Week #3: Vector Subspaces

- Vector subspaces, and simplified process for identifying them
- Linear combinations
- Span of a set of vectors as a subspace
- Implications of span for the solution of systems of equations

No clicker questions today (Wed)

Section 3 - Vector Subspaces

Recall: Vector Spaces are

- **A set V** combined with:
- An **addition** map/operator called "+"; and
- A **scalar multiplication** map/operator called "·".

These spaces must also satisfy a set of 8 axioms around their addition and scalar multiplication operators, e.g.

- There must be a zero element, **0**, such that $0 + x = x + 0 = x$.
- Each element x must have a negative/inverse of itself, called $-x$, such that $x + (-x) = \mathbf{0}$.
- addition is commutative.
- ...

Given (V, +, ·) combo

misses any 1 or more of axioms → not a vector space.

satisfies all 8 axioms → (V, +, ·) is a vector space

Vector Spaces	Not Vector Spaces

In Qlicker

$\mathbb{R}$

$\mathbb{R}^3$

$\mathbb{R}^n$

C^∞

P_n

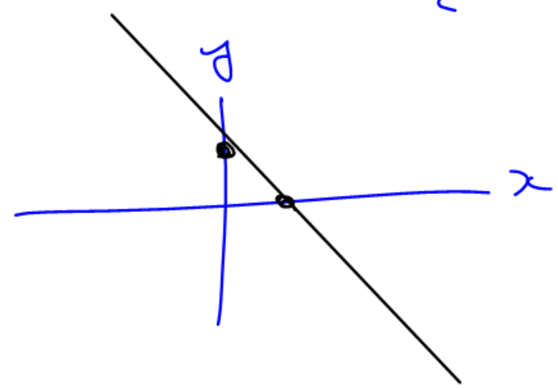
$\mathbb{Z} \to$ b/c

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 3\}$ real × integer scalar mult $\to$ can get a non integer

eg $\pi \cdot 2 \to \pi \cdot 2 \notin \mathbb{Z}$

$\mathbb{R} \cdot \mathbb{Z}$

so scalar mult is not a valid function from $\mathbb{R} \times \mathbb{Z} \to \mathbb{Z}$



$U_1 = \{(x,y) \in \mathbb{R}^2 : x + y = 1\}$.

$\to U_1$ not a vector space.

We now look at a somewhat simpler question: if we **already have** a vector space, when will a **subset** (with the same addition and scalar multiplication) also be a vector space?

we know $(\mathbb{R}^2, +, \cdot)$ is a vector space.

exactly our V_1 scenario.

Example: We have seen $\mathbb{R}^2$ is a vector space, with the usual addition and scalar multiplication.

Is the subset of $\mathbb{R}^2$ defined by
$V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = \mathbf{1}\}$ also a vector space?

Option 1: go through all 8 vector space axioms like last week.

Option 2: because we are just looking at a subset of a known vector space, we can take a short-cut!

Vector Subspace. Definition: Let $(\mathbf{V}, +, \cdot)$ be a real vector space. Now let $\mathbf{W}$ be a subset of $\mathbf{V}$, i.e. $\mathbf{W} \subset \mathbf{V}$, and working with the same $+$ and $\cdot$. $(\mathbf{W}, +, \cdot)$ will also be vector space, or more commonly called a **vector subspace** of $(\mathbf{V}, +, \cdot)$ if the following axioms are satisfied:

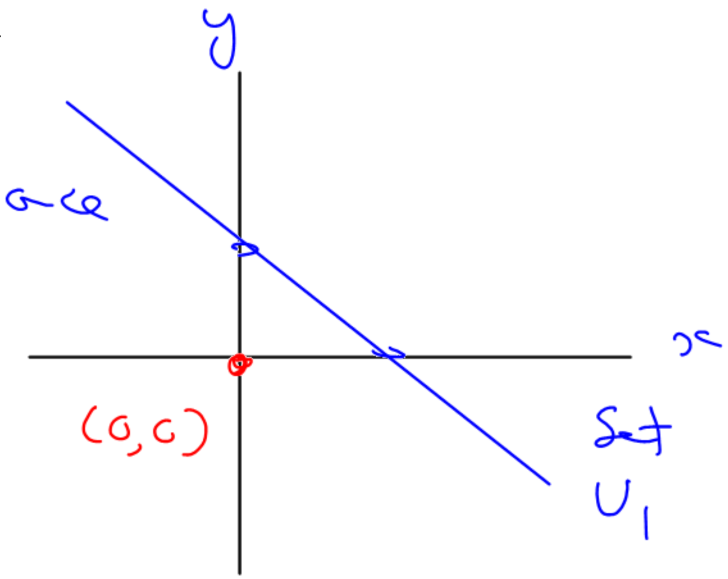
(1) The subset $\mathbf{W}$ contains the zero element $\mathbf{0}$ of $(\mathbf{V}, +, \cdot)$.

(2) $\mathbf{W}$ is closed under addition. I.e. for any $x, y \in \mathbf{W}$, the sum $x + y \in \mathbf{W}$ too.

(3) $\mathbf{W}$ is closed under scalar multiplication. I.e. for any real α and element $x \in \mathbf{W}$, the product $\alpha x \in \mathbf{W}$ too.

components of our $\mathbb{R}^2$ vector

Based on that shorter set of tests, is the subset of $\mathbb{R}^2$ defined by
$V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = \mathbf{1}\}$ a vector subspace?



V_1 is $\subset \mathbb{R}^2$ so it could be a vector subspace of $\mathbb{R}^2$

1) Is $\vec{0}$ from $\mathbb{R}^2$ in V_1?

The $\vec{0}$ in $\mathbb{R}^2$ (w/ usual vector $+, \cdot$) is $(0,0)$

No: $(0,0)$ is not in V_1, b/c $0 + 0 = 0$, not $= 1$
(x comp, y comp)

$\Rightarrow$ b/c axiom 1) for a vector subspace is not satisfied $\Rightarrow$ V_1 is not a vector subspace.

(1) The subset $\mathbf{W}$ contains the zero element $\mathbf{0}$ of $(\mathbf{V}, +, \cdot)$.
(2) $\mathbf{W}$ is closed under addition. I.e. for any x, y [u, v] $\in \mathbf{W}$, the sum $x + y$ [u+v] $\in \mathbf{W}$ too.
(3) $\mathbf{W}$ is closed under scalar multiplication. I.e. for any real α and element $x \in \mathbf{W}$, the product $\alpha x \in \mathbf{W}$ too.

$$V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = \mathbf{1}\}$$

(1) The subset **W** contains the zero element **0** of $(\mathbf{V}, +, \cdot)$.
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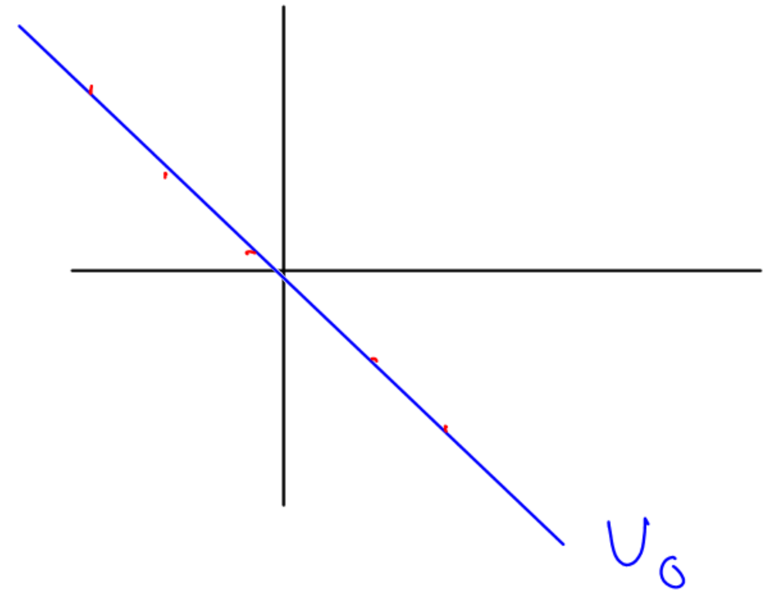
Example: Is the subset of $\mathbb{R}^2$ defined by $y = -x$

$V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$ a vector subspace?



(1) Is $\vec{0}$ from $\mathbb{R}^2$ in V_0?

Yes! $(0,0)$ is $\vec{0}$ in $\mathbb{R}^2$

and $0 + 0 = 0$ (x ↙, ↘ y) $\rightarrow$ so $(0,0) \in V_0$

V_0 membership condition

(1) The subset $\mathbf{W}$ contains the zero element $\mathbf{0}$ of $(\mathbf{V}, +, \cdot)$.

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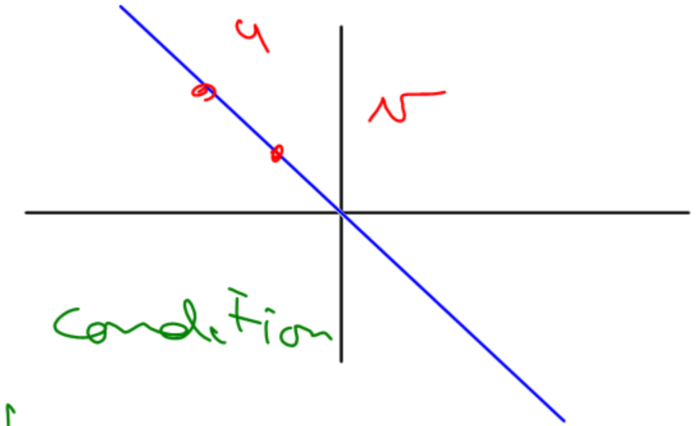
$V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$



Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$

b/c $u \in V_0 \rightarrow x_1 + y_1 = 0$ membership condition for V_0

and b/c $v \in V_0 \longrightarrow x_2 + y_2 = 0$

regular $\mathbb{R}^2$ addition

Look at $u + v = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \; y_1 + y_2)$

x comp y comp

Is $u + v$ in V_0? Yes! b/c $(x_1 + x_2) + (y_1 + y_2)$

$= (x_1 + y_1) + (x_2 + y_2)$ b/c $u, v \in V_0$

$= 0 + 0 = 0$

(1) The subset $\mathbf{W}$ contains the zero element $\mathbf{0}$ of $(\mathbf{V}, +, \cdot)$.

