

Week #5: Linear Independence

- Application of Linear Independence to Systems of Linear Equations
- Strategies and Patterns in Solving Linear Systems

Vector space - 8 axioms

would be given 1 or 2, then asked to prove

Vector subspace - 3 axioms ✓

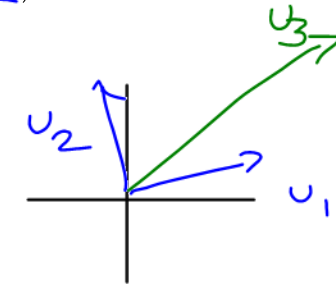
Qlicker later today (Wed)

Midterm 1 2019 - Practice Question

#5 (c) Suppose that \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 are elements of a vector space \mathbf{V} , and we know that

not necessarily in \mathbb{R}^2
Sketch for intuition

- the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, and also that
- the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.



Show that $\mathbf{v}_3 \in S(\mathbf{v}_1, \mathbf{v}_2)$.

is in the span of $\{v_1, v_2\}$.

i.e. show that $v_3 = a v_1 + b v_2$ for some $a, b \in \mathbb{R}$

Know $\alpha v_1 + \beta v_2 = \vec{0}$ has only $\alpha = 0, \beta = 0$ as a sol'n. (lin indep)

while $c v_1 + d v_2 + e v_3 = \vec{0}$ has a sol'n w/ at least one of $c, d, e \neq 0$. (lin depend)

If $e \neq 0$ in so ~

then $e v_3 = -c v_1 - d v_2$

Continued.

$$\text{or } v_3 = -\frac{c}{e} v_1 - \frac{d}{e} v_2 \quad (\text{ok/safe b/c } e \neq 0)$$

$\Rightarrow v_3$ is a lin' comb'n of v_1, v_2

$\Rightarrow v_3$ is in the span of $\{v_1, v_2\}$.

But if $e = 0$?

\Rightarrow

$$c v_1 + d v_2 + \cancel{0} v_3 = \vec{0} \quad \text{w/ one of } c, d \text{ non-zero.}$$

but $\{v_1, v_2\}$ is lin'ly indep

\Rightarrow only c, d that satisfy

$$c v_1 + d v_2 = \vec{0} \quad \text{as } c = 0, d = 0$$

(contradiction)

$\Rightarrow e = 0$ assumption cannot be true

$\Rightarrow e \neq 0$ only possible case $\Rightarrow v_3 \in \text{span}\{v_1, v_2\}$

Section 6 - Linear Independence and Systems of Equations

Last week, we tied systems of linear equations back to a relationship about the span of a set of vectors. x_1, x_2 are unknowns

we want to

solve for.

$$x_1 + 3x_2 = 1$$

$$2x_1 + x_2 = 3$$

$$4x_1 - x_2 = -3$$



$$x_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

equivalent.

$$x_1 v_1 + x_2 v_2 = w$$

Represent the question “Does the system of equations on the left have a solution?” in new terms, using span.



Is RHS/w in the span of $\{v_1, v_2\}$?



w is a lin combin of v_1, v_2 .

Example: compare the **number of solutions** to the following systems:

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \rightarrow x_1 &= 0 \\ x_2 &= 3 \\ x_1 + x_2 &= 4 \quad \times \end{aligned}$$

\Rightarrow no solution.

$\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$ not in span of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 3 \\ x_1 + x_2 &= 4 \quad \checkmark \\ 1 + 3 &= 4 \end{aligned}$$

\Rightarrow single solution

$$\overset{1}{x_1} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \overset{3}{x_2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \overset{0}{x_3} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

0^+ 4^+ 1^+

$x_1=1, x_2=3, x_3=0$ is a sol'n \checkmark
 but so is
 $x_1=0, x_2=4, x_3=1 \quad \checkmark$

\vdots

There are an infinite # of solutions.

