Week #4: Linear Independence

- Linear Combinations - Continued
- Definitions of Linear Independence and Linear Dependence

Qlicker question open now.

Qlicker enrolment code:

QM3CXK

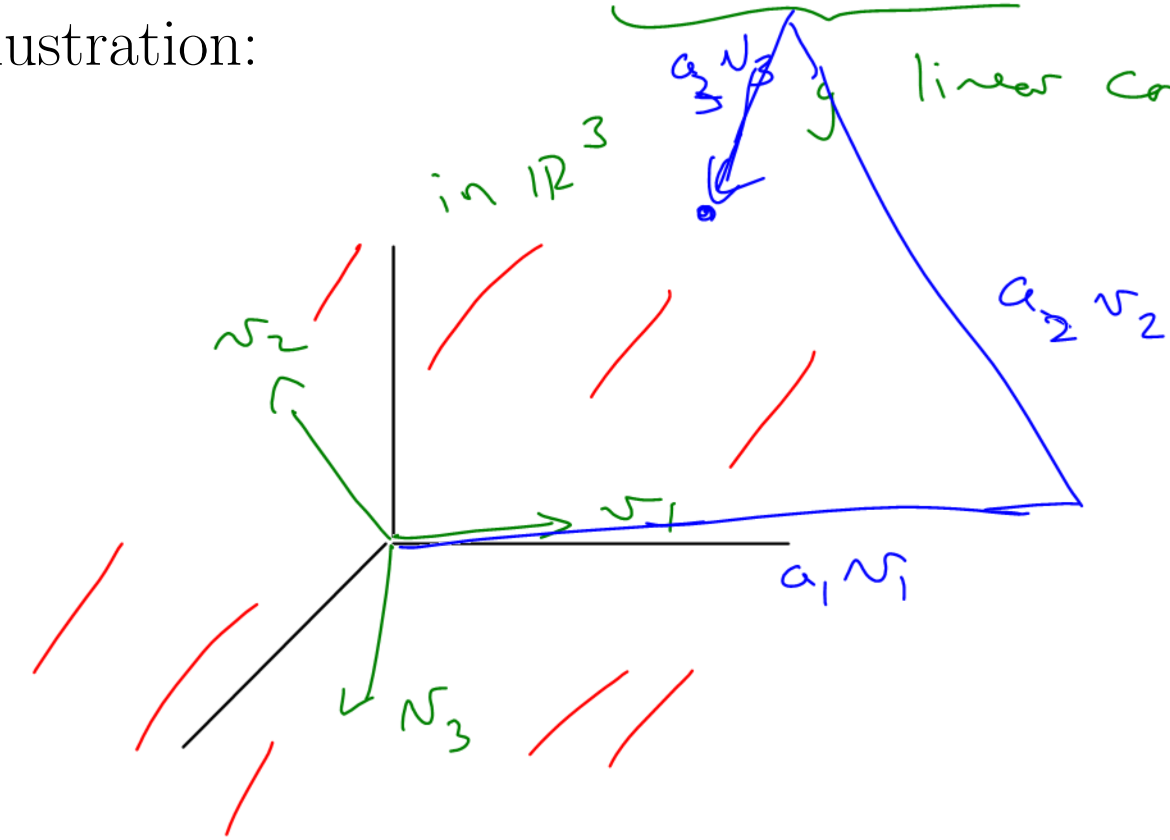
Section 4 - Linear Combinations and Span - Continued

Recall the relationship between building blocks

- a vector set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and
- the **span** of that vector set,
  $S = \{\mathbf{w} \in \mathbf{V} : \mathbf{w} = \alpha_1\mathbf{v}_1 + \ldots + \alpha_p\mathbf{v}_p\}$ for some $\alpha_i \in \mathbb{R}$.

linear combination of building blocks

Illustration:



span of $\{v_1, v_2, v_3\}$

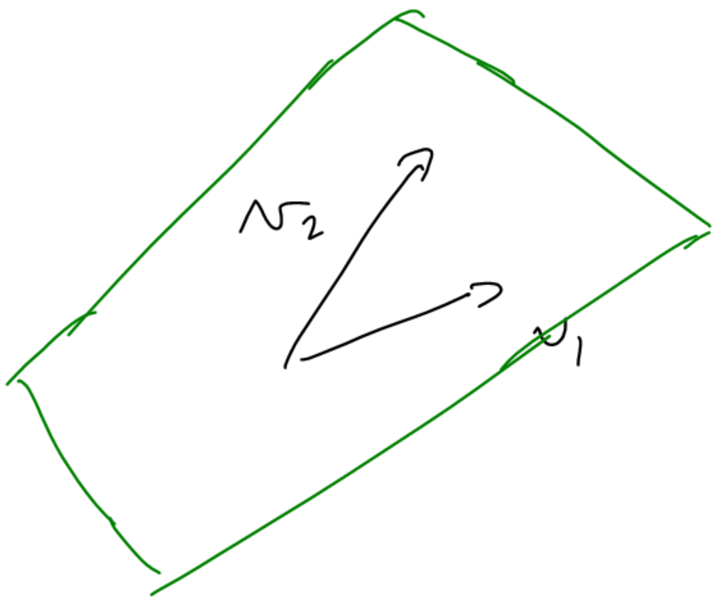
Example: Consider the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ below.
Is the vector $\mathbf{w} \in \mathbb{R}^4$ in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ -2 \end{pmatrix} \qquad \boxed{\mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ -3 \\ 0 \end{pmatrix}}$$

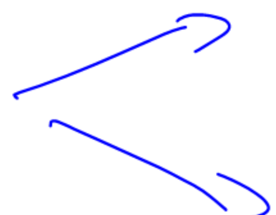
Qlicker

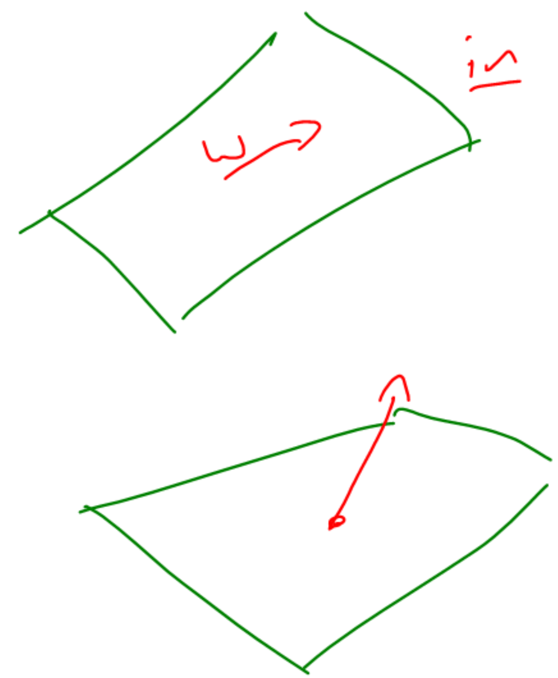
Is w a linear comb'n of $\{v_1, v_2\}$?