(2) $\mathbf{W}$ is closed under addition. I.e. for any x, y (u, v) $\in \mathbf{W}$, the sum $x + y$ ($u + v$) $\in \mathbf{W}$ too.

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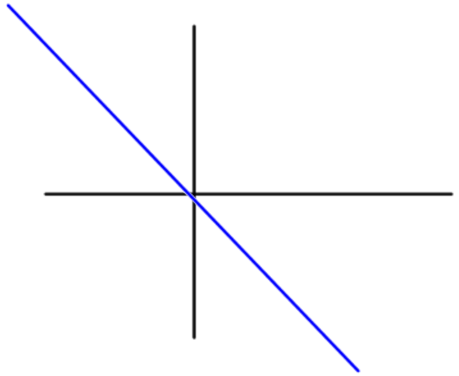
$V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$



(3) Let $u \in V_0$ w/ $u = (x, y) \to x + y = 0$
and $\alpha \in \mathbb{R}$

Look at $\alpha \cdot u = \alpha(x, y)$ ← regular $\mathbb{R}^2$ scalar mult of a vector
$= (\alpha x, \alpha y)$
x comp, y comp

and $\alpha \cdot u \in V_0$ b/c $\alpha x + \alpha y = \alpha(x + y)$ ← b/c $u = (x, y) \in V_0$
$= \alpha \cdot 0$
$= 0$

so $\alpha u \in V_0$
$\Rightarrow V_0$ is closed under scalar mult.

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Because U_0 is a subset of $\mathbb{R}^2$ (a known vector space)

<u>and</u> U_0 satisfied all 3 subspace axioms

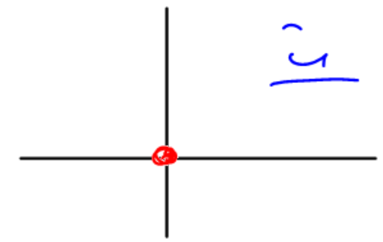
$\Rightarrow$ U_0 is a vector subspace

(U_0 " " " space)

Example: Is the subset of any vector space $(\mathbf{V}, +, \cdot)$ defined simply by $W = \{\mathbf{0}\}$ a vector subspace?

eg $\mathbb{R}^2$ big space

Yes $\{0\}$ is a vector subspace



Qlicker

(1) is $\vec{0}$ in W? Yes!

(2) closed under addition $\Rightarrow$ take sum of any 2 elements of W: is the sum also in W? Yes $\vec{0} + \vec{0} = \vec{0} \in W$

(3) $\alpha \cdot \vec{0} = \vec{0} \in W$ — W is closed under scalar mult

Example: Is the empty set, i.e. which is a subset of any vector space $(\mathbf{V}, +, \cdot)$, a vector subspace?

ϕ

Qlicker

No! (1) requires that $\vec{0} \in W$ but W is empty so $\vec{0} \notin W$

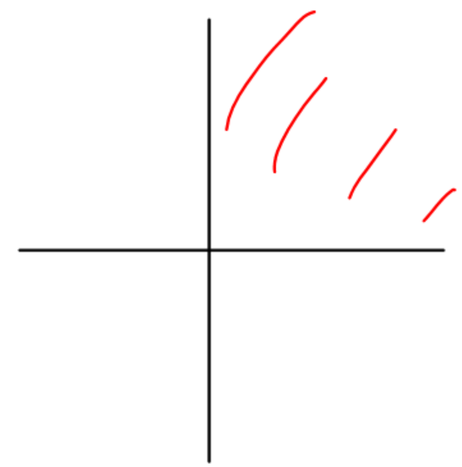
$\Rightarrow W = \phi =$ empty set is not a vector subspace

$\vec{0} \in \mathbb{R}^2$ is $(0,0)$ under our usual $+, \cdot$

↳ defined by $\vec{u} + \vec{0} = \vec{u}$

- different from $0 \in \mathbb{R}$, a single real number

$\vec{0}$ in $\mathbb{R}^2$ w/ <u>different</u> $+, \cdot$ <u>may not</u> be $(0,0)$!

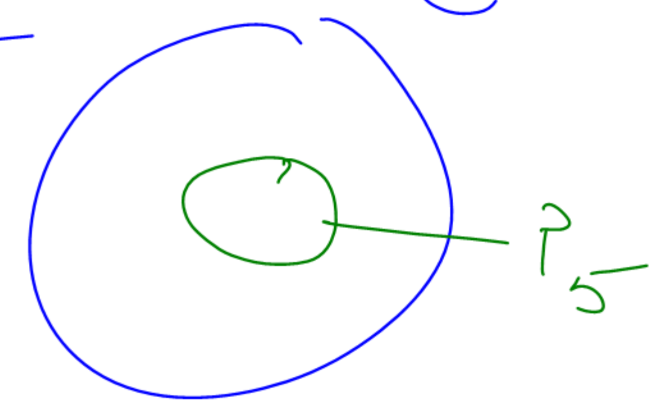


W_2 $\vec{0} = (1,1)$

Example: Is the set of all polynomials of degree 5 or less, $P_5(\mathbb{R})$, a vector subspace of the C^∞ vector space?

continuous, infinitely diff'ble functions e.g. $x^5 - 3x + 2 \in P_5$

$\sin(x) \notin P_5$



Yes! The $\vec{0}$ for C^∞ is the zero-valued function $f(x) = 0$ for all 0.

and $\vec{0}$ = all zero-valued function is in P_5. $\Rightarrow$ $\vec{0} = 0x^5 + 0x^4 + 0x^3 + 0x^2 + 0x + 0$

(1) The subset $\mathbf{W}$ contains the zero element $\mathbf{0}$ of $(\mathbf{V}, +, \cdot)$.
(2) $\mathbf{W}$ is closed under addition. I.e. for any $x, y \in \mathbf{W}$, the sum $x + y \in \mathbf{W}$ too.
(3) $\mathbf{W}$ is closed under scalar multiplication. I.e. for any real α and element $x \in \mathbf{W}$, the product $\alpha x \in \mathbf{W}$ too.

$2x + y + 5(z - 2) = 0$

(2) P_5 is closed under addition

add any two poly'ls of degree 5 or less,

and the sum is also a poly'l degree 5 or less:

Let $f = a_5 x + a_4 x^4 \dots + a_1 x + a_0$ $a_0, 1 \dots \in \mathbb{R}$

and $g = b_5 x^5 + b_4 x^4 \dots + b_1 x + b_0 \to b_0, b_1 \dots \in \mathbb{R}$

$\Rightarrow f+g = (a_5+b_5) x^5 + (a_4+b_4) x^4 + \dots + a_0 + b_0 \in P_5$

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(3) P_5 is closed under scalar mult'n:

Let $f = a_5 x^5 + a_4 x^4 \cdots + a_1 x + a_0 \in P_5$

and $\alpha \in \mathbb{R}$

$\alpha f = (\alpha a_5) x^5 + (\alpha a_4) x^4 + \cdots + \alpha a_0 \in P_5$ too.