Example: compare the **linear independence** and **span** of the LHS vectors, in the same systems:

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$$

RHS = $\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$ is not in the span of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is lin'ly independent
 (eq'n had exactly 1 solution)

Both have
 RHS \in Span

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Note: $v_1 - v_2 = v_3$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ is lin'ly dependent
 (eq'n had an infinite # of solns)

Determining the number of solutions to a system of linear equations

Extract the column vectors of coefficients from the linear system.

Example:

of unknowns

equation ↓

$$\begin{aligned} x_1 + 3x_2 &= 1 \\ 2x_1 + x_2 &= 3 \\ 4x_1 - x_2 &= -3 \end{aligned}$$

$$x_1 \underbrace{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}_{\mathbf{v}_1} + x_2 \underbrace{\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}}_{\mathbf{v}_2} = \underbrace{\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}}_{\mathbf{w}}$$

In general, we can convert any system of linear equations in x_1, \dots, x_p into a new set of equations with vectors:

$$u_1 \dots u_p, w \in \mathbb{R} \quad (\# \text{ eq'ns})$$

$$x_1 u_1 + x_2 u_2 + \dots + x_p u_p = w$$

unknowns = # of x's.

Cases:

Goal: solve for x 's.

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{w}$$

(1) \mathbf{w} is **not** in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

\Rightarrow no solution.

(the ^{linear} dependence/indep of $\{v_1, \dots, v_p\}$ is irrelevant)

(2) \mathbf{w} is in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent.

\Rightarrow exactly one solution for x_1, x_2, \dots, x_p .

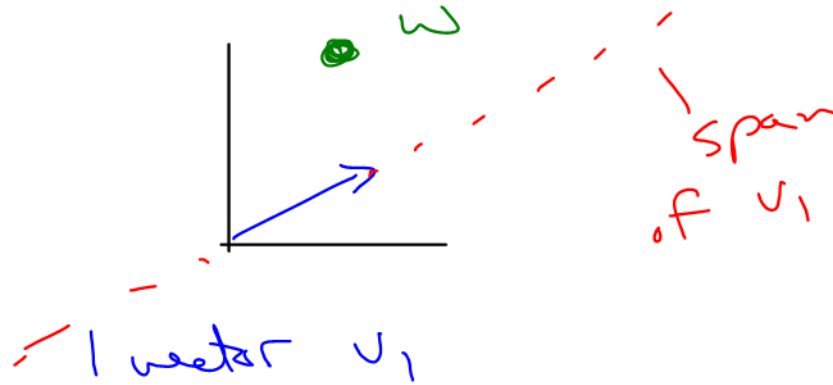
(3) \mathbf{w} is in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent.

\Rightarrow Infinite # of solutions for x_1, x_2, \dots, x_p .

