

Week #6: Solving to RREF, Interpreting Solutions from RREF, Generating Sets and Basis Sets

Qlicker

Systems of Equations and Augmented Matrices

$$3x + 7y - 4z = 2$$

$$6x - 2y + z = 6$$

traditional

← duality

column indicates the
associated variable.

$$\begin{bmatrix} 3 & 7 & -4 & | & 2 \\ 6 & -2 & 1 & | & 6 \end{bmatrix}$$

x y z RHS
augmented matrix
of coeff's + RHS values

Ease of Identifying Solutions In Different Forms

These 3 systems are equivalent, and have the same solution.

$$\begin{array}{c} x \quad y \quad z \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right] \end{array}$$

↓ back to traditional format

$$\begin{array}{l} x = 3 \\ y = 1 \\ z = -2 \end{array}$$

Solution can just be read directly from matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array} \right] \begin{array}{l} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{array}$$

$$\boxed{z = -2}$$

$$y + 3z = -5 \textcircled{2}$$

$$y + 3(-2) = -5$$

$$\boxed{y = 1}$$

$$\textcircled{1} \rightarrow x + 2y - z = 7$$

$$x + 2(1) - (-2) = 7$$

$$\boxed{x = 3}$$

All 3 eq's/rows,
3 variables/3 col's before/

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 1 & -3 & 2 & -4 \\ -3 & 3 & -2 & -2 \end{array} \right] \begin{array}{l} \text{left to} \\ \text{vertical} \\ \text{line} \end{array}$$

mess → need to do a lot of cancelling/adding before we identify a sol'n.

Names for Matrix Forms

Row reduced echelon form

RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

easiest to analyze for sol'n

Row echelon form

upper triangular

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$x = \dots$
 $y = \dots$
 $z = \dots$

straight forward to get solution using

"back substitution"

def'n in \checkmark online textbook

general / no specific name.

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 1 & -3 & 2 & -4 \\ -3 & 3 & -2 & -2 \end{array} \right]$$

hard to analyze for sol'n

Gaussian Elimination (GE)

Gaussian elimination is the process of applying *matrix row operations* until a matrix is in *row-reduced echelon form (RREF)*.

Matrix Row Operations

analogous to reordering, add/mult traditional equations

The allowable row operations are:

- Switching, swapping or permuting the rows.
- Multiplying or dividing any row by a non-zero number.
- Adding or subtracting a multiple of row i to/from row j , and putting the result in row j .

Row Reduced Echelon Form RREF

A matrix is in RREF **if**:

- In each row, there should be a leading 1.
 - From left to right, the first non-zero value should be a 1.
 - All 0's is okay.
- If a column has a leading 1, all other values in that column should be 0's.
- The leading rows form a staircase from top left to bottom right.

$$\begin{matrix} 0 \\ 0 \\ 00 \\ 0 \\ 0 \end{matrix} \begin{matrix} 1 \\ \dots \end{matrix}$$

Example: Solve the following system in x_1, x_2, x_3 using Gaussian elimination.

$$R_1 \begin{bmatrix} 2 & 6 & 1 & | & 8 \end{bmatrix}$$

$$R_2 \begin{bmatrix} 3 & 9 & 2 & | & 14 \end{bmatrix}$$

(Qlicker part-way through)

RREF? Not yet

$$R_1/2 \begin{bmatrix} 1 & 3 & 1/2 & | & 4 \end{bmatrix}$$

$$3R_1 - 2R_2 \begin{bmatrix} 0 & 0 & -1 & | & -4 \end{bmatrix}$$

$$\rightarrow -1 \cdot R_2 \begin{bmatrix} 1 & 3 & 1/2 & | & 4 \\ 0 & 0 & 1 & | & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & | & 4 \\ 1 & 3 & 0 & | & 2 \end{bmatrix} \times$$

$$\rightarrow R_1 - \frac{1}{2}R_2 \begin{bmatrix} 1 & 3 & 0 & | & 2 \\ 0 & 0 & 1 & | & 4 \end{bmatrix}$$

THIS IS
in RREF!

$$x_1 + 3x_2 = 2$$

$$x_3 = 4$$

Infinite # of sol^{ns} w/ $x_1 = 2 - 3x_2$
 $x_3 = 4$

Example: Solve the following system in x_1, x_2, x_3 after first reducing the system to RREF.

$$\begin{array}{l}
 R_1 \\
 R_2 \\
 R_3
 \end{array}
 \begin{array}{ccc|c}
 \overset{x_1}{2} & \overset{x_2}{-1} & \overset{x_3}{7} & -15 \\
 5 & -4 & 13 & -48 \\
 \textcircled{1} & -1 & 2 & -11
 \end{array}$$

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3 eq / rows, 3 unknowns / cols to left of |

$$-48 - 5(-11) = -48 + 55$$

Not RREF yet!