- obvious? Not usually.



span of $\{v_1, v_2\}$





in the hyper-plane

w not in hyperplane

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ -2 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ -3 \\ 0 \end{pmatrix}$$

Usually we need a calculation to determine if $w \in \text{span}\{v_1, v_2\}$.

Does

$$a \underbrace{\begin{pmatrix} 1 \\ \boxed{0} \\ -2 \\ 1 \end{pmatrix}}_{v_1} + b \underbrace{\begin{pmatrix} -1 \\ \boxed{1} \\ 3 \\ -2 \end{pmatrix}}_{v_2} = \begin{pmatrix} 2 \\ \boxed{1} \\ -3 \\ 0 \end{pmatrix}$$

have a solution w/ $a, b \in \mathbb{R}$?

components:

$$a \cdot 1 + b(-1) = 2 \quad (1)$$

$$\boxed{b = 1} \quad (2)$$

$$a(-2) + b(3) = -3 \quad (3)$$

$$a(1) + b(-2) = 0 \quad (4)$$

$a + (1)(-1) = 2 \to a = 2+1 = 3$

Check $a = 3, b = 1$ works for (3), (4)

(3) $\to$ LHS $= (3)(-2) + 1 \cdot 3 = -3$ ✓ "RHS"

(4) $\to$ LHS $= 3 + 1(-2) = 1 \neq 0 =$ RHS

$\Rightarrow$ no solution a, b that satisfies all equations

$\Rightarrow$ w is <u>not</u> a lin comb'n of $v_1\ v_2$ $\Rightarrow$ not in span $\{v_1, v_2\}$.

Example: Consider the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ below.
Is the vector $\mathbf{w} \in \mathbb{R}^{\cancel{4}\,3}$ in the Span$(\mathbf{v}_1, \mathbf{v}_2)$?

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 0 \\ 12 \\ -4 \end{pmatrix}$$

i.e. is there a solution $a, b \in \mathbb{R}$ such that

$$\underbrace{a \underbrace{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}}_{v_1} + b \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}_{v_2}}_{\text{lin comb'n of } \{v_1, v_2\}} = \underbrace{\begin{pmatrix} 0 \\ 12 \\ -4 \end{pmatrix}}_{w}$$

components:

$$a(-2) + b(1) = 0 \quad (1) \quad \rightarrow \quad b = 2a \quad (4)$$

$$a(-1) + b(2) = 12 \quad (2) \qquad (4) \rightarrow (2) \qquad -a + 2(2a) = 12$$

$$3a = 12$$

$$a(1) + b(-1) = -4 \quad (3)$$

$$a = 4$$

$$\hookrightarrow (4) \quad b = 2a = 8$$

Check (3): $\text{LHS} = 4 + (-1)8 = -4 = \text{RHS}$ ✓

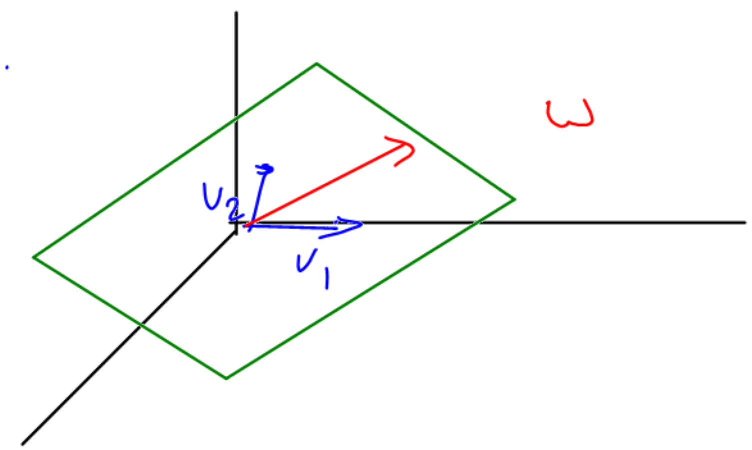
$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 0 \\ 12 \\ -4 \end{pmatrix}$$

we found

$$\underbrace{4}_{a}\underbrace{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}}_{v_1} + \underbrace{(8)}_{b}\underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}_{v_2} = \underbrace{\begin{pmatrix} 0 \\ 12 \\ -4 \end{pmatrix}}_{w} \checkmark$$

so w is a lin comb'n of $\{v_1, v_2\}$

$\Rightarrow$ w is in the span $\{v_1, v_2\}$.



Example: Is the function (vector) $\mathbf{w} \in C^\infty(\mathbb{R})$ in the span of the two functions (vectors) $\mathbf{v}_1$ and $\mathbf{v}_2$ shown below?

$$\mathbf{v}_1(x) = 3 + x^2$$
$$\mathbf{v}_2(x) = -2x + x^2 + x^3$$
$$\mathbf{w}(x) = 3 + 2x - x^3$$

Is there a solution $a, b \in \mathbb{R}$ for $a v_1 + b v_2 = w$?

$$a\begin{pmatrix} 3 \\ +\boxed{0}x \\ +x^2 \\ +0x^3 \end{pmatrix} + b\begin{pmatrix} 0 \\ \boxed{-2}x \\ +x^2 \\ +x^3 \end{pmatrix} = \begin{pmatrix} 3 \\ \boxed{+2}x \\ +0x^2 \\ -x^3 \end{pmatrix}$$

components/coefficient equalities:

const: $a \cdot 3 + b \cdot 0 = 3 \to a = +1$

x: $a \cdot 0 + b(-2) = 2 \to (-1)(-2) = 2$ ✓

x^2: $a \cdot 1 + b \cdot 1 = 0 \to 1 \cdot 1 + (-1) \cdot 1 = 0$ ✓

x^3: $\cancel{a \cdot 0} + \underline{b \cdot 1 = -1}$ $\quad b = -1$

$$\mathbf{v}_1(x) = 3 + x^2$$
$$\mathbf{v}_2(x) = -2x + x^2 + x^3$$
$$\mathbf{w}(x) = 3 + 2x - x^3$$

we found
$$\underbrace{(1)}_{a}\underbrace{(3+x^2)}_{v_1} + \underbrace{(-1)}_{b}\underbrace{(-2x+x^2+x^3)}_{v_2} = \underbrace{3+2x-x^3}_{w}$$

so w <u>IS</u> a lin comb'n of v_1, v_2

$\Rightarrow$ w <u>IS</u> in the span of $\{v_1, v_2\}$.