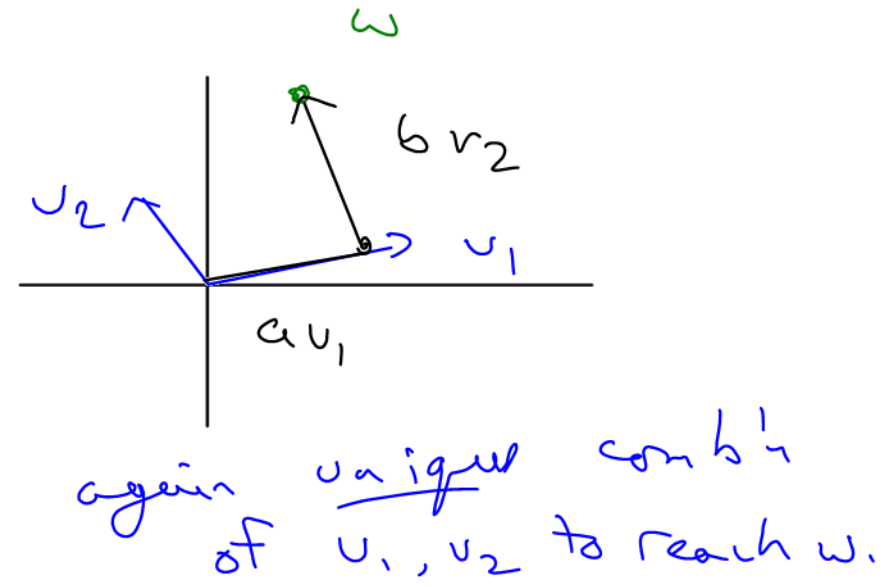
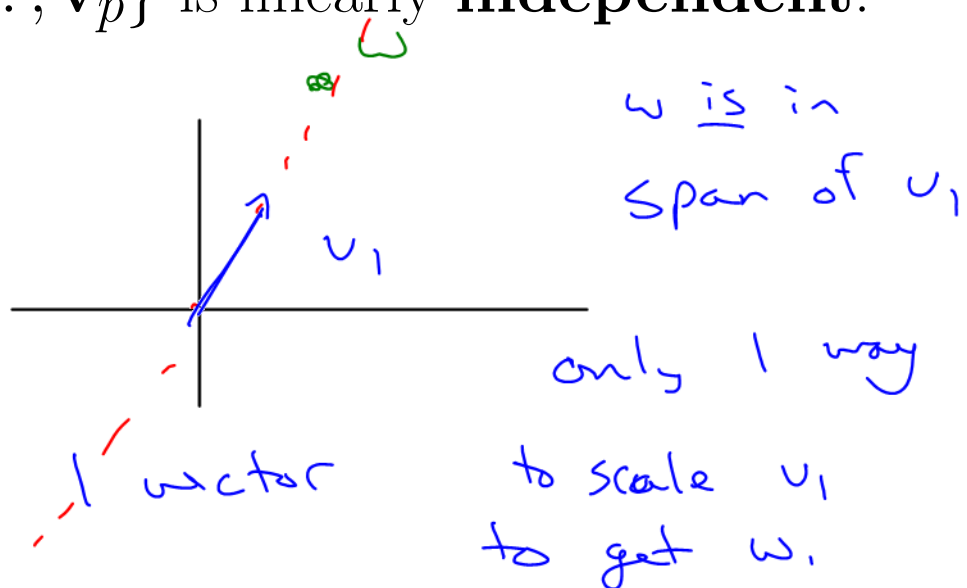
$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{w}$$