$$\begin{array}{l}
 R_3 \\
 R_1 - 2R_3 \\
 R_2 - 5R_3
 \end{array}
 \left[\begin{array}{ccc|c}
 1 & -1 & 2 & -11 \\
 0 & 1 & 3 & 7 \\
 0 & 1 & 3 & 7
 \end{array} \right]$$

$$\begin{array}{l}
 R_1 \\
 R_2 \\
 R_2 - R_3
 \end{array}
 \left[\begin{array}{ccc|c}
 \textcircled{1} & \textcircled{-1} & 2 & -11 \\
 0 & \textcircled{1} & 3 & 7 \\
 0 & 0 & 0 & 0
 \end{array} \right]$$

is RREF

$$\begin{array}{l}
 R_1 + R_2 \\
 R_2 \\
 R_3
 \end{array}
 \left[\begin{array}{ccc|c}
 \textcircled{1} & 0 & 5 & -4 \\
 0 & \textcircled{1} & 3 & 7 \\
 0 & 0 & 0 & 0
 \end{array} \right]$$

$x_1 + 5x_3 = -4$
 $x_2 + 3x_3 = 7$
 x_3 is "free variable"
 x_1, x_2 "dependent variables"
 Infinite # of solns

pick $x_3 = 1$ for example

$$\rightarrow x_1 = -4 - 5x_3 = -4 - 5 = -9$$

and $x_2 = 7 - 3x_3 = 7 - 3 = 4$

once we picked x_3 (free var), x_1 and x_2 (dependent vars)

had no choice / were fixed.

$$\left[\begin{array}{ccc|c} 2 & -1 & 7 & -15 \\ 5 & -4 & 13 & -48 \\ 1 & -1 & 2 & -11 \end{array} \right]$$

Example: Solve the following system in x_1, x_2, x_3, x_4 using Gaussian elimination.

3 eq's

$$\begin{bmatrix} 7 & 14 & 70 & -6 & | & -11 \\ 1 & 2 & 10 & -1 & | & -2 \\ 3 & 7 & 33 & -3 & | & -8 \end{bmatrix}$$

(Note: In the original image, the first three rows and columns are circled in red, and the first three rows are underlined in red.)

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- ✓ — no solution
- ~~exactly 1 sol~~
- ✓ — infinite #

4 vectors in 3D
 \Rightarrow lin'ly dep.

$$\begin{array}{l} R_2 \\ 3R_2 - R_3 \\ R_1 - 7R_2 \end{array} \left[\begin{array}{cccc|c} \textcircled{1} & 2 & 10 & -1 & -2 \\ 0 & -1 & -3 & 0 & 2 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right] \xrightarrow{\text{tidy}} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{cccc|c} \textcircled{1} & \boxed{2} & 10 & \boxed{-1} & -2 \\ 0 & \textcircled{1} & 3 & 0 & -2 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right] \quad \underline{\underline{RREF}}$$

$$\begin{array}{l} R_1 - 2R_2 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 4 & \boxed{-1} & 2 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{R_1 + R_3} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{cccc|c} \textcircled{1} & 0 & 4 & 0 & 5 \\ 0 & \textcircled{1} & 3 & 0 & -2 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right]$$

(Note: In the original image, the variables x_1, x_2, x_3, x_4 are labeled under the columns of the final matrix.)

$$x_1 + 4x_3 = 5$$

$$x_2 + 3x_3 = -2$$

$$x_4 = 3$$

x_1, x_2, x_4 are dependent

This has an infinite
of sol's

$$\left[\begin{array}{cccc|c} 7 & 14 & 70 & -6 & -11 \\ 1 & 2 & 10 & -1 & -2 \\ 3 & 7 & 33 & -3 & -8 \end{array} \right]$$

v 's

\Rightarrow completely known after the free variables are set.

x_3 is a free variable

\Rightarrow cols are lin'ly dep

and RHS is in the span of v 's.

$$\left[\begin{array}{cccc|c} 7 & 14 & 70 & -6 & -11 \\ 1 & 2 & 10 & -1 & -2 \\ 3 & 7 & 33 & -3 & -8 \end{array} \right]$$

Interpreting RREF forms

Example: Determine the number of solutions and the general solution for the following system in RREF.

$$\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right] \end{array}$$

$$\begin{aligned} \rightarrow \quad & 1x_1 + 0x_2 = 1 \\ & 0x_1 + x_2 = 3 \\ & 0x_1 + 0x_2 = 1 \end{aligned}$$

$$0 = 1$$

in general,

row of

$$\left[\begin{array}{cccc|c} \dots & \dots & \dots & \dots & \neq \text{not zero} \\ 0 & 0 & 0 & \dots & 0 \end{array} \right]$$

↓

no solution

impossible! \Rightarrow no solution

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

→ RHS $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ is not in span of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Example: Determine the number of solutions and the general solution for the following system in RREF.

$$\begin{array}{c} x_1 \quad x_2 \\ \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 7 \\ \hline 0 & 0 & 0 \end{array} \right] \end{array}$$