B/c P_5 satisfied all 3 subspace axioms,

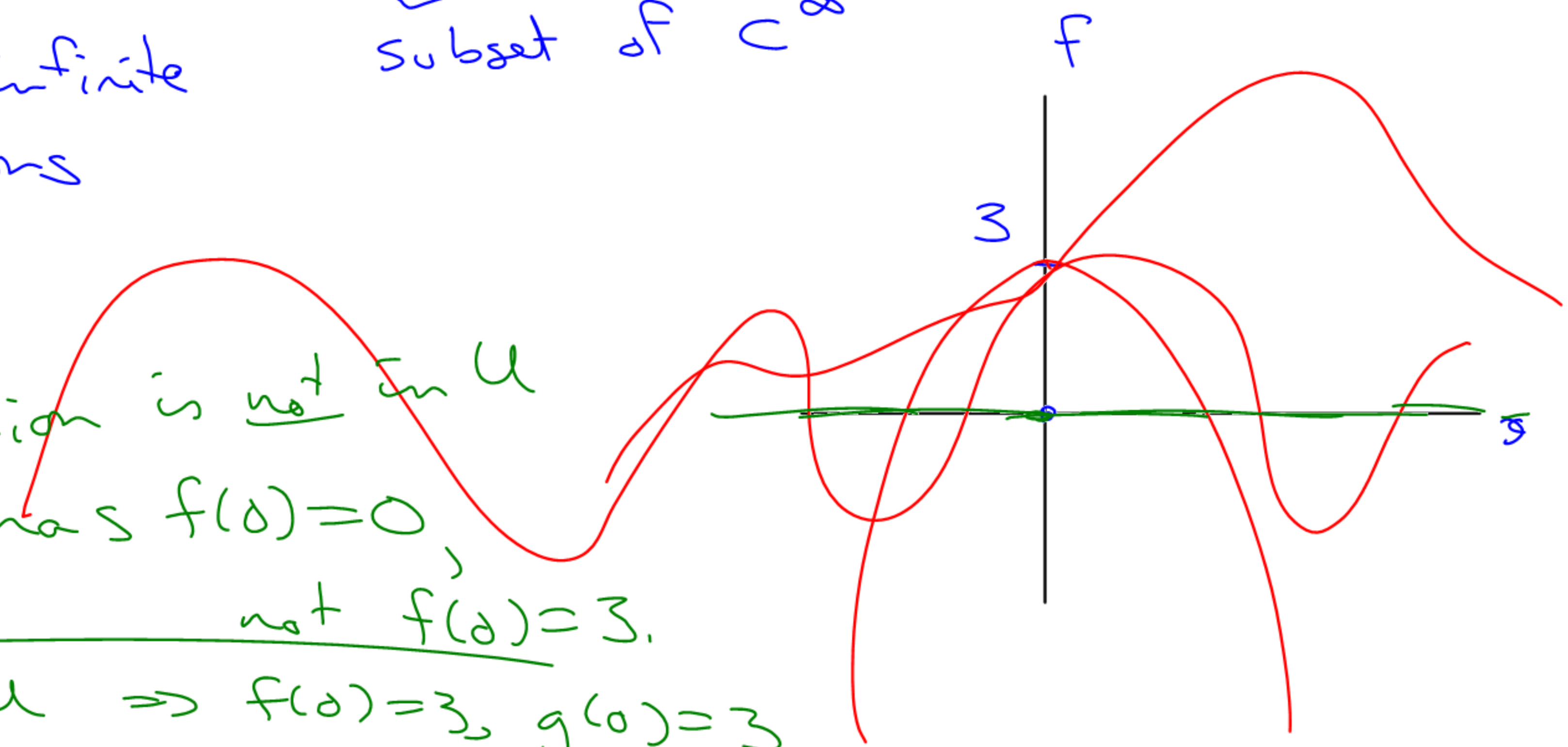
$\Rightarrow$ P_5 is a vector subspace of C.

usual function f, · is a vector space

Example: What about the set $\mathbf{U} = \{f \in C^\infty : f(0) = 3\}$: is it a vector subspace of C^∞?

contains an infinite # of functions

subset of C^∞



(1) Fails b/c the $\vec{0}$ in C^∞ or the constant 0 function is not in U b/c " " " has $f(0)=0$, not $f(0)=3$.

(2) Fails pick $f, g \in U \Rightarrow f(0)=3,\ g(0)=3$

but $(f+g)(0) = f(0)+g(0) = 3+3 = 6 \neq 3$ so $f+g \notin U$

$f = 0$ in C^∞

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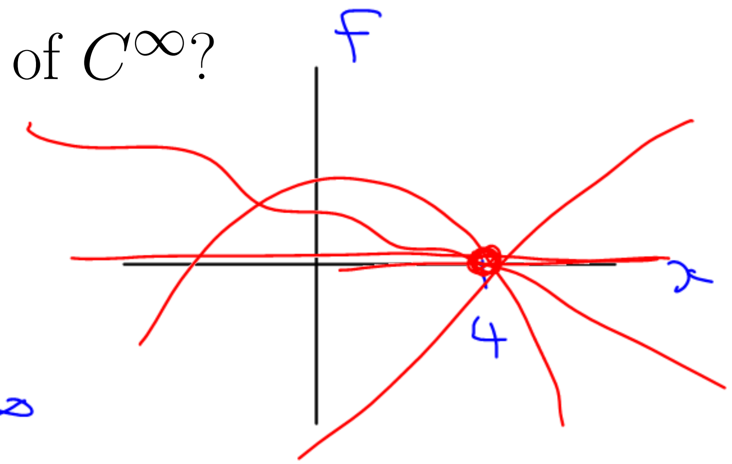
Is there a similar type of set that **would** be a vector subspace of C^∞?



Let $V = \{ f \in C^\infty : f(4) = 0 \}$

This V is a vector subspace

b/c (1) It contains the $\vec{0}$ for C^∞

i.e. $\underbrace{f(x) = 0}_{\vec{0}}$ satisfies $f(4) = 0$

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Let $V = \{f \in C^\infty : f(4) = 0\}$

Also (2) for any 2 functions $f, g \in V$

$f+g$ is also in V:

Test $(f+g)(4) = f(4) + g(4)$

$= 0 + 0$

$= 0$

so $f+g \in V$ too

b/c f, g both in V
so $f(4)=0$, $g(4)=0$

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also (3) for any real α, and $f \in V$

αf is also in V:

Test: $(\alpha f)(4) = \alpha[f(4)]$ | $f(4)=0$ b/c $f \in V$

$= \alpha \cdot 0$

$= 0$

so $(\alpha f) \in V$

so V is closed under scalar mult'n.

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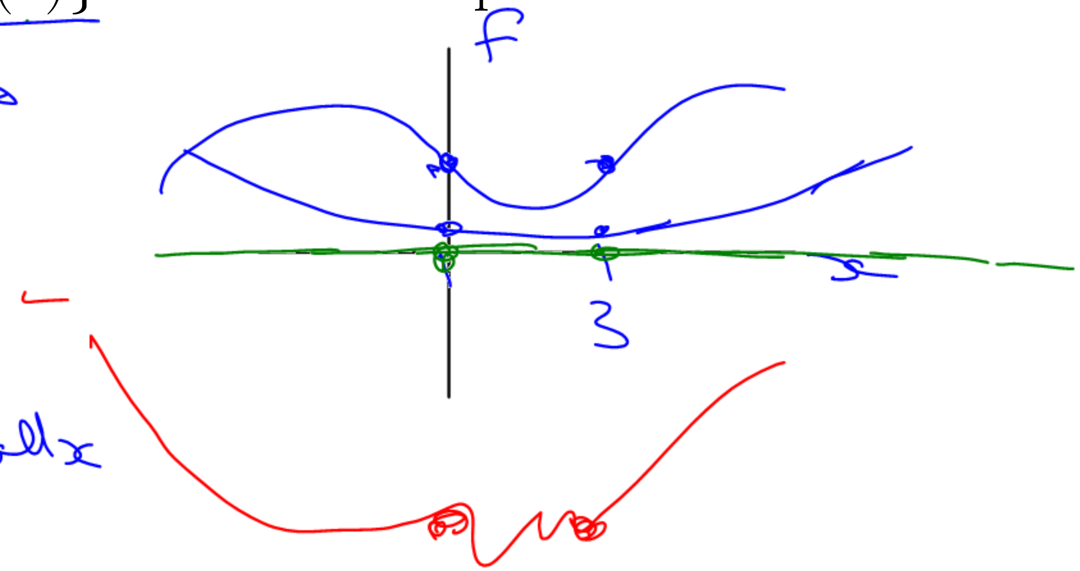
two y/output values

Example: Is the set $\mathbf{W} = \{f \in C^\infty : f(0) = f(3)\}$ a vector subspace of the C^∞ vector space? (Note the f on the RHS this time!)



W is a vector subspace of C^∞

(1) satisfied b/c

$\vec{0}$ in C^∞ is $f(x) = 0$ for all x

and so $f(0) = 0$ } equal

and $f(3) = 0$ → $\vec{0} \in W$

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(2), (3) same logic

(2). Take $f, g \in W \Rightarrow f(0) = f(3)$

and $g(0) = g(3)$

Test $(f+g)$: $(f+g)(0) = f(0) + g(0)$

$= f(3) + g(3)$

$= (f+g)(3)$

| $f(0) = f(3)$, and $g(0) = g(3)$

def'n of $f+g$

so $(f+g)$ satisfies $(f+g)(0) = (f+g)(3)$

so $(f+g) \in W$ too. So W is closed under addition.

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differential equation

Example: Is the set $\mathbf{W} = \{f \in C^\infty : \frac{d}{dx}f(x) = f(x)\}$ a vector subspace of C^∞?

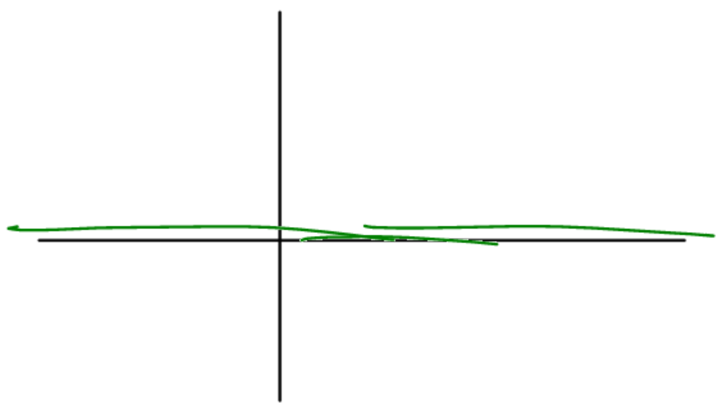
f = its own derivative

(1) $\vec{0}$ is in W b/c if f(x) = 0 for all x's → constant

f'(x) = 0 for all x's.

⇒ f(x) = f'(x) for $\vec{0}$ of C^∞

⇒ $\vec{0} \in W$



$\vec{0} \Rightarrow f(x) = 0$ funct

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$\mathbf{W} = \{f \in C^\infty : \frac{d}{dx}f(x) = f(x)\}$

(1) (2)

(2) Pick $f, g \in W \to f'(x) = f(x)$ and $g'(x) = g(x)$

Test $(f+g)$: $\frac{d}{dx}(f+g)(x) = \left(\frac{d}{dx}f\right)(x) + \left(\frac{d}{dx}g\right)(x)$

$= f'(x) + g'(x)$

$= f(x) + g(x)$ (1), (2)

$= (f+g)(x)$

so $f+g \in W \Rightarrow W$ is closed under addition.