Geometric Intuition in \mathbb{R}^2

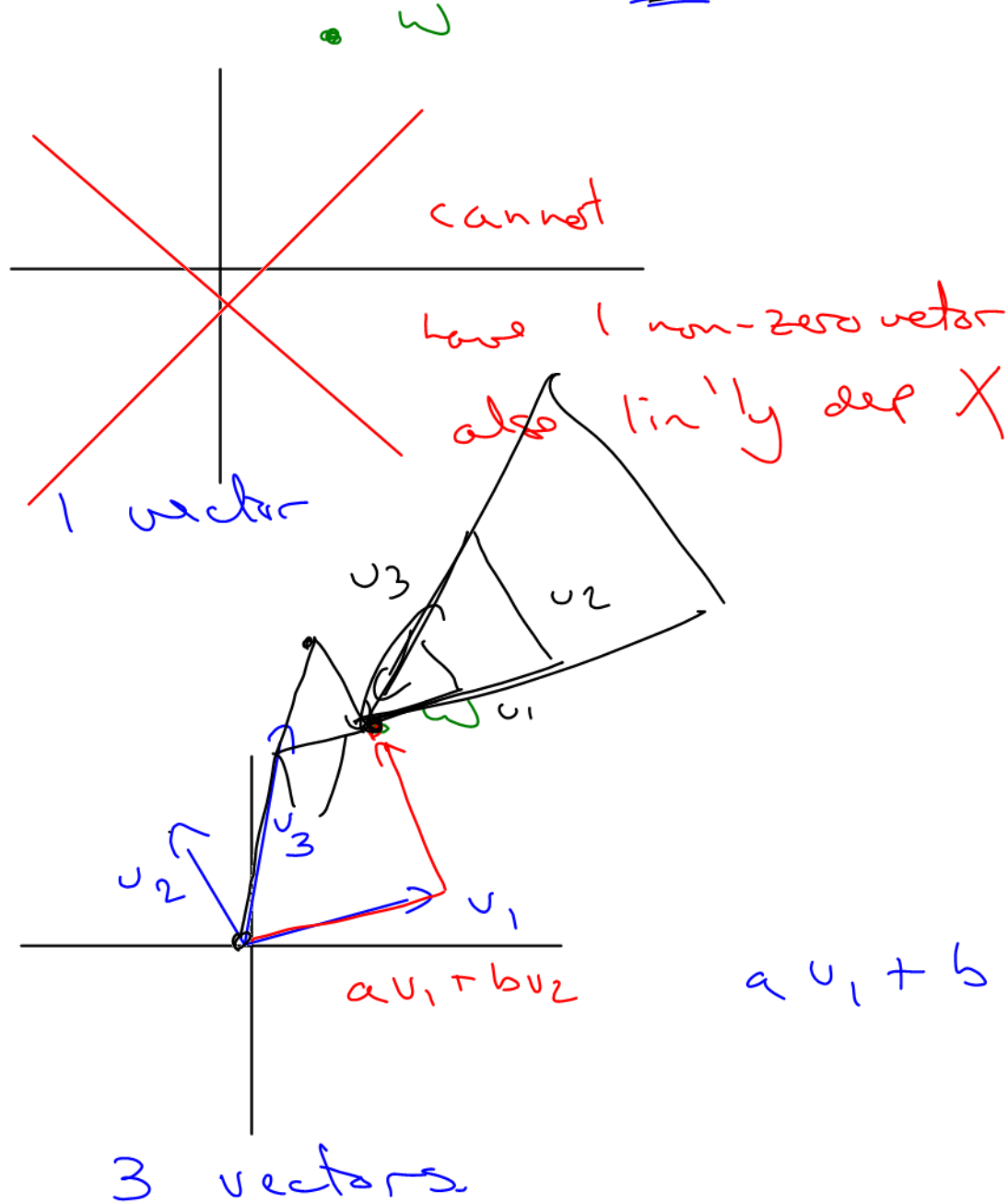
(1) \mathbf{w} is **not** in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.



(2) \mathbf{w} is in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$;
 $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly **independent**.



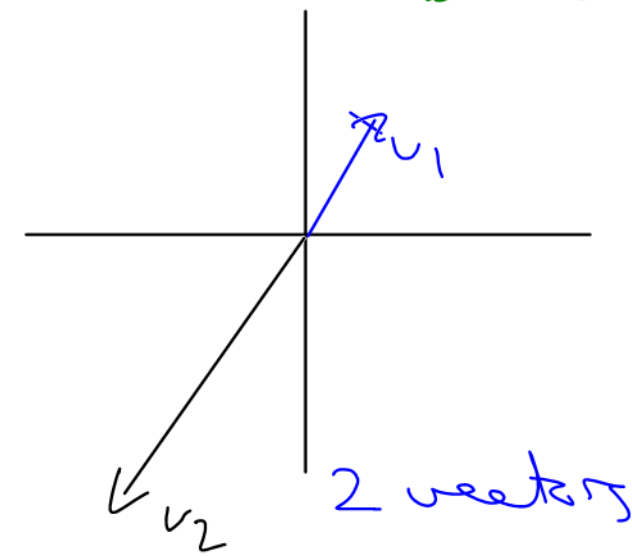
(3) \mathbf{w} is in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$;
 $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent.