Note: Last week, we saw that
"Does this system of linear equations have a solution?"
could be turned into
"Is the RHS (as a vector) in the span of the LHS (as vectors)?"

is this

$$\begin{aligned} 1x_1 + 3x_2 &= 1 \\ 2x_1 + x_2 &= 3 \\ 4x_1 - x_2 &= -3 \end{aligned}$$

$$x_1 \underbrace{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}_{v_1} + x_2 \underbrace{\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}}_{v_2} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

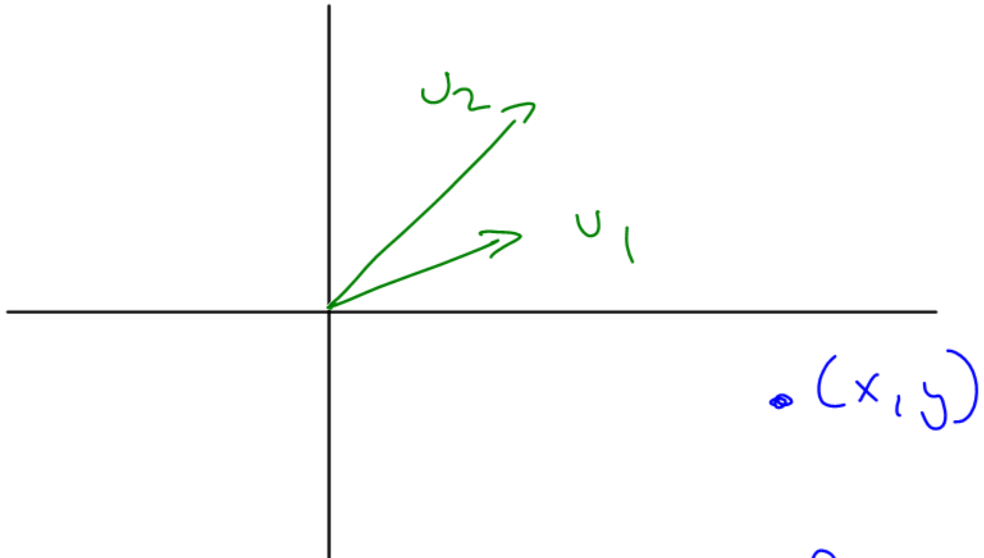
in span of $\{v_1, v_2\}$?

Q: does it have 0, 1 or ∞ # of sol'ns?

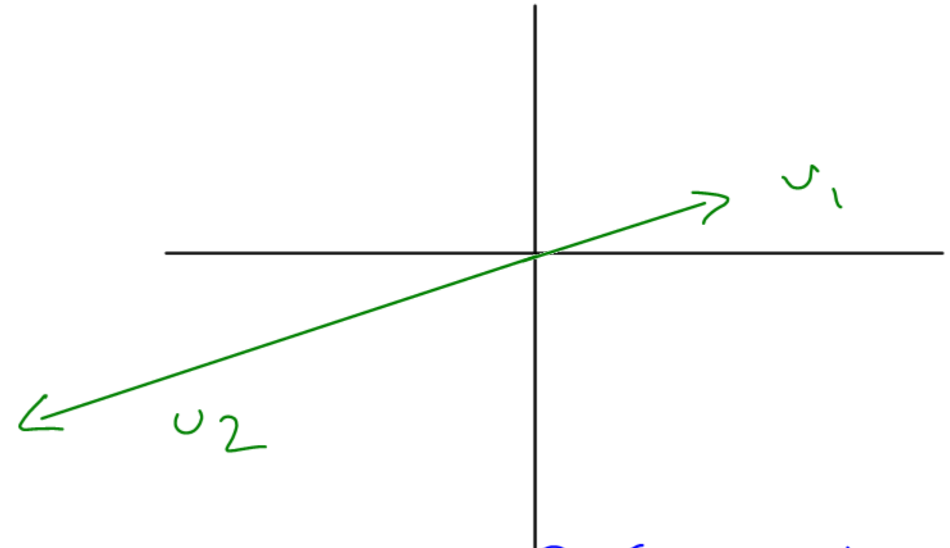
Now we are seeing that, to answer the question "is the vector $\mathbf{w}$ in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$?", we set up and try to solve a linear system of equations!

Section 5 - Linear Independence and Linear Dependence

Recall: if we pick a set of two vectors in $\mathbb{R}^2$, when will the **span** of those vectors **not** be all of $\mathbb{R}^2$?



vs



span $\{v_1, v_2\}$ is all of $\mathbb{R}^2$

Proof would be solving

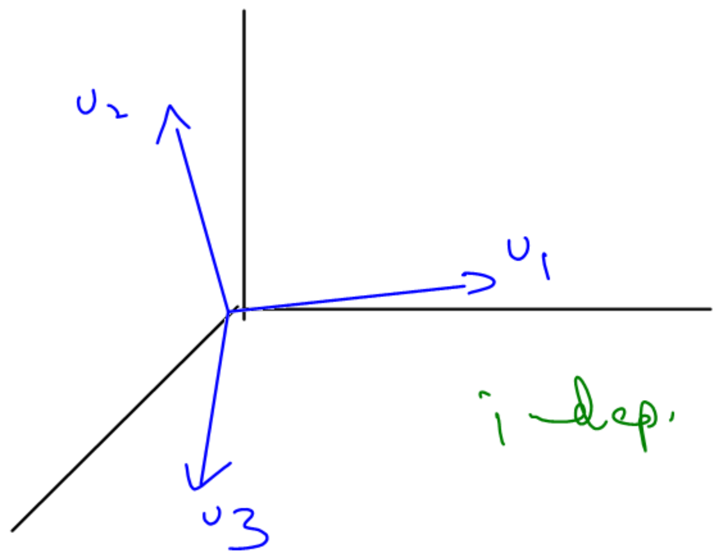
$\to$ $a(v_1) + b(v_2) = \begin{pmatrix} x \\ y \end{pmatrix}$