RHS 0

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$$\begin{array}{l} x_1 = 5 \\ x_2 = 7 \end{array}$$

$$0x_1 + 0x_2 = 0$$

unique solution.

can ignore

not an issue

all-zero rows

Example: Determine the number of solutions and the general solution for the following system in RREF.

leading 1's

$$\begin{bmatrix} 1 & 3 & 0 & | & 5 \\ 0 & 0 & 1 & | & 2 \\ \hline 0 & 0 & 0 & | & 0 \end{bmatrix}$$

x_1 x_2 x_3

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$$x_1 + 3x_2 = 5$$

$$x_3 = 2$$

has multiple solutions

ignore b/c
all 0's, including RHS

column w/o leading 1
Let $t \in \mathbb{R}$
Let $x_2 = t$

\Rightarrow eq'ns become

$$x_1 + 3t = 5$$

$$x_3 = 2$$

$$\begin{aligned} x_1 &= 5 - 3t \\ x_2 &= t \\ x_3 &= 2 \end{aligned}$$

general solution

x_2 is a free/independent variable

x_1, x_3 is "dependent variable"

pick $t \rightarrow$ get
a full x_1, x_2, x_3 sol'n.

General Procedure for writing down all the solutions (from RREF)

Special Case: a row like $\left[\begin{array}{cccc|c} \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \textcircled{1} \end{array} \right]$: no solution case.
or any non-zero

(Remember back to the span of the columns...)

More general case, **with** a solution:

$$\begin{array}{cccccc|c}
 x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS.} \\
 \hline
 1 & 4 & 0 & 0 & 3 & 0 & 2 \\
 0 & 0 & 1 & 0 & -5 & 0 & 1 \\
 0 & 0 & 0 & 1 & 2 & 0 & 8 \\
 0 & 0 & 0 & 0 & 0 & 1 & 6
 \end{array}$$

- Write the variable names on the top of the matrix.
 - circle leading 1 in each row.
- Columns **without** leading 1's: **independent** variables, can take on **any value**.
 - For each of these, pick a new free variable, usually t or s.
- Columns with leading 1's: **dependent** variables, will get $x_i = \dots$ equations.

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$$x_1 + 4x_2 + 3x_5 = 2$$

$$x_3 - 5x_5 = 1$$

$$x_4 + 2x_5 = 8$$

$$x_6 = 6$$



$$x_1 = 2 - 4t - 3s$$

$$x_2 = t$$

$$x_3 = 1 + 5s$$

$$x_4 = 8 - 2s$$

$$x_5 = s$$

$$x_6 = 6$$

If there is a free variable / a column w/out a leading 1 \Rightarrow there are an infinite # of solutions

$$\begin{bmatrix} \boxed{1} & 4 & 0 & 0 & 3 & 0 & | & 2 \\ 0 & 0 & \boxed{1} & 0 & -5 & 0 & | & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 & 0 & | & 8 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & | & 6 \end{bmatrix}$$

let $x_2 = t$

let $x_5 = s$

general solution.

pick t & s values $\in \mathbb{R}$

\rightarrow get a full solution.

Example: Find the vector form of the solution to the system below.

$$\begin{array}{cccccc|c}
 x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
 \hline
 1 & 0 & -2 & 0 & 5 & 0 & -1 \\
 0 & 1 & 4 & 0 & -6 & 0 & 7 \\
 0 & 0 & 0 & 1 & 8 & 0 & -5 \\
 \hline
 0 & 0 & 0 & 0 & 0 & 1 & -1
 \end{array}$$

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Let $x_3 = t$

and $x_5 = s$

vector form of the general solution.

general solution

$$x_1 = -1 + 2t - 5s$$

$$x_2 = 7 - 4t + 6s$$

$$x_3 = t$$

$$x_4 = -5 - 8s$$

$$x_5 = s$$

$$x_6 = -1$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 0 \\ -5 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -5 \\ 6 \\ 0 \\ -8 \\ 1 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & -2 & 0 & 5 & 0 & -1 \\ 0 & 1 & 4 & 0 & -6 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