2 vectors:
 one of $a, b \neq 0$

$$a u_1 + b u_2 = \vec{0}$$

$$\rightarrow b u_2 = -a u_1$$



multiple ways to combine u_1, u_2 to reach w .

$$a u_1 + b u_2 + c u_3 = \vec{0}$$

infinite # of routes that end at w

\Rightarrow infinite # of solutions.

Proofs.

(1) Prove that if \mathbf{w} is **not** in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, then there exists no solutions to the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{w}$.

The span of $\{u_1, \dots, u_p\}$ is

the set of all vectors of the form

$$v = x_1 u_1 + \dots + x_p u_p, \text{ where } x_i \text{'s} \in \mathbb{R}.$$

but w is not in span of $\{u_1, \dots, u_p\}$

\Rightarrow there does not exist x_i 's $\in \mathbb{R}$

that satisfy

$$w = x_1 u_1 + \dots + x_p u_p$$

There are no x_i 's that satisfy the equation $x_1 u_1 + \dots + x_p u_p = w$.

(2) Prove that if

- \mathbf{w} is in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, and
- the set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly **independent**, then there is **exactly one solution** to the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{w}$.

Similar to "only one $\vec{0}$ in a vector space" WKZ.

Assume that there are 2 solutions to equation

i.e. (1) $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_p v_p = w$

and (2) $\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_p v_p = w$ are both solutions.

$$\begin{array}{l} \text{LHS (1)} \\ - \text{LHS (2)} \end{array} \cdot \frac{(\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \dots + (\alpha_p - \beta_p)v_p = \vec{0}}{\text{lin comb'n of } \{v_1, \dots, v_p\}} \quad \begin{array}{l} \text{RHS (1)} \\ - \text{RHS (2)} \\ \text{RHS} = \vec{0} \end{array}$$

Because $\{v_1, \dots, v_p\}$ is linearly independent

\Rightarrow all $(\alpha_i - \beta_i)$ coefficients must be 0.

$$\text{If } \alpha_i - \beta_i = 0$$

then all the $\alpha_i = \beta_i$

Or 2 solutions are in fact identical / the same solution.

\Rightarrow the solution α_i 's must be unique.

(3) Prove that if

- \mathbf{w} is in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, and
 - the set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent, then
- there are infinitely many solutions to the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{w}.$$

Here, we will start w/ one solution, and then show it can be a foundation for building an infinite number more.

Let $\alpha_1, \dots, \alpha_p$ be a sol'n (know at least 1 sol'n exists b/c $\mathbf{w} \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$)

$$\Rightarrow \textcircled{1} \alpha_1 \mathbf{v}_1 + \dots + \alpha_p \mathbf{v}_p = \mathbf{w}$$

Now b/c $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is lin'ly dependent

there exist β_1, \dots, β_p s.t. w/ at least one $\beta_i \neq 0$

$$\textcircled{2} \quad \underline{k(\beta_1 \mathbf{v}_1 + \dots + \beta_p \mathbf{v}_p) = \mathbf{0} \cdot k} \quad \text{some } k \in \mathbb{R}$$

$$\text{Add } \textcircled{1} + \textcircled{2} \quad (\alpha_1 + \beta_1) \mathbf{v}_1 + \dots + (\alpha_p + \beta_p) \mathbf{v}_p = \mathbf{w} + \mathbf{0}$$

$\Rightarrow (\alpha_1 + k\beta_1), \dots, (\alpha_p + k\beta_p)$ is also a solution! Continued.

and at least one of those β 's is not zero

\Rightarrow the solutions $\alpha_1, \dots, \alpha_p$ and $(\alpha_1 + k\beta_1), \dots, (\alpha_p + k\beta_p)$
are different! for any real k

\Rightarrow there are an infinite of solutions!