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$\mathbf{W} = \{f \in C^{\infty} : \frac{d}{dx}f(x) = f(x)\}$

(3) Let $a \in \mathbb{R}$, $f \in W$ $\Rightarrow f'(x) = f(x)$ ①

Test (af): $\frac{d}{dx}(af)(x) = a\left(\frac{d}{dx}f(x)\right)$ ↲ ①

$= a f(x)$

$= (af)(x)$

so $af \in W$ too

$\Rightarrow$ W is closed under scalar multiplication.

(1), (2) (3) all satisfied so W is a vector subspace of C^{∞}.

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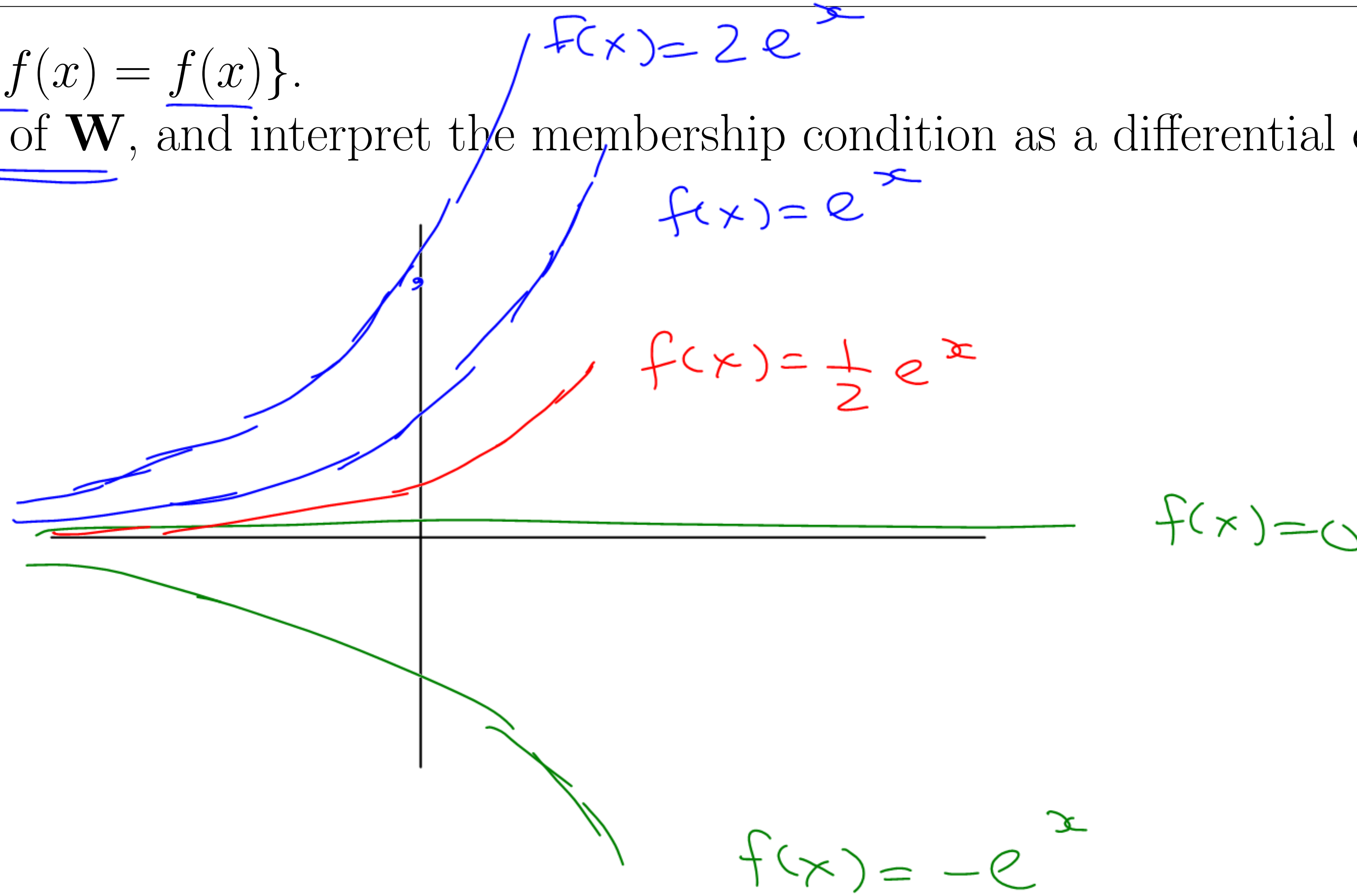
$\mathbf{W} = \{f \in C^\infty : \frac{d}{dx}f(x) = f(x)\}$.

Graph the members of $\mathbf{W}$, and interpret the membership condition as a differential equation.

$a \cdot f(x) \in W$

eg $2e^x$



Does the differential equation $x''(t) = -x(t)$ ring any bells from APSC 171?

sin(t), cos(t)

$mx'' = -x$

$k = 1$, $m = 1$ kg

Example: Is the set $\mathbf{W} = \{x(t) \in C^\infty : \frac{d^2}{dt^2}x(t) = -x(t)\}$ a vector subspace of C^∞?

(1) ✓

(2) ✓

(3) ✓

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$$\mathbf{W} = \{x(t) \in C^\infty : \frac{d^2}{dt^2}x(t) = -x(t)\}$$

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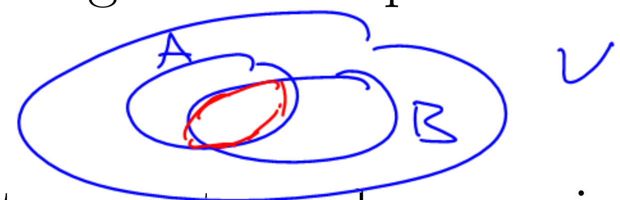
Return to the abstract case. We will be working with subspaces a lot, so having some quick rules we can use will be helpful.



Example: **Prove** that intersection of any two vector subspaces is also a vector subspace. (Ref: Theorem 3 on page 49 of the Lecture Notes.)

Let $A, B \subset V$, with both A, B being vector subspaces of V (a vector space)

Study $A \cap B$.

Need (1) → $\vec{0}$ from V must be in $A \cap B$

Proof: $\vec{0} \in A$ | A is a V.S. (vector space)

also $\vec{0} \in B$ | B " " "

so $\vec{0} \in A \cap B$ | b/c $\vec{0} \in A$ and $\vec{0} \in B$.

Continued: **Prove** that intersection of any two vector subspaces is also a vector subspace.

(2) Need if $u, v \in A \cap B$, then $u+v$ also in $A \cap B$.

$u, v \in A \Rightarrow u+v \in A$ | A is a V.S. closed under addition.

also $u, v \in B \Rightarrow u+v \in B$ | B is a V.S. too

so $u+v \in (A \cap B)$ b/c

$u+v \in A$ <u>and</u> $u+v \in B$.

$\Rightarrow A \cap B$ is closed under addition.

Continued: **Prove** that intersection of any two vector subspaces is also a vector subspace.

(3) If $a \in \mathbb{R}$, $u \in A \cap B$, then $a \circ u$ also $\in A \cap B$

Proof: $u \in A \Rightarrow a \cdot u \in A$ | A is V.S. closed under scalar mult.

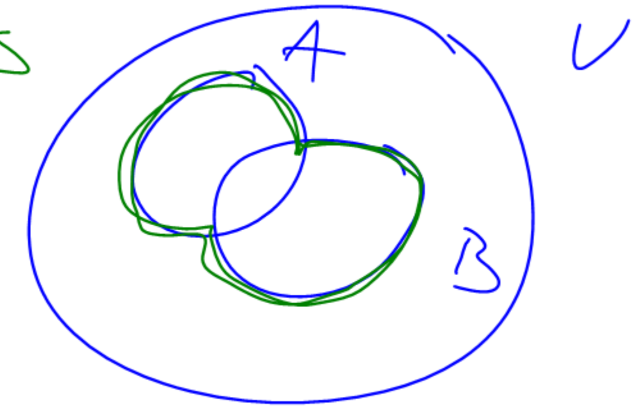
and $u \in B \Rightarrow a \cdot u \in B$ | B is a V.S.