Section 8 - Generating Sets, Bases and Coordinates

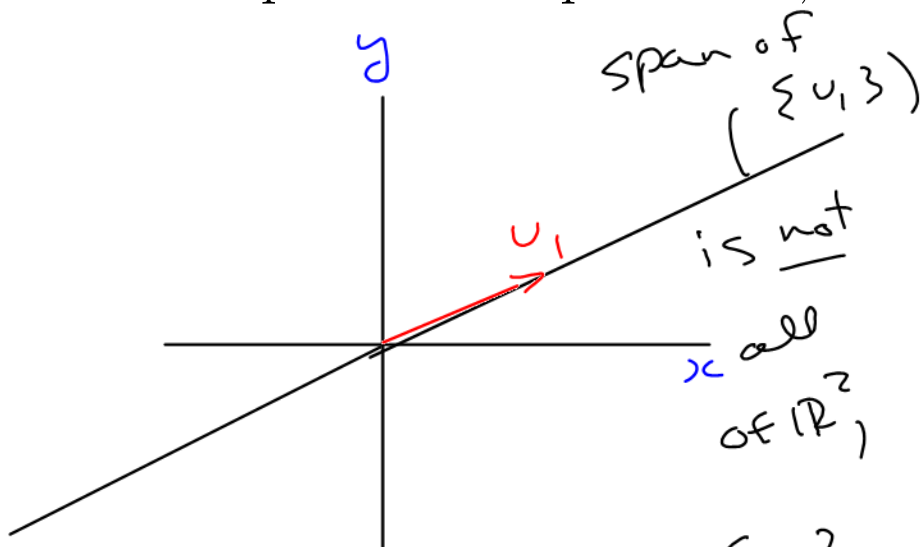
After looking at systems of equations for awhile, we now go (mostly) back to looking at vector sets and spaces.

Definition: Given a vector space \mathbf{V} , a generating or spanning set for \mathbf{V} is a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbf{V} whose span is all of \mathbf{V} .

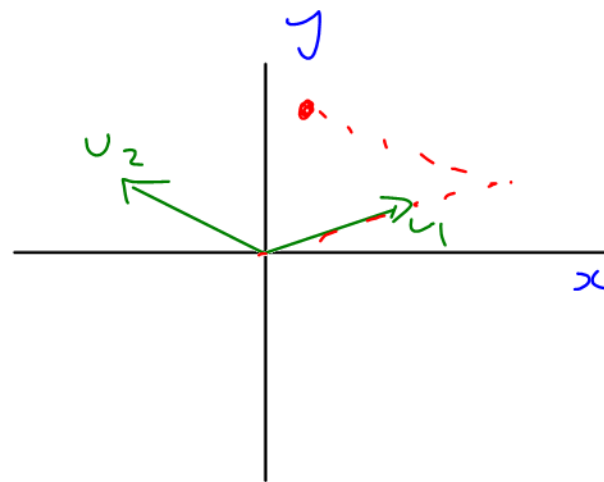
↳ building block vectors.

Equivalent: Every vector in \mathbf{V} is a linear combination of the set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

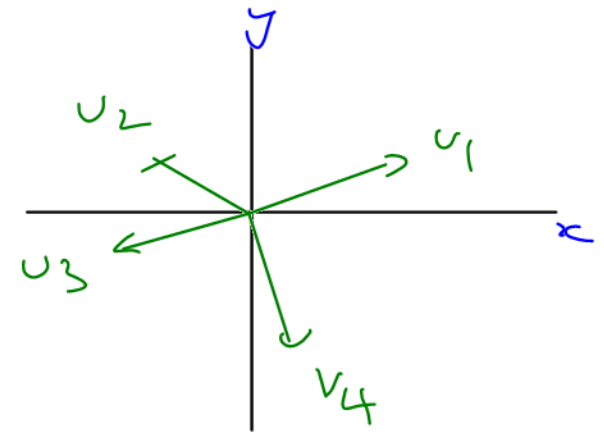
Example: In the plane \mathbb{R}^2 , what options are there for possible spanning sets?



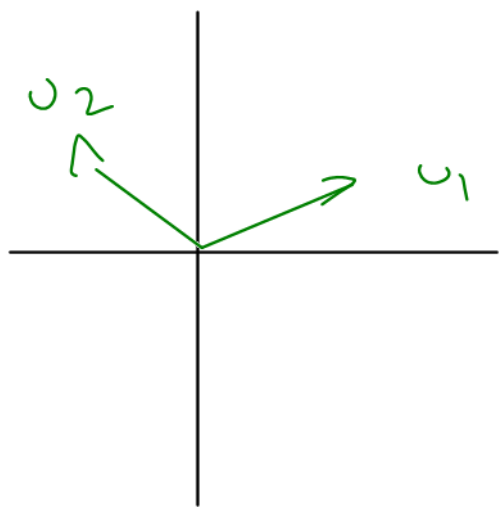
a single vector
 make any $w = a_1 u_1$
 so $\{u_1\}$
 is not
 a spanning
 set for \mathbb{R}^2



two vectors
 $w = a_1 u_1 + a_2 u_2$
can reach any
 point in \mathbb{R}^2
 \Rightarrow set $\{u_1, u_2\}$
is a spanning set
 or a generating set
 for \mathbb{R}^2 .