(each $k \rightarrow$ new solution)

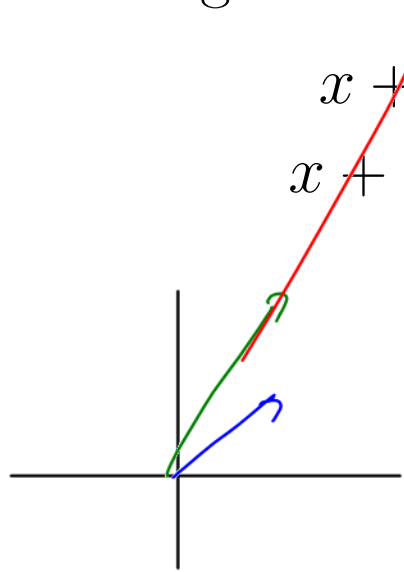
New facts:

- In solving systems of equations, the RHS (as a vector) must be in the **span** of the LHS for there to even be a solution.

- If there is a solution, the linear dependence or independence of the LHS vectors determines whether the solution is unique or not.

Examples: For each of the following systems of equations, determine the number of solutions using vector properties as much as possible.

$$\begin{aligned} x + y &= 5 \\ x + 2y &= 10 \end{aligned}$$



$$\rightarrow x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

$\alpha_1 v_1$ $\alpha_2 v_2$ w

is in span of

$\{v_1, v_2\}$

exactly 1

infinite

$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ is lin'ly indep.

\Rightarrow exactly 1 solution

for sets of 2 vectors

lin dep

\Updownarrow

scalar multiples/
parallel

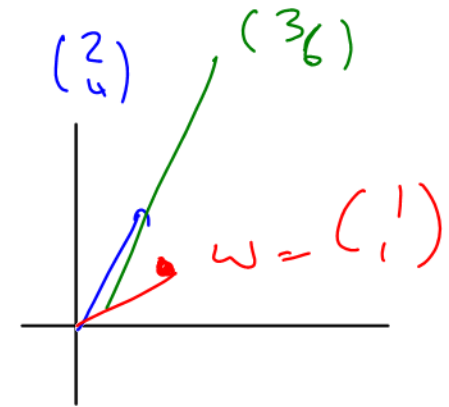
$$\begin{aligned} 2x + 3y &= 1 \\ 4x + 6y &= 1 \end{aligned}$$

$$x \begin{pmatrix} 2 \\ 4 \end{pmatrix} + y \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \right\}$ is lin'ly dependent

but $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in the span of $\left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \right\}$

\Rightarrow 0 solutions



$$2x - 3y = 5$$

$$4x - 6y = 10$$

$$x \begin{pmatrix} 2 \\ 4 \end{pmatrix} + y \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

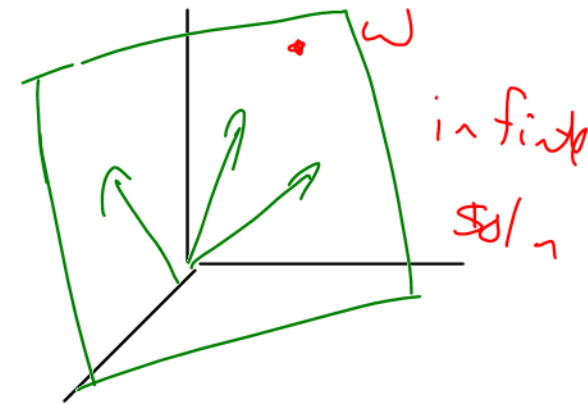
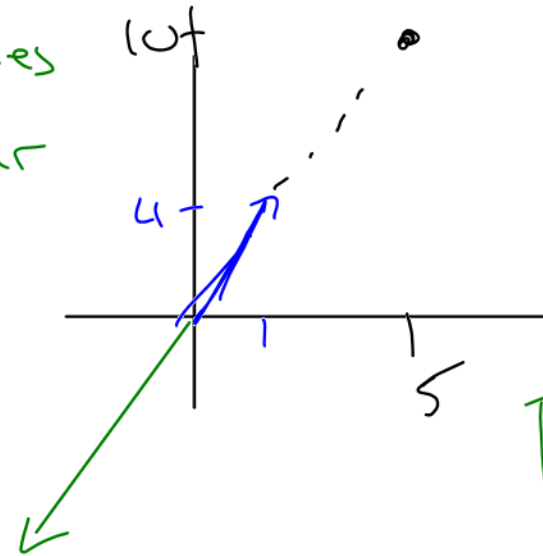
are multiples of each other