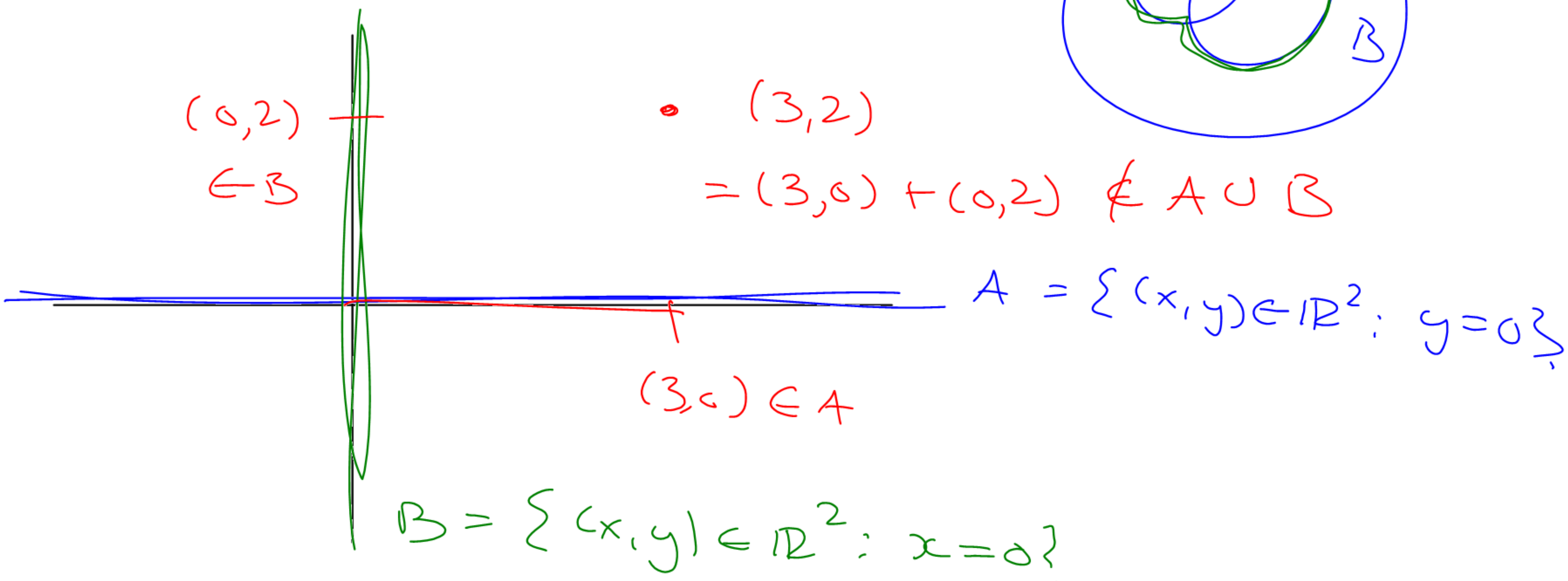
$\Rightarrow a \cdot u \in A \cap B$.

(1)(2) + (3) satisfied $\Rightarrow$ If A and B are known V.S. then $A \cap B$ will also be a V.S.

Example: We now have shown that the **intersection** of any two vector subspaces is also a vector subspace. However, the Notes also indicate that the **union** of two vector subspaces **may or may not be a subspace**. No guarantees
Illustrate this with examples of subsets from $\mathbb{R}^2$.

in $\mathbb{R}^2$





Add inverse your + operator

$$x + (-x) = \vec{0} \quad \text{axiom}$$

inverse

defined by your + operator

or

gives you

$$-1 \cdot x = (-x)$$

your scalar mult

additive invers

Section 4 - Linear Combinations

Once we have a real vector space $(\mathbf{V}, +, \cdot)$, we can safely:

- Add any elements of $\mathbf{V}$, and
- Multiply any elements of $\mathbf{V}$ by a real scalar.

If $\{v_1, v_2, \dots v_p\}$ are vectors/elements in $\mathbf{V}$, what kinds of new vectors can we create? Suppose $\alpha_i \in \mathbb{R}$.

set of single elements

$v_1 + v_2$ (add)

or $a_1 v_1$ for any $a_1 \in \mathbb{R}$

or $a_2 v_2$ " " $a_2 \in \mathbb{R}$

or $a_1 v_1 + a_2 v_2$

or $(a_1 v_1 + a_2 v_2) + a_3 v_2 + \dots a_p v_p$

Linear Combination. Definition: let $v_1, \ldots, v_p$ be a finite collection of elements of the vector space $\mathbf{V}$ (with $p \geq 1$). The expression

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p,$$

a set of

with $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$, is called a **linear combination** of the vectors $v_1, \ldots, v_p$.

Example: write a linear combination of $5 \in \mathbb{R}$ and $7 \in \mathbb{R}$.

a lin comb'n ex: $17 \cdot 5 + 23.794 \cdot 7$

$a_1 \quad v_1 \quad a_2 \quad v_2$

Example: write a linear combination of $[1, 0, 0]$, $[0, 0, 3]$, $[1, -1, 2]$, all $\in \mathbb{R}^3$.

ex $\quad 4[1,0,0] + 7.9[0,0,3] + 0[1,-1,2]$

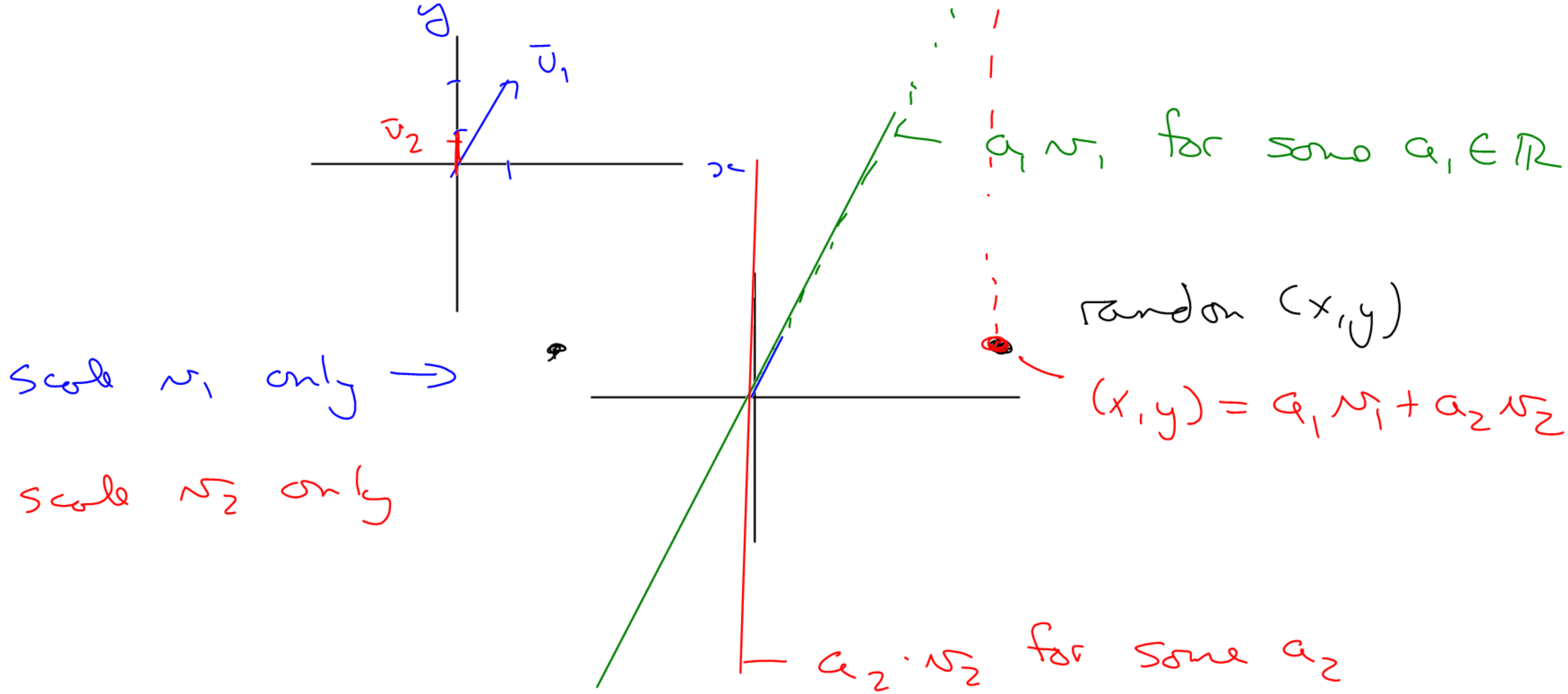
in general $\quad a_1[1,0,0] + a_2[0,0,3] + a_3[1,-1,2]$,

Example: write a linear combination of $\cos(t)$, $\sin(t)$, both $\in C^\infty$.

