Week #7: Dimension of a Vector Space, Linear Transformations

Review: Generating Sets, Bases

Concept Question:

Definition: Given a vector space $\mathbf{V}$, a **generating** or **spanning set** for $\mathbf{V}$ is a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbf{V}$ for which...

Select **all** that apply:

(A) The set of vectors is linearly independent.

(B) The set of vectors is linearly dependent.

(C) The span of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is equal to $\mathbf{V}$.

(D) The span of $\mathbf{V}$ is equal to $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

Concept Question

Definition: Given a vector space $\mathbf{V}$, a **basis for V** is a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that...

basis

= generating set

+

lin'ly independent.

Select **all** that apply:

(A) The set of vectors is linearly independent.

(B) The set of vectors is linearly dependent.

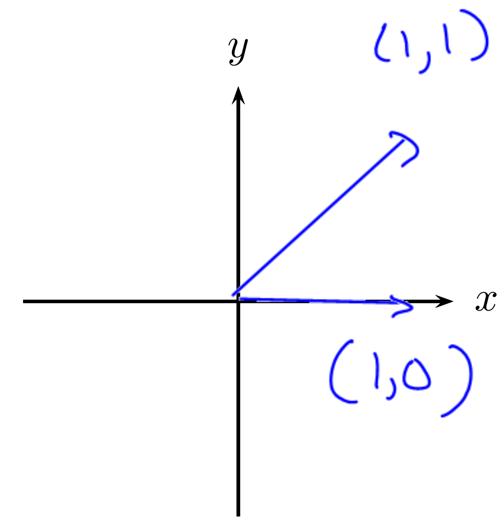
(C) The span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is equal to $\mathbf{V}$.

(D) The span of $\mathbf{V}$ is equal to $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example: Categorize each of the following vectors sets by whether they are generating sets, linearly independent, and/or a basis for $\mathbb{R}^2$.

(a) The set $\{(1, 0), (1, 1)\}$.

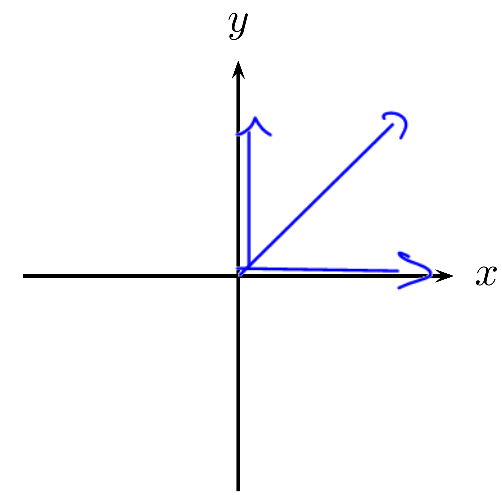
Generating Set	Linearly Indep.	Basis
Yes	Yes	Yes



(b) The set $\{(1,0), (0,1), (1,1)\}$.

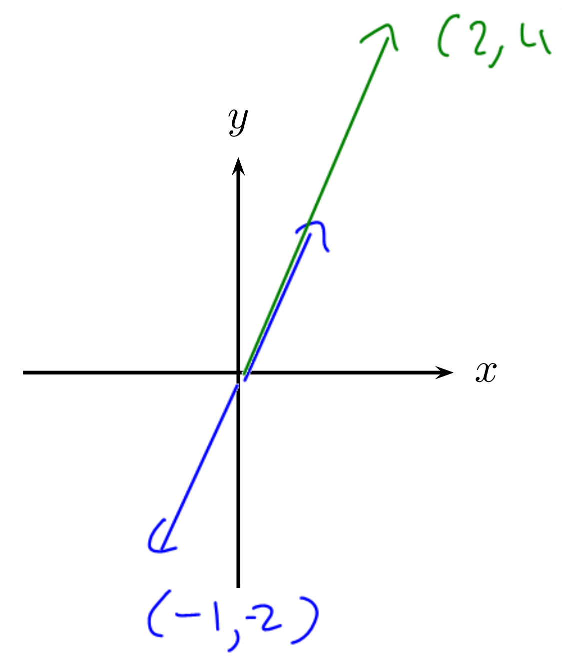
$(1,1) = (1,0) + (0,1)$

Generating Set	Linearly Indep.	Basis
Yes	No	No



(c) The set $\{(1,2), (2,4), (-1,-2)\}$.

Generating Set	Linearly Indep.	Basis
No	No	No



Dimension of a Vector Space

Provisional definition: Let **V** be a vector space. The *dimension* of **V**, or dim(**V**), is the number of vectors in a basis for **V**.

If **V** doesn't have a finite basis then dim(**V**) is infinite.

Example: what vector spaces have we seen with finite dimension?

$\mathbb{R}$ – 1 dim'l

$\mathbb{R}^2$ – 2 "

$\mathbb{R}^3$ – 3 "

P_2 : span $\{1, x, x^2\}$

3 dim'l

Example: what vector spaces have we seen with infinite dimension?

C^∞

Issue: we need to confirm that **any** two bases for the same vector space have the same number of elements.