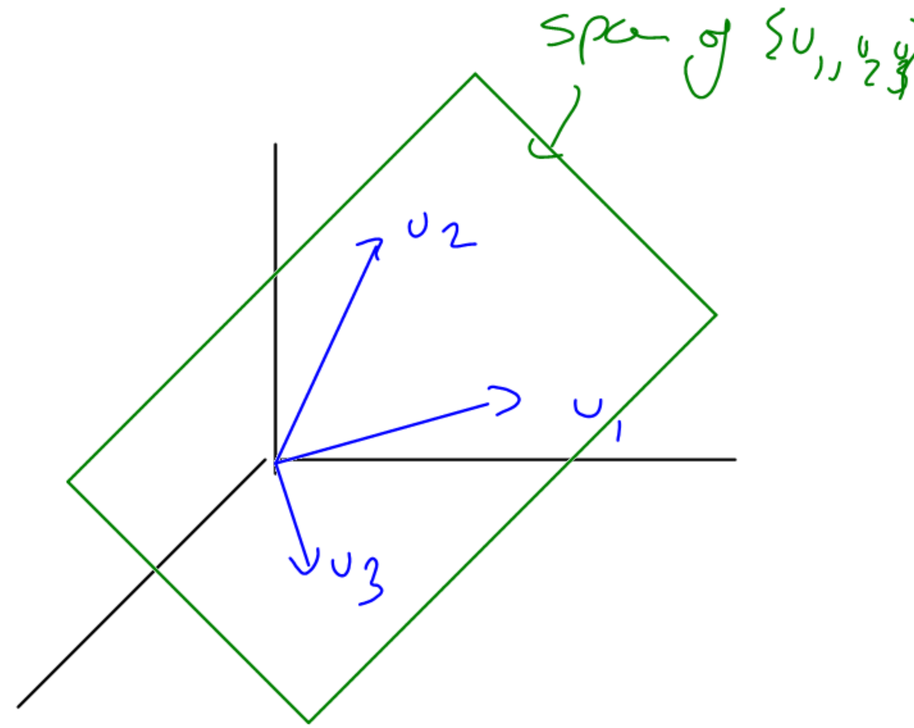
for any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

span of $\{v_1, v_2\}$ is not all of $\mathbb{R}^2$; it's just points on the line $\parallel$ to v_1 and/or v_2

Span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$: $S \subset \mathbf{V}$ consisting of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

Example: if we pick a set of **three** vectors in $\mathbb{R}^3$, when will the span of those vectors **not** be all of $\mathbb{R}^3$?





span of $\{u_1, u_2, u_3\}$ is all of $\mathbb{R}^3$

u_1, u_2, u_3 are all within a plane together

$\Rightarrow$ span of $\{u_1, u_2, u_3\}$ is not all of $\mathbb{R}^3$

Example: Compare the span of the following sets of functions from C^∞:

$\{1, 1+x, 1+x^2\}$

1

$1+x$

$1+x^2$

$\xrightarrow{-} x$

$\xrightarrow{+} x^2$

span $\{1, 1+x, 1+x^2\}$

= all of P_2 / all quadratic poly's

independent

$\{1, 1-x^2, 3+2x^2\}$

1

$1-x^2$

$3+2x^2$

at a glance, no x^1's in the set

$\Rightarrow$ span is not all of P_2

can't build eg. $3+x$

absent from $\{1, 1-x^2, 3+2x^2\}$

Linear Independence and Linear Dependence: generalizing the 2D ‘vectors are parallel’ idea.

Definition: The finite subset $p = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of a vector space $\mathbf{V}$ is **linearly independent** if, for any

$\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R}$, the relation

all cancel

try combine $= 0$

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_p\mathbf{v}_p = \mathbf{0}$$

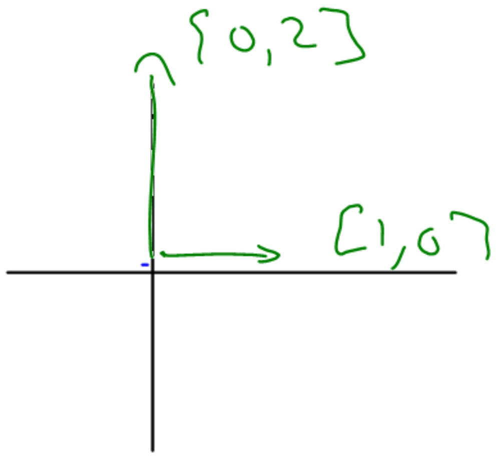
implies that all the $\alpha_1, \alpha_2, \ldots, \alpha_p$ must be zero.

ends up back at origin

multiplies

trajectory

Example: show that $\{[1,0],[0,2]\}$ is a linearly independent set.



claim: these are lin'ly independent

proof: solving

$$a[1,0] + b[0,2] = [0,0]$$

gives

$$a \cdot 1 + \cancel{b \cdot 0} = 0 \rightarrow a = 0$$

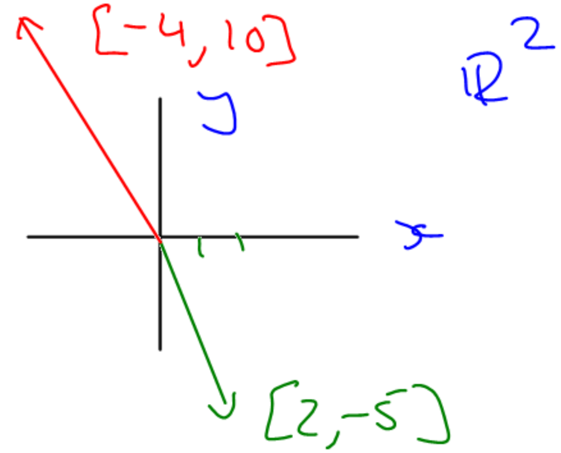
$$\cancel{a \cdot 0} + b \cdot 2 = 0 \rightarrow b = 0$$

is the only solution to $a v_1 + b v_2 = \bar{0}$

then $\{v_1, v_2\}$ are lin'ly indep't.

Alternative definition: a set of vectors is linearly **dependent** if there exists a non-zero set of coefficients α_i such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_p \mathbf{v}_p = \mathbf{0}$$



Example: show that $\{[2,-5],[-4,10]\}$ is a linearly dependent set.