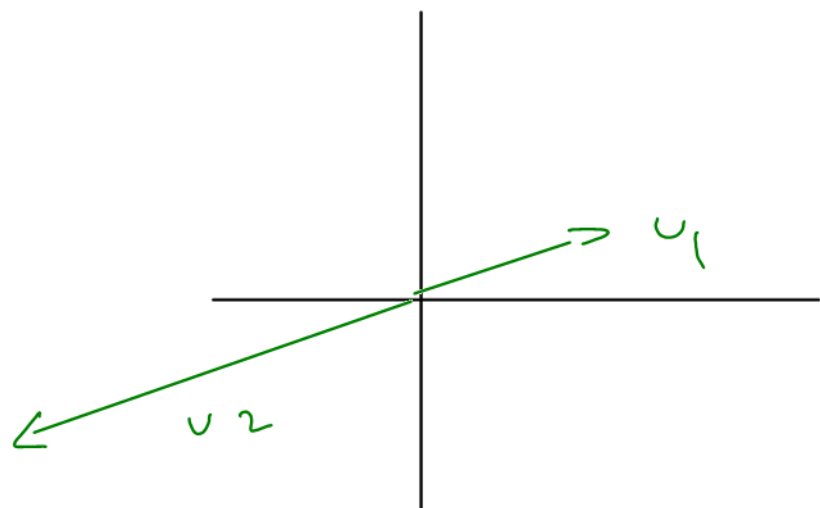


is also a
 spanning set for \mathbb{R}^2



2 vectors in \mathbb{R}^2

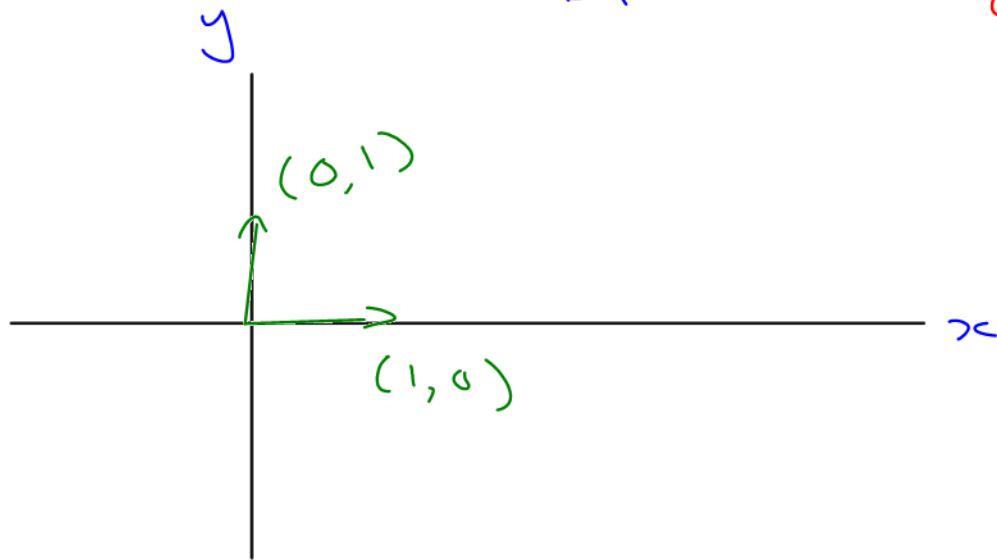
$\{u_1, u_2\}$ \leftarrow lin'ly independent
does span \mathbb{R}^2



$\{u_1, u_2\}$ \leftarrow lin'ly dependent
does not span \mathbb{R}^2

Example: In the plane \mathbb{R}^2 , is the set $\{(1, 0), (0, 1)\}$ a spanning set of \mathbb{R}^2 ?

Draw

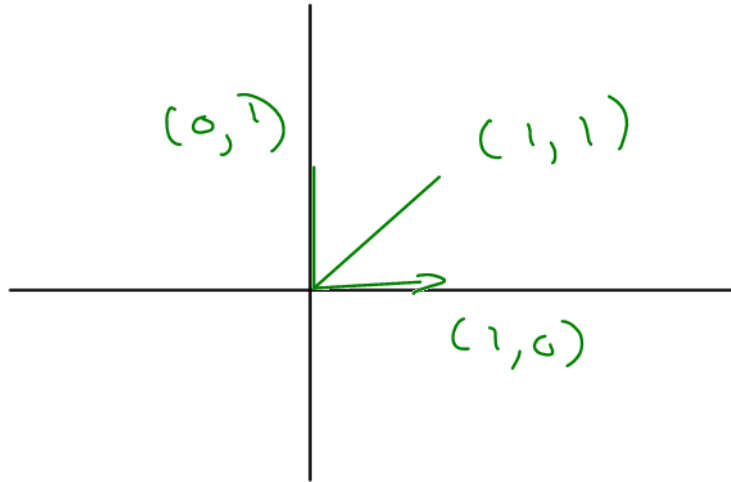


Yes, this pair $\{(1, 0), (0, 1)\}$

is a spanning set for \mathbb{R}^2

We can represent any (x, y) point as $w = x\hat{i} + y\hat{j}$
 $= x(1, 0) + y(0, 1)$.

Example: In the plane \mathbb{R}^2 , is the set $\{(1, 0), (0, 1), \underline{(1, 1)}\}$ a spanning set of \mathbb{R}^2 ?



is still a spanning
set for \mathbb{R}^2 .