$$\beta_1 v_1 + \beta_2 v_2 = \vec{0}$$

$$3 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \vec{0}$$

• $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ is in span of $\left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \end{pmatrix} \right\}$

• $\left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \end{pmatrix} \right\}$ is lin'ly dependent



⇒ there are an infinite # of sol's.

eg

$$\begin{aligned}
 & \alpha_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \alpha_2 \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \\
 & + 7\beta_1 \quad + 2\beta_2 \\
 & 7 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 3 \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}
 \end{aligned}$$

or

$$\begin{aligned}
 & 10 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 5 \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \\
 & 5.5 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} -3 \\ -6 \end{pmatrix}
 \end{aligned}$$

New goals:

- Can we more efficiently determine whether a vector is in the span?

- Can we more efficiently classify sets of vectors as linearly dependent or independent?

We can (soon)!

Reaching both of these goals will tie back to solving systems of linear equations, so we will start by building an efficient strategy for solving any linear system of equations!

Section 7 - Strategies and Patterns in Solving Linear Systems

Recipe for solving Systems of Linear Equations

- (1) Encode the system of linear equations in a matrix (the “augmented matrix of the system”).
- (2) Use row operations to get the matrix into a good form (“upper triangular”, or better “Row Reduced Echelon Form” or RREF)
- (3) With the matrix in this form, read off the solution(s).

We will spend some time seeing what each of these steps means.

Definition: A matrix is a rectangular array or a grid of numbers.

$$\begin{array}{c} \left[\begin{array}{ccc} 1 & 7 & 9 \\ 2 & \pi & 4 \end{array} \right] \\ \underline{2} \times \underline{3} \text{ matrix} \\ \# \text{ rows} \qquad \# \text{ cols.} \end{array}$$

$$\begin{array}{c} \begin{array}{c} \xrightarrow{2 \text{ cols}} \\ \left[\begin{array}{cc} 1 & e \\ -3 & 1 \\ 7 & -0.5 \\ 0.1 & 2 \end{array} \right] \\ \downarrow 4 \text{ rows} \\ \underline{4} \times \underline{2} \text{ matrix} \end{array} \end{array}$$

Qlicker

(1) Encode the system of linear equations in a matrix (the “augmented matrix of the system”).

Example:

$$\begin{aligned}
 7x_1 + 14x_2 + 70x_3 - 6x_4 &= 11 \\
 x_1 + 2x_2 + 10x_3 - 1x_4 &= -2 \\
 3x_1 + 7x_2 + 33x_3 - 3x_4 &= -8
 \end{aligned}$$

3 eqⁿ

4 unknowns

marker of transition

b/w LHS vs RHS

Represent these equations using a matrix.

out distill variables

$$\begin{bmatrix}
 7 & 14 & 70 & -6 & | & 11 \\
 1 & 2 & 10 & -1 & | & -2 \\
 3 & 7 & 33 & -3 & | & -8
 \end{bmatrix}$$

#

augmented part of the matrix

Patterns/Process:

distill

$$x_1 \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 14 \\ 2 \\ 7 \end{pmatrix} + x_3 \begin{pmatrix} 70 \\ 10 \\ 33 \end{pmatrix} + x_4 \begin{pmatrix} -6 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 11 \\ -2 \\ -8 \end{pmatrix}$$

Think about operations you usually use when solving equations like adding, subtracting multiples of rows. Is anything lost or gained by doing similar operations on the coefficients in matrix form?