Look at $\alpha_1 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + \alpha_2 \begin{bmatrix} -4 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Ask:
- is the only α_1, α_2 solution $\alpha_1 = 0$, $\alpha_2 = 0$
$\Rightarrow$ lin'ly indep.

components:

$$\begin{cases} \alpha_1 \cdot 2 + \alpha_2 \cdot (-4) = 0 & \times 5 \\ \alpha_1 (-5) + \alpha_2 (10) = 0 & \times 2 \end{cases}$$

$\alpha_1 10 + \alpha_2 (-20) = 0$

$+ \cancel{\alpha_1 (-10) + \alpha_2 (20) = 0}$

redundant

- any non-zero solution $\Rightarrow$ lin'ly dependent.

or $\div 10$

$\alpha_1 + (-2)\alpha_2 = 0$ $\quad \alpha_1 = 2\alpha_2$

pick $\alpha_2 = 1$
$\Rightarrow \alpha_1 = 2$ is a

Example: show that $\{[2,0,1],[3,1,1],[4,2,1]\}$ is a linearly dependent set.

Look at
$$\underset{1}{\cancel{a}}\begin{bmatrix}2\\0\\1\end{bmatrix}+\underset{-2}{\cancel{b}}\begin{bmatrix}3\\1\\1\end{bmatrix}+\underset{1}{\cancel{c}}\begin{bmatrix}4\\2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Look for possible a,b,c values

components:

$2a + 3b + 4c = 0$ (1)

$b + 2c = 0$ (2) $\rightarrow b = -2c$ (4) $\rightarrow b = -2a$ (5)

$a + b + c = 0$ (3)

$c = a$

(4) → (3): $a + (-2c) + c = 0 \rightarrow a - c = 0 \rightarrow c = a$ (6)

(5),(6) → (1): $2a + -6a + 4a = 0$

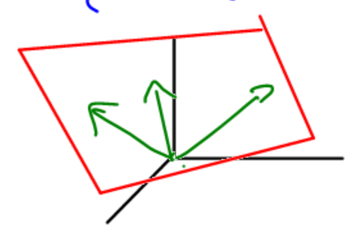
$0a = 0$ always true. redundant eq'n.

pick $a = 1 \rightarrow b = -2a = -2$ and $c = a = 1$ is a solution.

So $\{\ldots\}$ is lin'ly dependent

Are any of these vectors parallel? **Qlicker**

What part of $\mathbb{R}^3$ would be spanned by these three vectors? →



$\mathbb{R}^3$

all 3 vectors lie in same plane

Common interpretation:

If p is a linearly **dependent** subset of $\mathbf{V}$, then at least one of the elements of p can be written as a linear combination of the other elements of p.

Or: p is linearly dependent if at least one vector $\mathbf{v}_p$ is in the span of the others, $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$.

removed v_p.

Online textbook

From Lecture Notes, Section 5 Problem 2. All the $\mathbf{v}_i \in \mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Example: Is the set $\{\mathbf{v}_1\}$ linearly independent, or linearly dependent? **Qlicker**

Look at

$$a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

→ $a = 0$ only solution → linly independent!

→ some nonzero a works.

components $a \cdot 1 = 0 \rightarrow \boxed{a = 0}$

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix}2\\1\\0\\0\end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \quad \mathbf{v}_4 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \quad \mathbf{v}_5 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

Example: Is the set $\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5\}$ linearly independent, or linearly dependent? **Qlicker**

Look at

$$a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \vec{0} \text{ in } \mathbb{R}^4$$

components

$$\begin{aligned} a &= 0 \\ c &= 0 \\ b &= 0 \\ b + c &= 0 \end{aligned}$$

$a=0$, $b=0$, $c=0$ is the only solution

$\Rightarrow \{v_1, v_4, v_5\}$ is lin'ly independent.

in $\mathbb{R}^4$, w/ 5 vectors

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix}2\\1\\0\\0\end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \quad \mathbf{v}_4 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \quad \mathbf{v}_5 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

Example: Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ linearly independent, or linearly dependent?

Look at

$$\underset{3}{\cancel{a}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \underset{-2}{\cancel{b}}\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + \underset{1}{\cancel{c}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + \underset{(-1)}{\cancel{d}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \underset{(1)}{\cancel{e}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

5 unknowns / 4 eq'ns

pick $e = 1$

(see if I can make sol'n work)

$$\begin{aligned} a + 2b + c &= 0 \\ b + c + e &= 0 \\ c + d &= 0 \\ d + e &= 0 \end{aligned}$$

$\boxed{b = -c - e = -2}$

$e = 1 \to d = -1$

$c = +1$

$a = -c - 2b = -1 - 2(-2)$

$a = +3$

From Lecture Notes, Section 5, similar to Problem 3. All the $f_i \in C^\infty(\mathbb{R})$.

$$f_1(t) = 1 \qquad f_2(t) = 2 + t \qquad f_3(t) = \underline{1} + \underline{t} + \underline{t^2}$$

$$f_4(t) = t^2 + t^3 \qquad f_5(t) = t + t^3$$

Example: Is the set $\{f_1\}$ linearly independent, or linearly dependent?

Save for end of week.

$$f_1(t) = 1 \qquad f_2(t) = 2 + t \qquad f_3(t) = 1 + t + t^2$$
$$f_4(t) = t^2 + t^3 \qquad f_5(t) = t + t^3$$

Example: Is the set $\{f_1, f_4, f_5\}$ linearly independent, or linearly dependent?

$$f_1(t) = 1 \qquad f_2(t) = 2 + t \qquad f_3(t) = 1 + t + t^2$$
$$f_4(t) = t^2 + t^3 \qquad f_5(t) = t + t^3$$

Example: Is the set $\{f_1, f_2, f_3, f_4, f_5\}$ linearly independent, or linearly dependent?