Example: In $\underline{\mathbf{W}} = \{f(x) \in \underline{C^\infty(\mathbb{R})} : \underline{f''(x) = 0}\}$, what options are there for possible generating sets?

all functions whose second derivative = 0.

$1 \in \mathbf{W}$
constant

$10, \pi, 73 \in \mathbf{W}$

There are an infinite #
of functions in \mathbf{W} .

$f(x) = x + 1 \in \mathbf{W}$

$5x - 9 \in \mathbf{W}$

\downarrow
 $f'(x) = 1$

$f''(x) = 0$

but the set $\{1, x\}$ is a spanning
set for \mathbf{W} .

i.e. all functions w/ $f''(x) = 0$

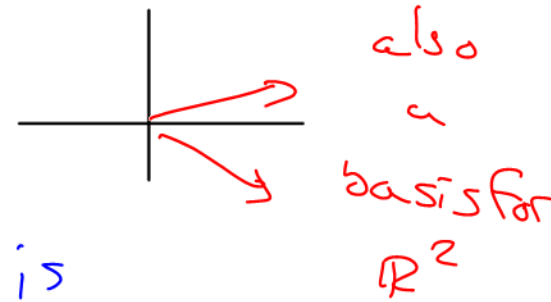
are of the form $f(x) = a \cdot 1 + b \cdot x$

Bases vs Spanning Sets

minimalist spanning set
↓

Definition: Given a vector space V , a **basis for V** is a set of vectors $\{v_1, \dots, v_p\}$ such that:

- $\{v_1, \dots, v_p\}$ is a generating or spanning set for V , **and**
- $\{v_1, \dots, v_p\}$ is linearly independent. ← extra condition.



Example: Contrast the sets of vectors

$\{(1, 0), (0, 1)\}$ and \Rightarrow spanning set + lin'ly indep \Rightarrow is a basis for \mathbb{R}^2

$\{(1, 0), (0, 1), (1, 1)\}$ in \mathbb{R}^2 .

↓
is a spanning set } is not a basis for \mathbb{R}^2
 but lin'ly dependent. } b/c not lin'ly indep.

$$1(1,0) + 1(0,1) + (-1)(1,1) = (0,0)$$

Note: in linear algebra, the plural of **basis** is **bases**.

Coordinates with Respect to a Basis Added!

A basis is useful because it provides us a unique way to represent any vector in V .

I.e. For a given basis $B = \{v_1, \dots, v_p\}$, a vector w has a p -tuple $(\alpha_1, \dots, \alpha_p)_B$ called **the coordinates of w with respect to the basis B** , if \Leftrightarrow

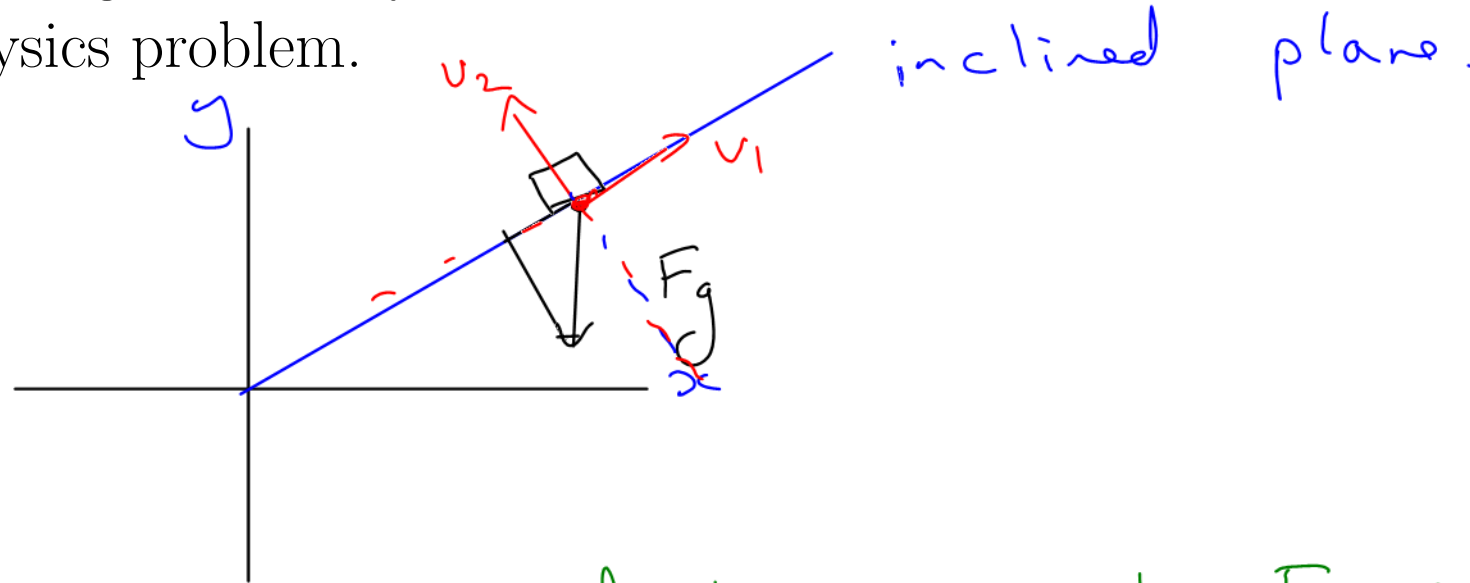
$$\underline{w} = \alpha_1 \underline{v_1} + \dots + \alpha_p \underline{v_p}$$