ex $\quad 3.5\cos(t) - 7\sin(t)$

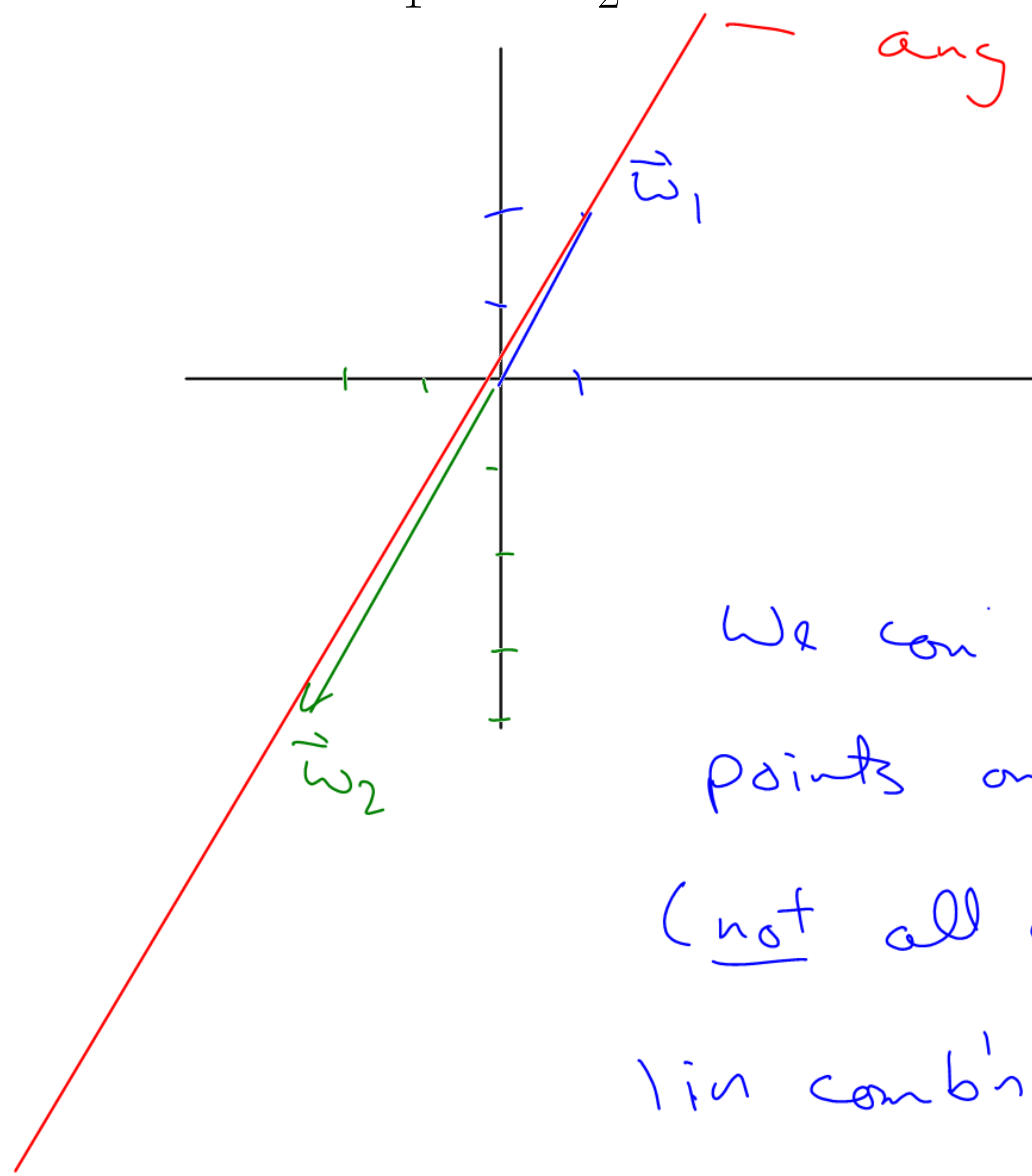
or $\quad$ general case $\quad A\cos(t) + B\sin(t)$

Example: If we take the vectors $v_1 = [1, 2], v_2 = [0, 1] \in \mathbb{R}^2$, what parts of $\mathbb{R}^2$ can we reach with linear combinations of v_1 and v_2?



We can reach <u>any</u> $(x, y) \in \mathbb{R}^2$ w/ a linear comb'n of $\{\vec{v}_1, \vec{v}_2\}$.

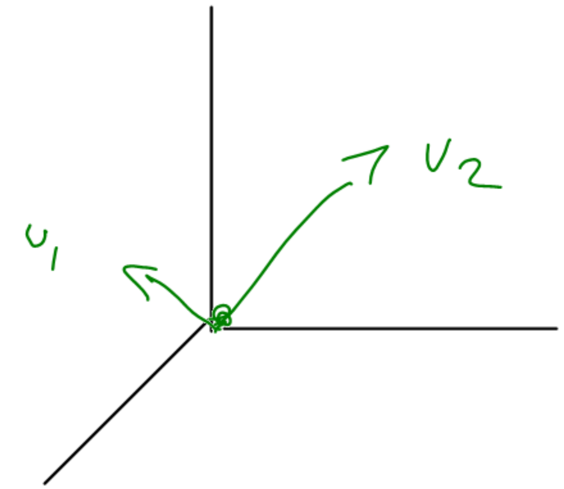
Example: If we take the vectors $w_1 = [1, 2], w_2 = [-2, -4] \in \mathbb{R}^2$, what parts of $\mathbb{R}^2$ can we reach with linear combinations of w_1 and w_2?



any $a_1 w_1 + a_2 w_2$ only on this line. :(

We can only reach points on the red line (<u>not</u> all of $\mathbb{R}^2$) w/ lin comb'ns of $\{w_1, w_2\}$.

Example: Follow the process below for $\mathbb{R}^3$.

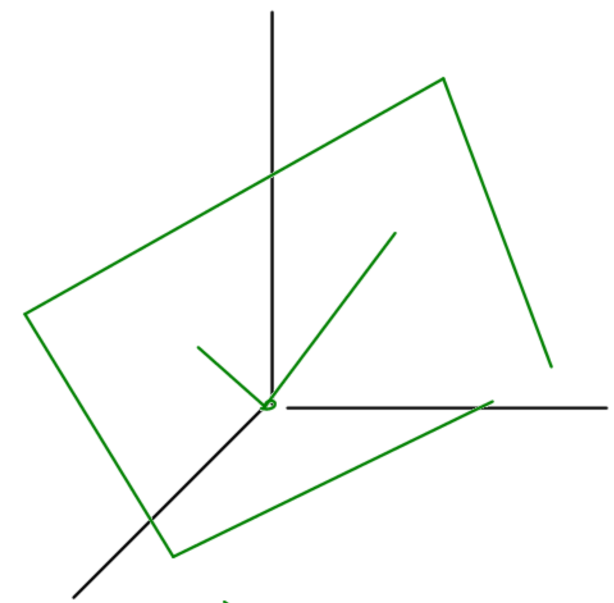


- We start with a known vector space $\mathbb{R}^3$.

- We pick a finite number of elements, $v_1, \ldots, v_p \in \mathbb{R}^3$ as 'seeds' or 'building blocks'.

  v_1, v_2

- We imagine the subset of $\mathbb{R}^3$ we can cover or reach with *linear combinations* of $v_1, \ldots, v_p$.



any $a_1 v_1 + a_2 v_2$

will lie on a plane.

(through $(0,0,0)$ w/ $0v_1 + 0v_2$)

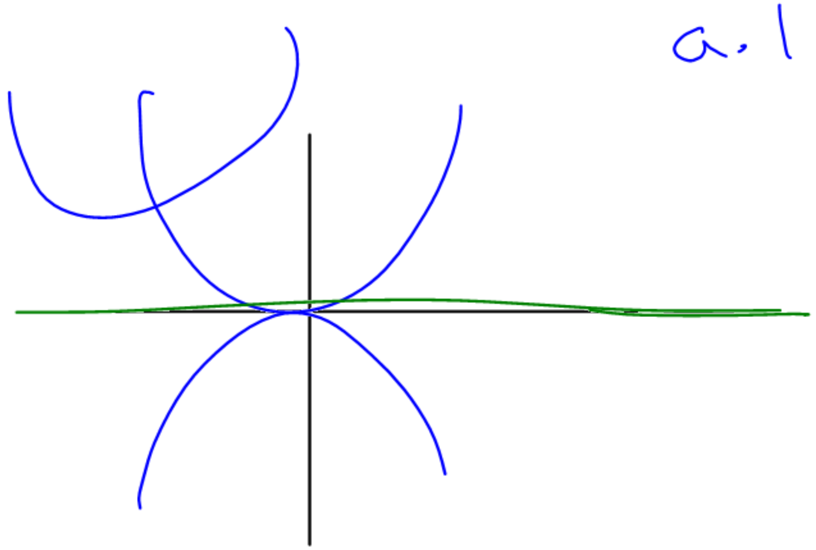
Follow the process below for C^∞.

- We start with a vector space C^∞.   functions continuous, inf'ly diff'ble

- We pick a finite number of elements, $v_1, \ldots, v_p \in C^\infty$ as 'seeds' or 'building blocks'. $\{1, x, x^2\}$

- We imagine the subset of C^∞ we can cover or reach with *linear combinations* of $v_1, \ldots, v_p$.

$a \cdot 1 + b x + c x^2 \rightarrow$ all possible quadratics.



$\vec{0}$ func $\in$ lin comb'ns

b/c $0 \cdot 1 + 0 \cdot x + 0 x^2$ is included.

Linear Combinations as Vector Subspaces. Proposition: Let $(\mathbf{V}, +, \cdot)$ be a real vector space, and $v_1, \ldots, v_p$ be a finite number of elements of the vector space $\mathbf{V}$ (with $p \geq 1$). The subset $S \subset \mathbf{V}$ consisting of all linear combinations of $v_1, \ldots, v_p$, is a vector subspace of the original $\mathbf{V}$.

The set S is called the **span** (or 'linear span') of the vectors $v_1, \ldots, v_p$.

lin comb'n of $\{v_1, v_2\}$ = span of $\{v_1, v_2\}$

$$S(v_1, v_2 \ldots v_p) = \{ w \in V : w = a_1 v_1 + a_2 v_2 + \ldots a_p v_p, \text{ for some } a_1, a_2 \ldots a_p \in \mathbb{R} \}.$$

Example: defining vector subspaces of $\mathbb{R}^n$.

Tu/Wed

Known vector space R^n

Today

Rule for selecting subset of R^n

eg $\{(x,y) \in \mathbb{R}^2 : x+y=1\}$ ✗

or $\{(x,y) \in \mathbb{R}^2 : x+y=0\}$ ✓

Pick some elements of R^n

pick $[1,2], [0,1]$ in $\mathbb{R}^2$

Verify 3 rules for vector subspace

(1) $\bar{0}$ in set

(2) closed add

(3) " scalar mult

Span will automatically be a vector subspace

$\{a_1[1,2] + a_2[0,1]$ for any $a_1, a_2 \in \mathbb{R}\}$

is a vector subspace. No 3 axioms req.

Example: defining vector subspaces of C^∞

Known vector space C^∞

Two ways

Rule for selecting subset of C^∞

$f(0) = 3$ ✗

$\frac{d}{dx} f = f$ ✓

Pick some elements of C^∞

$\{\sin(t), \cos(t)\}$

or

$\{1, x, x^2\}$

Verify 3 rules for vector subspace

(1)

(2)

(3)

Span will automatically be a vector subspace

all possible lin comb'ns.