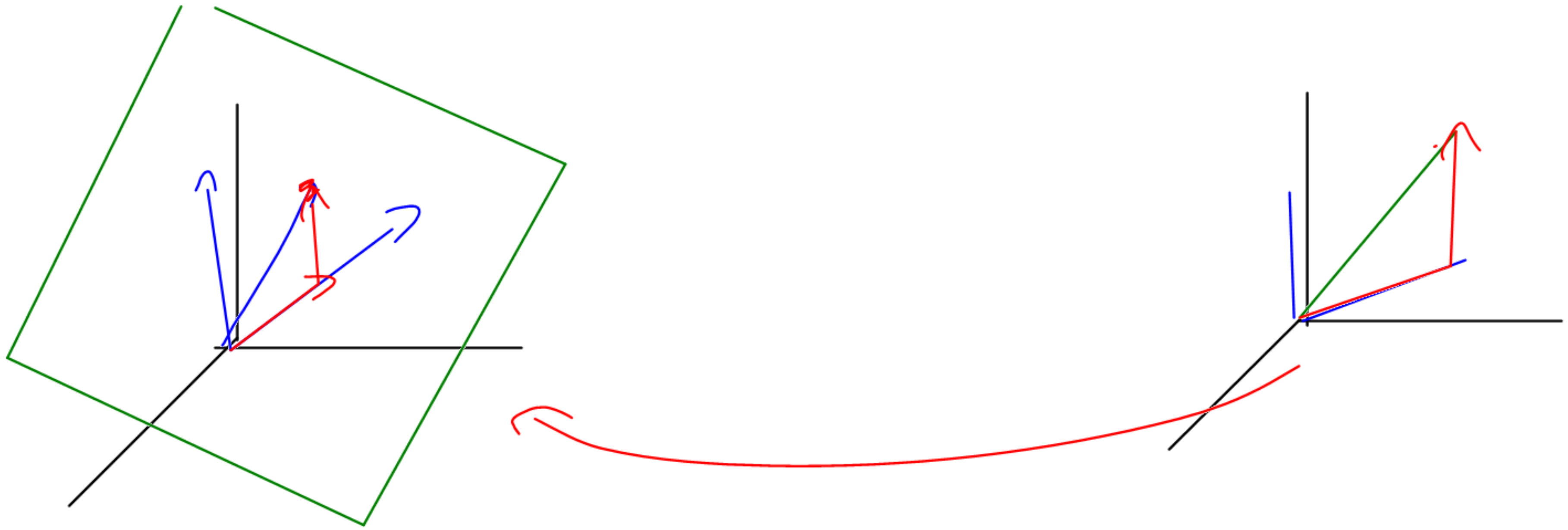
Intuition

If a set of vectors is linearly **independent**:

- ___No___ vectors can be removed without ~~changing~~ reducing the span.
- There ~~is~~/isn't any redundancy in the vectors, for making new vectors through sums, scaling.

If a set of vectors is linearly **dependent**:

- ___At least one of___ the vectors can be removed without changing the span.
- There is/is~~n't any~~ redundancy in the vectors, for making new vectors through sums, scaling.



Proof that linearly dependent $\implies$ redundancy. Show that if a set of vectors is linearly **dependent**, then one of the vectors **can** be written as a linear combination of the others.

Start w/ known property

Let set $\{v_1, v_2 .. v_p\}$ be linearly dependent

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + .. \alpha_p v_p = \bar{0}$ has

a sol'n w/ at least one of $\alpha_1, \alpha_2 .. \alpha_p \neq 0$

Let α_k be the first non-zero α.

safe b/c $\alpha_k \neq 0$

Consider $\dfrac{\alpha_1}{\alpha_k} v_1 + \dfrac{\alpha_2}{\alpha_k} v_2 + \dots \dfrac{\alpha_p}{\alpha_k} v_p = \bar{0}$

$\dfrac{\alpha_k}{\alpha_k} v_k$ all terms except v_k'th

v_k is a lin comb'n of $\{v_1 .. v_p\} \setminus v_k$

$$v_k = \underbrace{-\frac{\alpha_1}{\alpha_k} v_1 + -\frac{\alpha_2}{\alpha_k} v_2 + \dots - \frac{\alpha_p}{\alpha_k} v_p}$$

$\{u \ldots u_5\} \Leftarrow \; u_5 = u_1 \ldots u_4$

Proof that redundancy $\implies$ linearly dependent. Show that if one element in set of vectors **can** be written as a linear combination of the others, then the set is linearly **dependent**.

Let $\{u_1 \ldots u_p\}$ be a set of vectors

and $u_p = \alpha_1 u_1 + \alpha_2 u_2 \ldots + \alpha_{p-1} v_{p-1}$

i.e. u_p is a linear comb'n of the other vectors.

write in $\_\_ = \bar{0}$ form.

$$\bar{0} = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_{p-1} u_{p-1} - 1 u_p$$

$\bar{0}$ ✓

non-zero coeff

At least one coeff is non-zero $\Rightarrow$ $\{u_1 \ldots u_p\}$ is linearly dependent.

Example: single-vector edge cases.

Prove that the set made up of just the zero vector, $\{\mathbf{0}\}$, is linearly **dependent**.

$\{\bar{0}\}$ → look at

$\alpha \bar{0} = \bar{0}$

Q: what values of α satisfy this equation?

→ any non-zero α will work

⇒ $\{\bar{0}\}$ is lin'ly dependent.

Prove that a set made up of a single **non**-zero vector, $s = \{\mathbf{v}_1 \in \mathbf{V} : \mathbf{v}_1 \neq \mathbf{0}\}$, is linearly **independent**.

$\{v_1\}$

$\alpha v_1 = \bar{0}$

Q: what values of α satisfy this equation

only $\alpha = 0$ satisfies this eq'n

⇒ $\{v_1\}$ is lin'ly independent

Example: the zero vector with others. Consider any set $s = \{\mathbf{0}, \mathbf{v}_1, \ldots, \mathbf{v}_p\}$, $\mathbf{v}_i \in \mathbf{V}$ and $\mathbf{v}_i \neq \mathbf{0}$.
Prove that this s is linearly **dependent**.

Look at $\alpha_0 \bar{0} + \alpha_1 v_1 + \ldots \alpha_p v_p = \bar{0}$

Can pick any non-zero value for α_0, and 0 for all other α's.

$\Rightarrow$ at least 1 of the coeff's in eq'n is non-zero

$\Rightarrow$ $\{\bar{0}, v_1, \ldots v_p\}$ is lin'ly dependent

Given

Arguments with linear independence: Suppose that 4 vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cancel{\mathbf{v}_4}\}$ are linearly **in**dependent.
What can you say about the linear independence of the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Prove your answer.

The eq'n $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = \bar{0}$

has only one sol'n, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

Consider $\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = \bar{0}$ equation.