Question: the uniqueness of the representation relies on the term basis, rather than spanning set. Explain why using linear algebra terms.

$\{u_1 \dots u_p\}$
 linearly indep \Rightarrow linear eq'ns $\alpha_1 u_1 \dots + \alpha_p u_p = w$
 has either a unique solution

but a basis spans all of V or no solution
 \Rightarrow any $w \in V$ is a lin comb'n of $\{u_1, \dots, u_p\} \Rightarrow \exists$ solution. at least 1

Example: Show diagrammatically how the idea of a basis and coordinates in that basis might be used in a physics problem.



Need to represent F_g in
 component - parallel to plane.
 and \perp to plane.

equivalent to writing

$$\vec{F}_g = \underbrace{a_1}_{\substack{\text{parallel to} \\ \text{plane component}}} v_1 + \underbrace{a_2}_{\substack{\text{perp'r to plane} \\ \text{component}}} v_2$$

The set

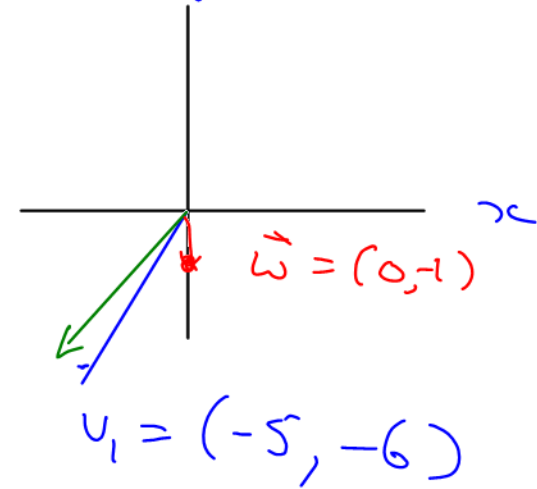
$$B = \left\{ \begin{bmatrix} -5 \\ -6 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \end{bmatrix} \right\}$$

is a basis for \mathbb{R}^2 . Find the coordinates of the vector $\vec{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ relative to the basis B .

$$[\vec{x}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

wk 7 weberwork

are lin'ly indep



$w = \text{lin comb'n of } \{u_1, u_2\} = (-5, -5)$

want $w = au_1 + bu_2$

$$\checkmark \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \cancel{1} \begin{bmatrix} -5 \\ -6 \end{bmatrix} + \cancel{1} \begin{bmatrix} -5 \\ -5 \end{bmatrix} \rightarrow \text{or } \left[\begin{array}{cc|c} -5 & -5 & 0 \\ -6 & -5 & -1 \end{array} \right]$$

augmented matrix

$R_1 / -5$
 $6R_1 - 5R_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -5 & 5 \end{array} \right]$$

$$\xrightarrow{R_2 / -5} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -1 \end{array} \right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array} \right]$$

equivalent

$$a = 1, b = -1$$

The set $B = \{4 + 2x^2, 12 - 4x + 6x^2, 37 - 8x + 18x^2\}$ is a basis for P_2 . Find the coordinates of $p(x) = 44 - 16x + 22x^2$ relative to this basis:

$$[p(x)]_B = \begin{bmatrix} a= \\ b= \\ c= \end{bmatrix}$$

all quadratic poly's.

wk 7 webwork.

i.e. $a(4 + 2x^2) + b(12 - 4x + 6x^2) + c(37 - 8x + 18x^2) = 44 - 16x + 22x^2$

const: $a \cdot 4 + b \cdot 12 + c \cdot 37 = 44$

coeffs: x : $a \cdot 0 + b(-4) + c(-8) = -16$

x^2 : $a \cdot 2 + b(6) + c(18) = 22$

$$\left[\begin{array}{ccc|c} 4 & 12 & 37 & 44 \\ 0 & -4 & -8 & -16 \\ 2 & 6 & 18 & 22 \end{array} \right]$$

sol'n is unique

There will be a unique sol'n - sol'n exists b/c $\{f_1, f_2, f_3\}$ spans P_2 \Rightarrow sol'n exists $\leftarrow \{f_1, f_2, f_3\}$ is lin'ly indep

Midterm 2 - 2016 - Problem 2

Consider the system of linear equations

$$3x_1 + x_2 - 2x_3 + x_4 = 3$$

$$2x_1 - 3x_2 + 4x_3 - 2x_4 = 2$$

$$-x_1 + 2x_2 + 3x_3 + 3x_4 = 8$$

$$4x_1 + x_2 + 15x_3 + 6x_4 = 31$$

where we wish to solve for the quadruple (x_1, x_2, x_3, x_4) of real numbers that satisfy this system.