Example: Prove, using the 3 requirements for a vector subspace, that the span of any finite set $v_1, \ldots, v_p \in \mathbf{V}$ will always be a vector subspace.

so our set $S = \{ a_1 v_1 + \ldots a_p v_p \in V : a_1 \ldots a_p \in \mathbb{R} \}$.

(1) Is $\vec{0} \in S$?

Yes: pick $a_1 = a_2 \ldots = a_p = 0 \ (\in \mathbb{R})$

so $a_1 \cdot v_1 = 0 \cdot v_1 = \vec{0}$ in V (proved that $0 \cdot v = \vec{0}$ in wk 2)

and $a_2 v_2 = 0 \cdot v_2 = \vec{0}$

so $a_1 v_1 + a_2 v_2 + \ldots a_p v_p = \bar{0} + \bar{0} + \bar{0} \ldots \bar{0} = \bar{0}$ is a lin comb of $\{v_1, \ldots v_p\}$

by $v + \bar{0} = v$ axiom ($v = \bar{0}$)

(1) The subset $\mathbf{W}$ contains the zero element $\mathbf{0}$ of $(\mathbf{V}, +, \cdot)$.
(2) $\mathbf{W}$ is closed under addition. I.e. for any $x, y \in \mathbf{W}$, the sum $x + y \in \mathbf{W}$ too.
(3) $\mathbf{W}$ is closed under scalar multiplication. I.e. for any real α and element $x \in \mathbf{W}$, the product $\alpha x \in \mathbf{W}$ too.

(2) Is S closed under addition? Yes!

Pick 2 elements of S, $w = (a_1 v_1 + a_2 v_2 + \dots a_p v_p)$ (some lin comb'n of v's)

and $r = (b_1 v_1 + b_2 v_2 + \dots b_p v_p)$ (another lin comb'n of v's)

Add them $w + r = a_1 v_1 + b_1 v_1 + a_2 v_2 + b_2 v_2 + \dots + a_p v_p + b_p v_p$

distributivity axiom

$= (a_1 + b_1) v_1 + (a_2 + b_2) v_2 + \dots + (a_p + b_p) v_p$

(each coefficient $\in \mathbb{R}$)

$\in S$ b/c it's a lin comb'n of $\{v_1, \dots v_p\}$

(1) The subset **W** contains the zero element **0** of $(\mathbf{V}, +, \cdot)$.
(2) **W** is closed under addition. I.e. for any $x, y \in \mathbf{W}$, the sum $x + y \in \mathbf{W}$ too.
(3) **W** is closed under scalar multiplication. I.e. for any real α and element $x \in \mathbf{W}$, the product $\alpha x \in \mathbf{W}$ too.

(3) Is S closed under scalar multiplication? Yes

Let $w = a_1 v_1 + \dots a_p v_p \in S$

and $k \in \mathbb{R}$

distributivity

Then $k \cdot w = k(a_1 v_1 + \dots + a_p v_p)$

$= \underbrace{(a_1 \cdot k)}_{\in \mathbb{R}} v_1 + \dots + \underbrace{(a_p \cdot k)}_{\in \mathbb{R}} v_p \in S$ too

Vector spaces give us a new way to think about solutions to systems of linear equations.

Example: Consider the system of linear equations below.

$$x_1 + 3x_2 = 1$$
$$2x_1 + x_2 = 3$$
$$4x_1 - x_2 = -3$$

3 eq'n

2 unknowns

Extract the x_i's as *coefficients* of $\mathbb{R}^3$ vectors:

1st compare → $x_1(1) + x_2(3) = 1$

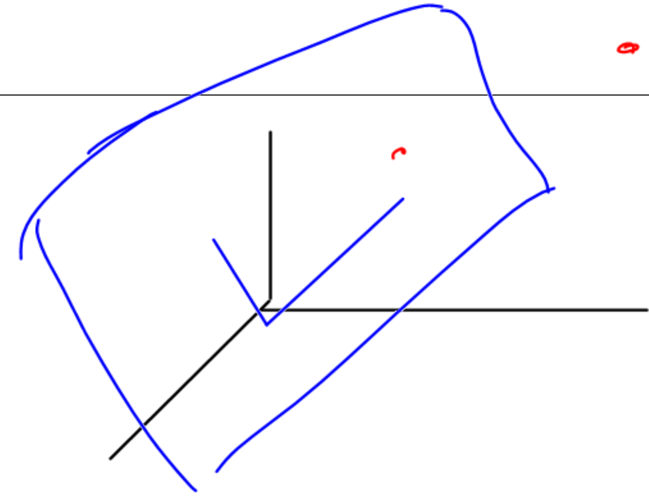
2nd $x_1(2) + x_2(1) = 3$ → $x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$

3rd $x_1(4) + x_2(-1) = -3$

What does the LHS of the equation now look like?

lin comb'n $a_1 v_1 + a_2 v_2$

$$x_1 + 3x_2 = 1$$
$$2x_1 + x_2 = 3$$
$$4x_1 - x_2 = -3$$



$$x_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

Represent the question "Does this system of equations have a solution?" in new terms, using span.

all possible x_1, x_2 s

$= \text{span of } \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \right\}$

Is $\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$ in the span of $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \right\}$?

output/RHS

Note: this hasn't yet helped us find an answer faster, but it *is* a new way to look at an old problem!

Example: Let us consider a system of equations related to $\mathbb{R}^2$.

$$\begin{aligned} x_1 + 2x_2 &= a \\ 4x_1 - 3x_2 &= b \end{aligned}$$

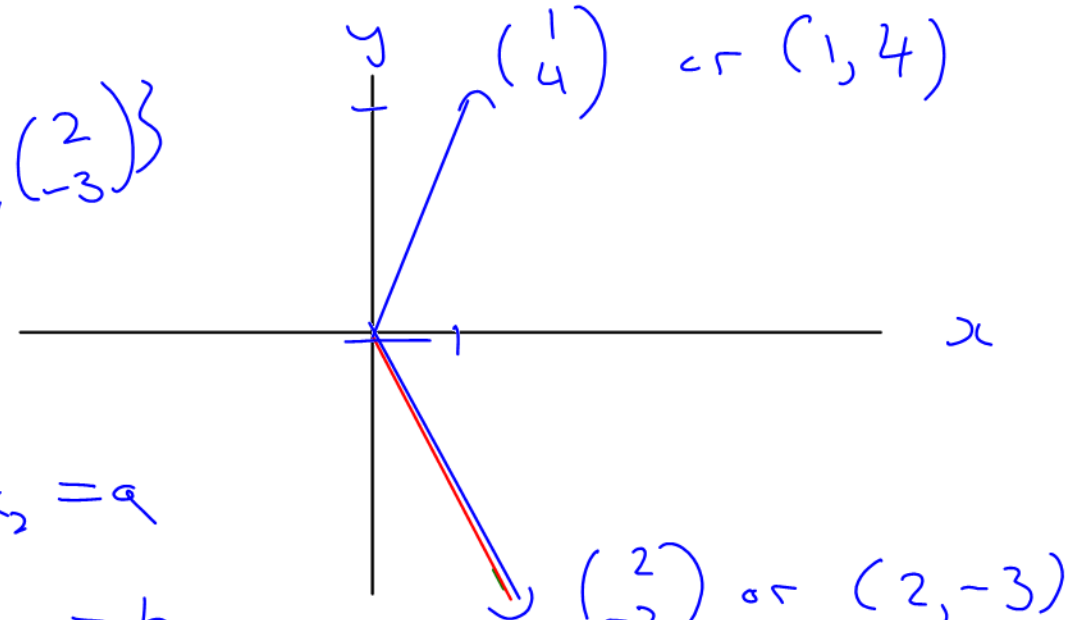
$\to$ is $\begin{pmatrix} a \\ b \end{pmatrix}$ in span $\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right\}$?

Use a span argument to prove that, no matter what values are picked for a and b, there **will** be a solution to this system.

The span of $\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right\}$ is all of $\mathbb{R}^2$

$$x_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$\Rightarrow$ for any $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix}$ is in the span of $\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right\}$



$\Rightarrow$ for any a, b, $\begin{aligned} x_1 + 2x_2 &= a \\ 4x_1 - 3x_2 &= b \end{aligned}$ has a solution.