(linear dep)

Proof by contradiction. Assume $\exists$ $\beta_1, \beta_2, \beta_3$ at least one $\neq 0$

"there exists"

such that $\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = \bar{0}$

at least one has $\beta_1, \beta_2, \beta_3 \neq 0$

But then $\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + 0 v_4 = \bar{0}$

$\Rightarrow \{v_1 .. v_4\}$ is lin'ly dependent

But we know $\{u_1 \ldots u_4\}$ is lin'ly independ

$\Rightarrow$ our assumption must be wrong

$\Rightarrow$ $\{u_1 \ldots u_3\}$ must also be lin'ly independent.

$$\{\quad\} \xrightarrow[\text{vector out}]{\text{take one}} \{\ \}$$

known indep

also indep.

Suppose that you find the relationship that $\mathbf{v}_5$ is in the span of $\{\mathbf{v}_1, v_2, v_3, v_4\}$. What can you say about the linear independence of the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$? Prove your answer.

$\Rightarrow \quad v_5 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$ for some α's.

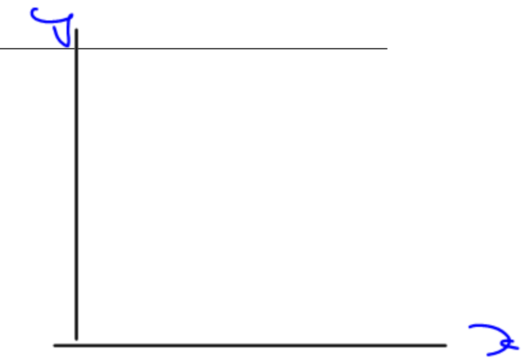
$\Rightarrow \quad \bar{0} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 - 1 v_5$

Is $\{v_1 \ldots v_5\}$ lin'ly indep or dep?

b/c $\alpha_1 v_1 \ldots + \alpha_4 v_4 (-1) v_5 = 0$

has a sol'n w/ at least one non-zero coeff

$\Rightarrow$ $\{v_1 \ldots v_5\}$ is lin'ly dependent.



Further examples: In $\mathbf{W}_2 = \{(x, y) : x > 0, y > 0\}$, are $\mathbf{v}_1 = (2, 3)$, $\mathbf{v}_2 = (4, 9)$ linearly independent or linearly dependent?

Does $\alpha_1 v_1 + \alpha_2 v_2 = \bar{0}$ have a non-zero α_1 or α_2 solution?

$$\alpha_1 (x_1, y_1) + \alpha_2 (x_2 y_2) = (1, 1)$$

our specific v_1, v_2

$$\alpha_1 (2, 3) + \alpha_2 (4, 9) = (1, 1)$$

$$(2^{\alpha_1}, 3^{\alpha_1}) + (4^{\alpha_2}, 9^{\alpha_2}) = (1, 1)$$

$$(2^{\alpha_1} \cdot 4^{\alpha_2}, 3^{\alpha_1} \cdot 9^{\alpha_2}) = (1, 1)$$

component:

- $2^{\alpha_1} 4^{\alpha_2} = 1$

Try $\alpha_1 = 2$ (so $2^2 = 4$)

$\alpha_2 = -1$, (so $4^{-1} = \frac{1}{4}$)

- $3^{\alpha_1} 9^{\alpha_2} = 3^2 9^{-1}$

$= 9 \, (1/9) = 1$

so $\alpha_1 = 2, \alpha_2 = -1$ is a non-zero sol'n $\Rightarrow$ in W_2, $\{(2,3), (4,9)\}$ is lin'ly dependent.

$\times 2 = (4, 9)$

$$+ : (W_2 \times W_2) \to W_2$$

$$(x, y) + (a, b) \to (xa, yb)$$

$$\cdot : (\mathbb{R} \times W_2) \to W_2$$

$$\alpha \cdot (x, y) \to (x^{\alpha}, y^{\alpha})$$

Continued

$$+ :(W_2 \times W_2) \to W_2$$
$$(x, y) + (a, b) \to (xa, yb)$$

$$\cdot :(\mathbb{R} \times W_2) \to W_2$$
$$\alpha \cdot (x, y) \to (x^\alpha, y^\alpha)$$

Example: In $C^{\infty}(\mathbb{R})$: are $f_1 = \sin(x)$ and $f_2 = \cos(x)$ linearly independent or linearly dependent?

Consider

$$\alpha_1 f_1 + \alpha_2 \cdot f_2 = \bar{0}$$

or $\alpha_1 f_1(x) + \alpha_2 f_2(x) = 0$ for all $x \in \mathbb{R}$

i.e These 2 functions (added & scaled) = 0 constant function

Complicated if they are dep, but straight forward for indep sets.

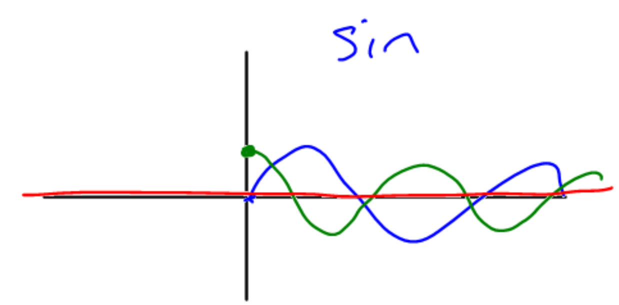
$\Downarrow$

pick some particular x values

e.g pick $x = 0 \rightarrow \cancel{\alpha_1 \sin(0)}^{\,0} + \alpha_2 \cos(0) = 0$