$$\begin{aligned} \underline{3x} + 7y - \underline{4z} &= \underline{2} & \textcircled{1} \\ \underline{6x} - 2y + z &= 6 & \textcircled{2} \end{aligned}$$

$$\begin{array}{l} R_1 \\ R_2 \end{array} \left[\begin{array}{ccc|c} 3 & 7 & -4 & 2 \\ 6 & -2 & 1 & 6 \end{array} \right]$$

$$2 \times \textcircled{1} - \textcircled{2} \quad \cancel{0x} + 16y - 9z = -2$$

completely
equivalent

$$\begin{array}{l} R_1 \\ 2 \times R_1 - R_2 \end{array} \left[\begin{array}{ccc|c} 3 & 7 & -4 & 2 \\ 0 & 16 & -9 & -2 \end{array} \right]$$

Matrix Row Operations

The allowable row operations are:

- Switching, swapping or permuting the rows.
- Multiplying or dividing any row by a non-zero number.
- Adding or subtracting a multiple of row i to/from row j , and putting the result in row j .

Example: If we had the matrix form below for variables x , y and z , how would we interpret it?

$$\begin{array}{ccc} x & y & z \\ \left[\begin{array}{ccc|c} 1 & 3 & 0 & 3 \\ \underline{0} & 0 & 1 & 4 \end{array} \right] \end{array}$$

$$\leftrightarrow \begin{array}{l} x + 3y = 3 \\ z = 4 \end{array}$$

$$\leftrightarrow z = 4$$

$$x = 3 - 3y$$

and y could be any \mathbb{R}

infinite # of solutions

easy to identify
values of x, y, z .

Compare that with the following matrix:

$$\begin{array}{ccc} x & y & z \\ \left[\begin{array}{ccc|c} 4 & 0 & 2 & 3 \\ 2 & 1 & 1 & 4 \end{array} \right] \end{array}$$

← con stans / R HS

$$\leftrightarrow \begin{array}{l} 4x + 2z = 3 \\ 2x + y + z = 4 \end{array}$$

hard to identify

specific values of x, y, z .

Example: Use row operations to simplify the system of equations in matrix form below.
 Use the variables x_1, x_2, x_3 .

$$1x_1 + 2x_2 - x_3 = -1$$

3 rows $\left\{ \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \right. \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array} \right] \rightarrow \begin{array}{l} 2R_1 - R_2 \\ 3R_1 - R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 2 & -3 & -3 \\ 0 & 1 & -1 & -2 \end{array} \right]$

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goal: more 0's in 1st col

Goal: more 0's in 2nd col.

$$\begin{array}{l} R_1 - 2R_3 \\ R_3 \\ R_2 - 2R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -1 & 1 \end{array} \right]$$

$$\begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \\ R_3 \times -1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

$$\begin{array}{l} x_1 = 4 \\ x_2 = -3 \\ x_3 = -1 \end{array}$$

ahh!

can read off the solution!

RREF.

Continued.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array} \right]$$

We will stop here, and do further examples and analysis in Week 6.

Today, we will switch over to past midterms.

Example: Use row operations to simplify the system of equations in matrix form below. Use the variables x_1, x_2, x_3, x_4 .

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 0 & 0 \end{array} \right]$$

Continued.

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 0 & 0 \end{array} \right]$$

Example: Use row operations to find a solution to the system of equations in matrix form below. Use the variables x_1, x_2, x_3, x_4 .

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array} \right]$$

Continued.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array} \right]$$

Continued.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array} \right]$$

Row Reduced Echelon Form

Formal goal: use row operations to put our equations' matrix into **RREF**.

A matrix is in RREF **if**:

- In each non-zero row, there is a **leading 1**.
 - From left to right, the first non-zero value should be a 1.
 - All 0's is okay.
- If a column has a leading 1, all other values in that column are 0's.
- The leading ones form a staircase from top left to bottom right.