(a) Write the augmented matrix for this system.

$$\left[\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 3 \\ 2 & -3 & 4 & -2 & 2 \\ -1 & 2 & 3 & 3 & 8 \\ 4 & 1 & 15 & 6 & 31 \end{array} \right]$$

$x_1 \quad x_2 \quad x_3 \quad x_4$

(b) Transform the augmented matrix into reduced row-echelon form (RREF) using a sequence of elementary row operations (and clearly indicate the elementary row operations you use at each step).

$$\left[\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 3 \\ 2 & -3 & 4 & -2 & 2 \\ -1 & 2 & 3 & 3 & 8 \\ 4 & 1 & 15 & 6 & 31 \end{array} \right]$$

$\xrightarrow{-R_3}$
 $\xrightarrow{R_2+2R_3}$
 $\xrightarrow{4R_3+R_4}$
 $\xrightarrow{R_1+3R_3}$

$$\left[\begin{array}{cccc|c} 1 & -2 & -3 & -3 & -8 \\ 0 & 1 & 10 & 4 & 18 \\ 0 & 9 & 27 & 18 & 63 \\ 0 & 7 & 7 & 10 & 27 \end{array} \right]$$

get 0's in col 2

Tidy R_3 , get 0 in R_1 col 2

$\xrightarrow{R_1+2R_2}$
 $\rightarrow R_2$
 $R_3/3$
 R_4

$$\left[\begin{array}{cccc|c} 1 & 0 & 17 & 5 & 28 \\ 0 & 1 & 10 & 4 & 18 \\ 0 & 3 & 9 & 6 & 21 \\ 0 & 7 & 7 & 10 & 27 \end{array} \right]$$

$\xrightarrow{R_1}$
 $\xrightarrow{R_2}$
 $\xrightarrow{3R_2-R_3}$
 $\xrightarrow{7R_2-R_4}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 17 & 5 & 28 \\ 0 & 1 & 10 & 4 & 18 \\ 0 & 0 & 21 & 6 & 33 \\ 0 & 0 & 63 & 18 & 99 \end{array} \right]$$

Tidy/scale R_3, R_4



$$\begin{array}{l}
 R_1 \\
 \rightarrow R_2 \\
 R_3/3 \\
 R_4/9
 \end{array}
 \left[\begin{array}{cccc|c}
 1 & 0 & 17 & 5 & 28 \\
 0 & 1 & 10 & 4 & 18 \\
 0 & 0 & 7 & 2 & 4 \\
 0 & 0 & 7 & 2 & 11
 \end{array} \right]$$

hey! duplicates!
 One row will get cancelled.

$$\left[\begin{array}{cccc|c}
 3 & 1 & -2 & 1 & 3 \\
 2 & -3 & 4 & -2 & 2 \\
 -1 & 2 & 3 & 3 & 8 \\
 4 & 1 & 15 & 6 & 31
 \end{array} \right]$$

need 0's here.

In RREF!

$$\begin{array}{l}
 R_1 \\
 R_2 \\
 R_3/7 \\
 R_3 - R_4
 \end{array}
 \left[\begin{array}{cccc|c}
 1 & 0 & 17 & 5 & 28 \\
 0 & 1 & 10 & 4 & 18 \\
 0 & 0 & 1 & 2/7 & 1/7 \\
 0 & 0 & 0 & 0 & 0
 \end{array} \right]$$

$$\begin{array}{l}
 R_1 - 17R_3 \\
 R_2 - 10R_3
 \end{array}
 \left[\begin{array}{cccc|c}
 1 & 0 & 0 & 1/7 & 9/7 \\
 0 & 1 & 0 & 8/7 & 16/7 \\
 0 & 0 & 1 & 2/7 & 1/7 \\
 0 & 0 & 0 & 0 & 0
 \end{array} \right]$$

(c) Using (b), find the set of all solutions to the original system.

$$\left[\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 3 \\ 2 & -3 & 4 & -2 & 2 \\ -1 & 2 & 3 & 3 & 8 \\ 4 & 1 & 15 & 6 & 31 \end{array} \right]$$

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline 1 & 0 & 0 & 1/7 & 9/7 \\ 0 & 1 & 0 & 8/7 & 16/7 \\ 0 & 0 & 1 & 2/7 & 11/7 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

free variable. Let $x_4 = t$.

$$x_1 = 9/7 - 1/7 t$$

$$x_2 = 16/7 - 8/7 t$$

$$x_3 = 11/7 - 2/7 t$$

$$x_4 = t$$