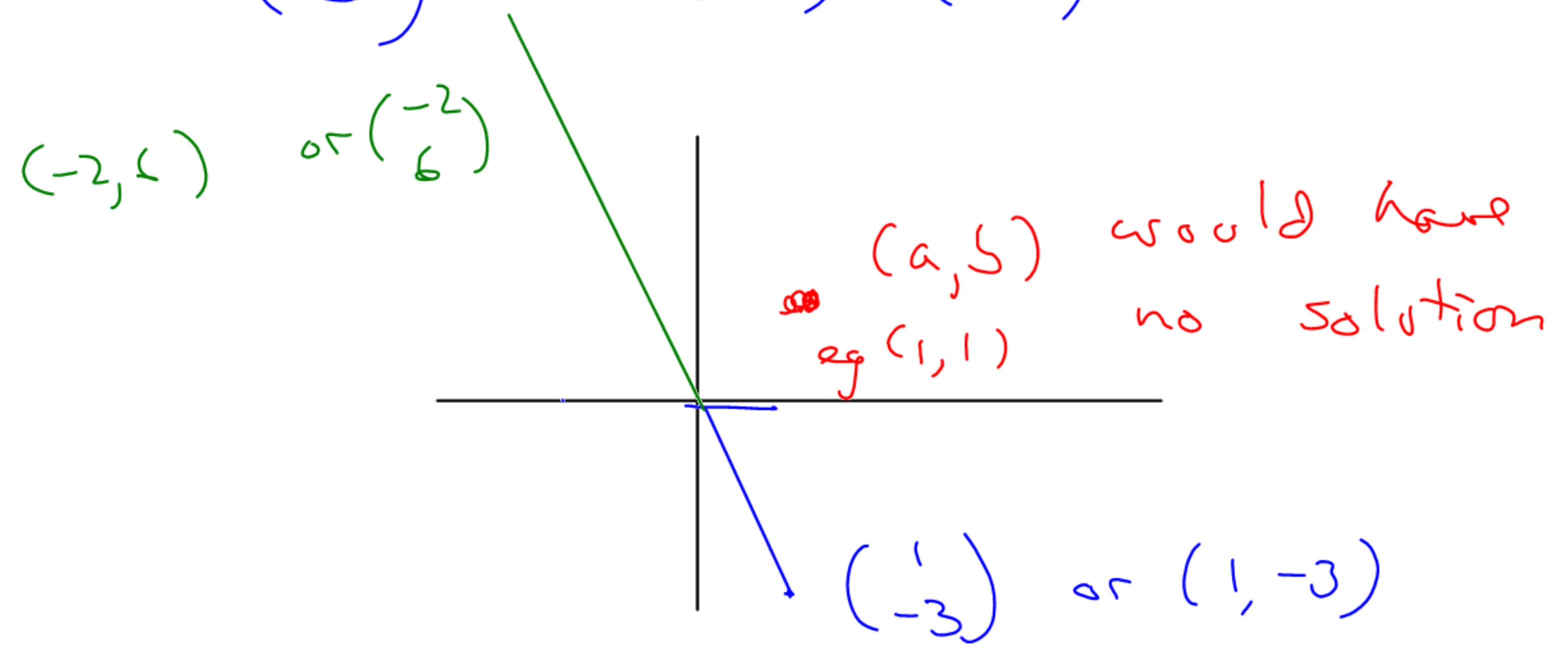
Example continued.

Example: Let us consider another system of equations related to $\mathbb{R}^2$.

$$\begin{aligned} 1\, x_1 - 2x_2 &= a \\ -3x_1 + 6x_2 &= b \end{aligned}$$

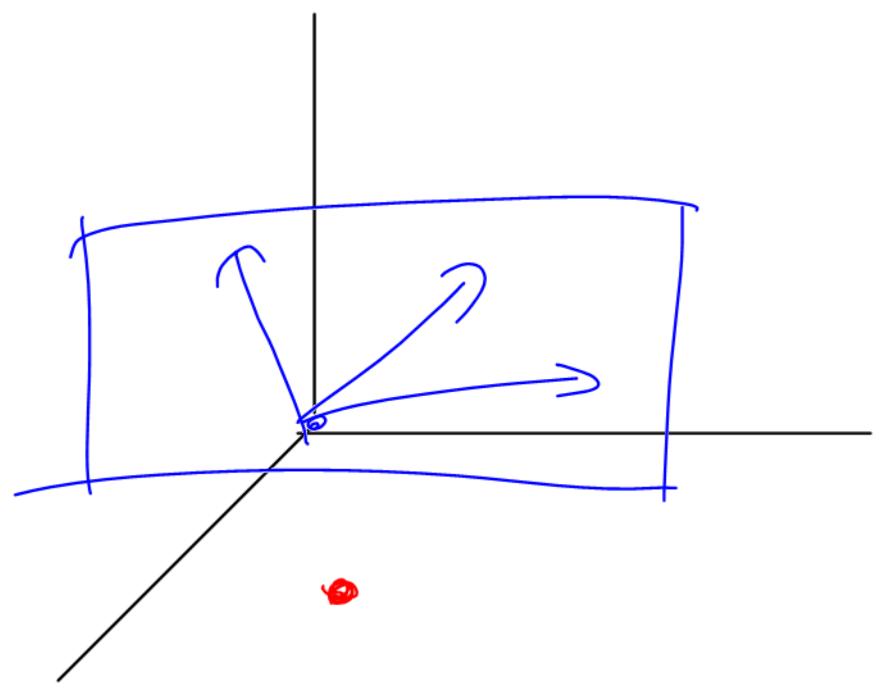
Use a span argument to prove that, for some a and b values for this system, there will **not** be a solution.

$$x_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$



$$a[1,0,0]+b[0,1,0]+c[0\ 0\ 1]=[\ ,\ ,\ ]$$

$$a(1)+b(x)+c\cdot x^2 = 3+7x+8x^2$$



$1+x^2$

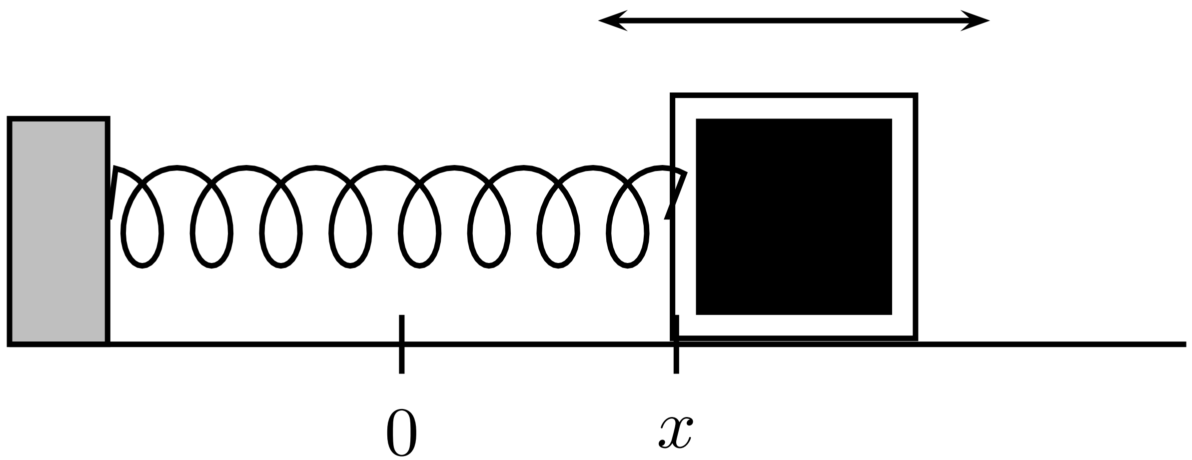
$1+x$

$x+x^2$

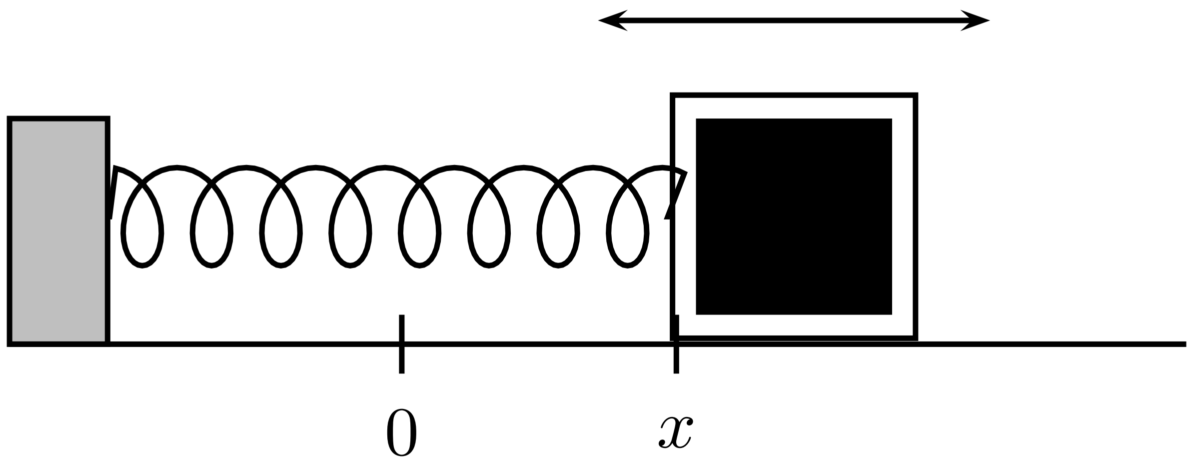
Example continued.

Vector spaces can also help us to better understand the solutions to the spring/mass differential equation from APSC 171 and APSC 112.

Example: Consider the spring system shown below.



What is the differential equation that the motion of the mass must follow?



$$mx''(t) + kx(t) = 0$$

What does a **solution**, $x(t)$, to the differential equation represent?

Prove that the set of all solutions to $mx''(t) + kx(t) = 0$ is a vector subspace of C^∞:
$\mathbf{W} = \{x(t) \in C^\infty \ : \ x(t) \text{ satisfies } mx''(t) + kx(t) = 0\}$

$$mx''(t) + kx(t) = 0$$

Continued.

$$mx''(t) + kx(t) = 0$$

$$\text{or } x''(t) = -\frac{k}{m}x(t)$$

Now, consider two **known** solutions to the differential equation (found in APSC 171):

- $x_1(t) =$

- $x_2(t) =$

Describe how the following two sets are related:

- set of all possible solutions to differential equation
$$mx'' + kx = 0, \text{ and}$$
- the span of the two simple solutions
$$\left\{ \sin\left(\sqrt{\frac{k}{m}}\, t\right), \cos\left(\sqrt{\frac{k}{m}}\, t\right) \right\}.$$

If time available, further examples or proofs of subspace and span relationships.