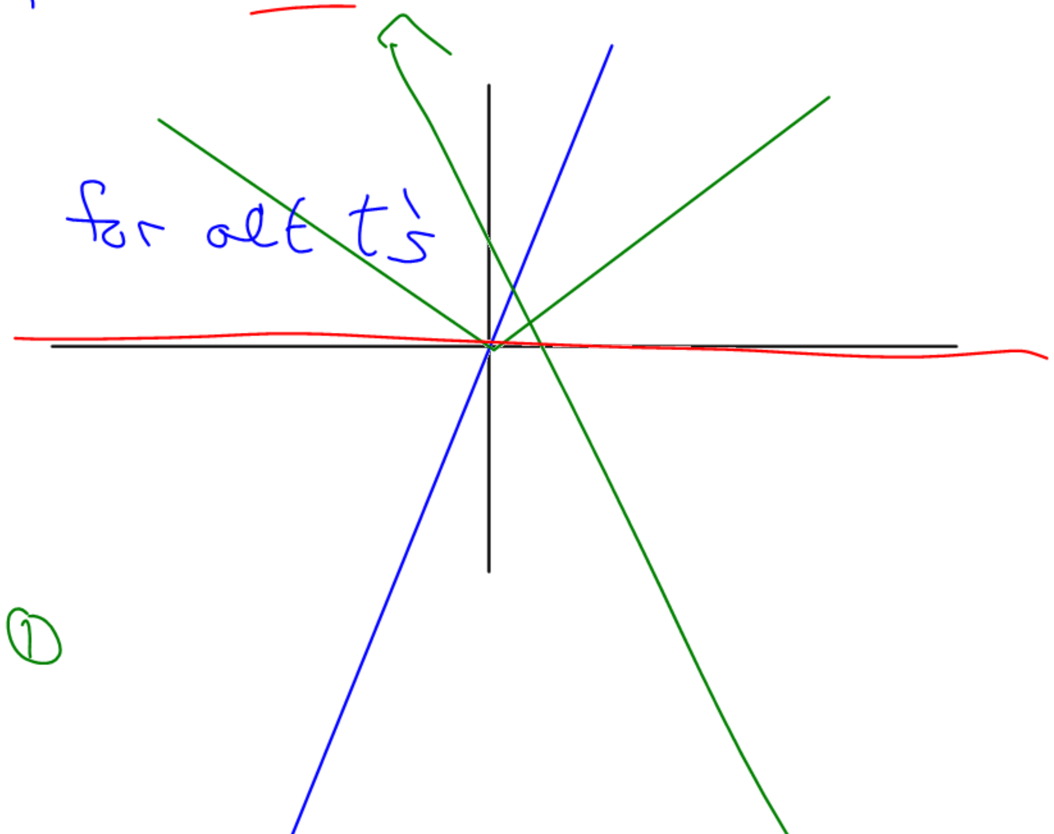
$\cos(0) = 1$

$\alpha_2 = 0$



$f(t) = 3t$, $g(t) = |t|$

in-dep



need $\underline{a}(3t) + b|t| = 0$ for all t's

pick t's: $t = +1$

$a \cdot 3 + b \cdot 1 = 0$ (1)

try $t = -1$ also

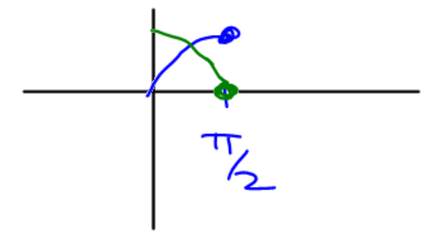
$a \cdot (-3) + b\,1 = 0$ (2)

$6a = 0$ (1)−(2)

$\boxed{a = 0} \xrightarrow{(1)} 3 \cdot 0 + b = 0 \to \boxed{b = 0}$

only possible solution.
$a = 0,\ b = 0$

pick $x = \frac{\pi}{2} \Rightarrow \alpha_1 \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} + \cancel{\alpha_2 \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}} = 0$ $\{\sin(x), \cos(x)\}$



so need $\alpha_1 = 0$ as well

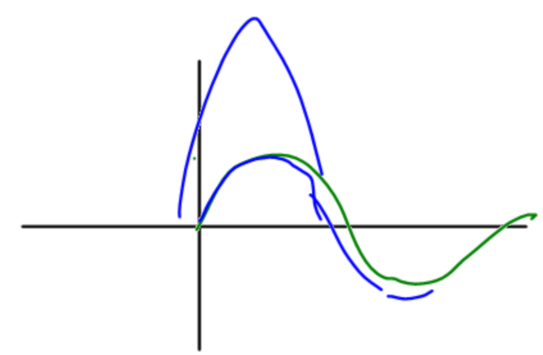
Only solution to $\alpha_1 \sin(x) + \alpha_2 \cos(x) = 0$ for $x = 0, \pi/2$

is $\alpha_1 = \alpha_2 = 0$

$\Rightarrow \{\sin(x), \cos(x)\}$ is lin'ly independent.

$\{4\sin(x), \cos(x - \pi/2)\}$

lin'ly dep



$\{\sin(x), \cos(x)\}$

Eg. of lin'ly dependent functions would be eg $\{1, x^2, 3+2x^2\}$

$f_1 \quad f_2 \quad f_3$

$$3+2x^2 = 3 \cdot 1 + 2 \cdot x^2$$

$$f_3 = \alpha_1 f_1 + \alpha_2 f_2$$

So f_3 is a lin comb'n of $\{f_1, f_2\}$

or f_3 is in the span of $\{f_1, f_2\}$.

$\Rightarrow \{f_1, f_2, f_3\}$ is a lin'ly dependent set.

vector space

8 axioms

recognize, be able to prove 1 or 2 if given the axiom

vector sub space

$W \subset V$

known vector space

2.

- 3 axioms

- do need to be memorized.

Example: In $C^\infty(\mathbb{R})$: are $f_1 = e^x$ and $f_2 = e^{3(x-1)}$ linearly independent or linearly dependent?

$$\{e^{3x}, e^{3(x-1)}\}$$

Example: In $C^\infty(\mathbb{R})$: is the set $\{x^2, 4|x|^2\}$ linearly independent or linearly dependent?

$$\{e^{3x}, e^{3(x-1)}\}$$

Defining subspaces with span and with conditions.

Consider the two sets:

- $\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$
- $\mathbf{S} = \text{ Span}([1, 0, 2], [-2, 1, 0])$

Describe both sets.

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$$
$$\mathbf{S} = \text{Span}(\mathbf{v}_1 = [1, 0, 2], \mathbf{v}_2 = [-2, 1, 0])$$

Verify that both $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbf{W}$.

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$$
$$\mathbf{S} = \text{ Span}(\mathbf{v}_1 = [1, 0, 2], \mathbf{v}_2 = [-2, 1, 0])$$

Knowing that both $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbf{W}$, what do you think the relationship is between the sets $\mathbf{S}$ and $\mathbf{W}$?

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$$
$$\mathbf{S} = \text{ Span}(\mathbf{v}_1 = [1, 0, 2], \mathbf{v}_2 = [-2, 1, 0])$$

Show that $\mathbf{S} \subset \mathbf{W}$.

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$$
$$\mathbf{S} = \text{ Span}(\mathbf{v}_1 = [1, 0, 2], \mathbf{v}_2 = [-2, 1, 0])$$

Show that $\mathbf{W} \subset \mathbf{S}$.

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$$
$$\mathbf{S} = \text{Span}(\mathbf{v}_1 = [1, 0, 2], \mathbf{v}_2 = [-2, 1, 0])$$

What have you proven about the relationship between $\mathbf{S}$ and $\mathbf{W}$?