-(x, y)]} = \vec{0}$$

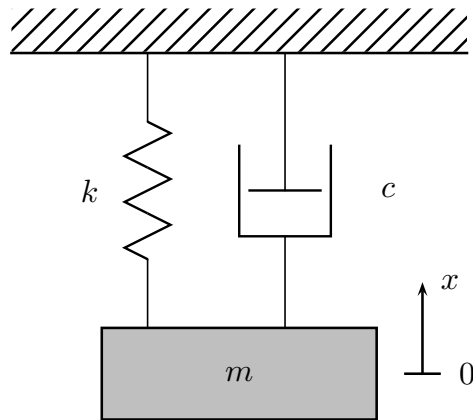
$(a, b)$

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Comment: All these theorems above can feel, well, *obvious* if we just think of  $\mathbb{R}$  and  $\mathbb{R}^n$ : we just assume these based on our experience.

However, as we identify or encounter new vector spaces, it can be profoundly helpful to have this rock-solid bed of properties like these that we can immediately rely on, even in unfamiliar spaces.

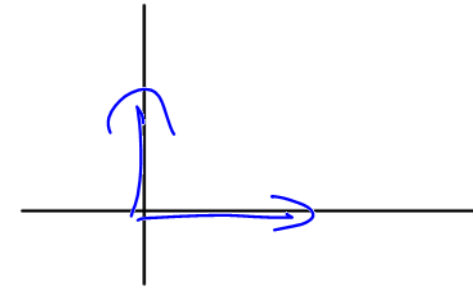
Example: from the spring/mass system in APSC 171.



$$ma = \sum F$$

or

$$mx'' = -kx - cx'$$



In APSC 171, we found two simple solutions, e.g.  $x_1(t) = e^{(a+bi)t}$  and  $x_2(t) = e^{(a-bi)t}$ .

We then made the statement “any linear combination of those two solutions is also a solution”, e.g.  $x(t) = z_1 e^{(a+bi)t} + z_2 e^{(a-bi)t}$

Relation to vector spaces:

$z_1 \cdot e^{(a+bi)t}$  +  $z_2 \cdot e^{(a-bi)t}$

*scalar multipl.*      *vector space*      *odd n*      *Euler's form*

$e^{ibt} = \cos bt + i \sin(bt)$

rational =  $\mathbb{Q}_+$

"  
ratios

$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12}$$

$$h\left(\frac{m}{n}\right) = 2^m \cdot 3^n$$

$$h\left(\frac{3}{4}\right) = 2^3 \cdot 3^4$$

$$\hookrightarrow h\left(\frac{6}{8}\right) = 2^6 \cdot 3^8$$

multiple  
↓  
outputs  
for  
single  
input.

valid function  $\rightarrow$  works/defined for all  $(m, n) \in \mathbb{Q}_+$

$$h: \mathbb{Q}_+ \rightarrow \mathbb{N}$$

$$h: [\mathbb{N}^+ \times \mathbb{N}^+] \rightarrow \mathbb{N}$$

inject  $\rightarrow$  each output is unique.

$$(m, n) \rightarrow 2^m \cdot 3^n$$

surjective  $\rightarrow$  covers all of  $\mathbb{N}$

{1, 2, 3, 4, 5, 6, 7, ...}

No.

there is no  $\frac{m}{n}$  s.t.  $h(m, n) = 2 = 2^1 = 2^1 \cdot 3^0$

not ~~a~~ function

$$f: U \rightarrow V$$

$$\underline{x} \in U \rightarrow f(x) \in V$$

- defined for all  $x$ 's in  $U$



eg. failing  $f: \mathbb{R} \rightarrow \mathbb{R}$

$$x \rightarrow \sqrt{x}$$

not defined for

eg  $x = -2$

- a single output for each  $x$

failing

$$x \rightarrow \pm \sqrt{x}$$

2 outputs

for 1 input