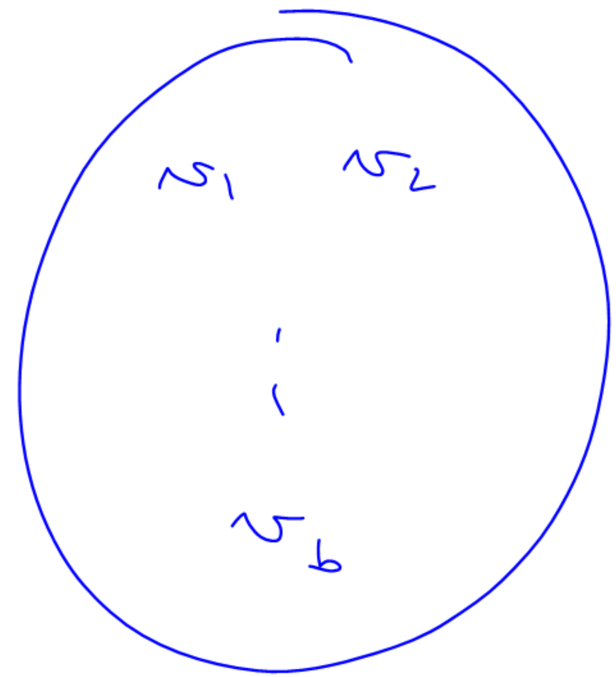
E.g. that in $\mathbb{R}^3$, it isn't possible to get a basis with 2 vectors, or with 4, if we choose them just right.

We will prove this with the help of a "lemma".
From Wikipedia: *In mathematics, a lemma* [...] *is a generally minor, proven proposition which is used as a stepping stone to a larger result. For that reason, it is also known as a "helping theorem" or an "auxiliary theorem".*

Key Lemma: Let $\mathbf{V}$ be a vector space, and suppose that $\{u_1, u_2, \ldots, u_a\}$ are a set of linearly independent vectors in $\mathbf{V}$, and that $\{v_1, v_2, \ldots, v_b\}$ are a spanning set for $\mathbf{V}$.
Then $a \leq b$.
Illustrate this with a diagram.

$a = \#$ vectors
lin'ly indep

$a \leq b$

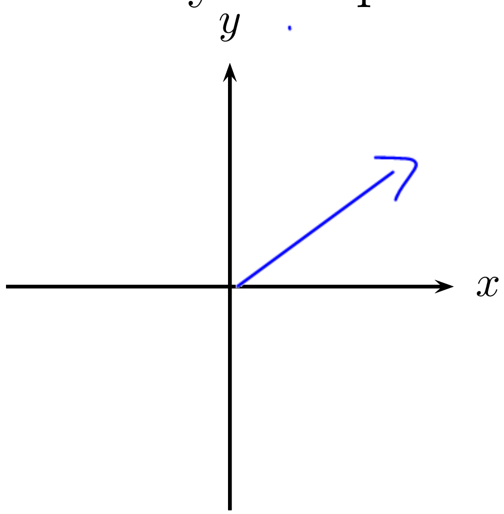


$b = \#$ of vectors
spanning set

Key Lemma: Let $\mathbf{V}$ be a vector space, and suppose that $\{u_1, u_2, \ldots, u_a\}$ are a set of linearly independent vectors in $\mathbf{V}$, and that $\{v_1, v_2, \ldots, v_b\}$ are a spanning set for $\mathbf{V}$.
Then $a \leq b$.

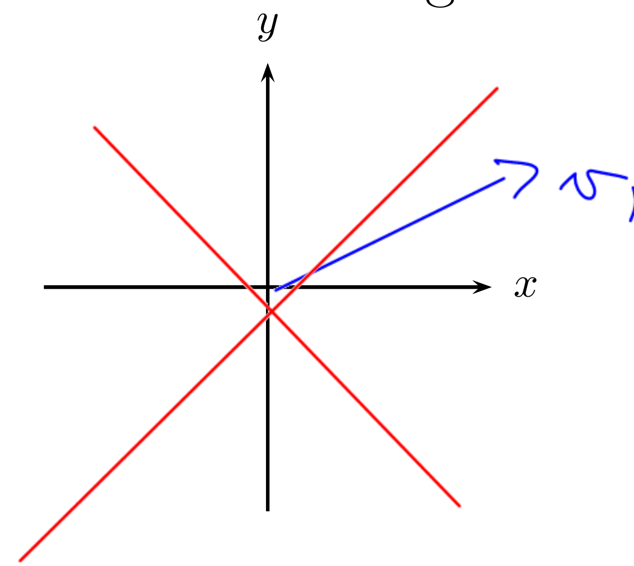
Example: Illustrate this with examples of vector sets in $\mathbb{R}^2$.

Linearly Independent



$a = 1$

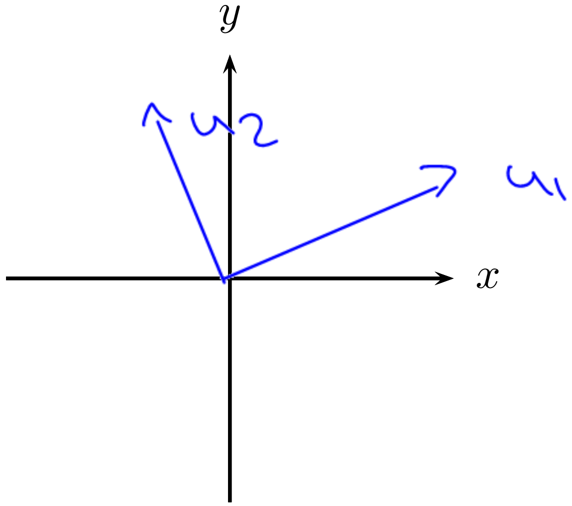
Generating Set



$b = 1$

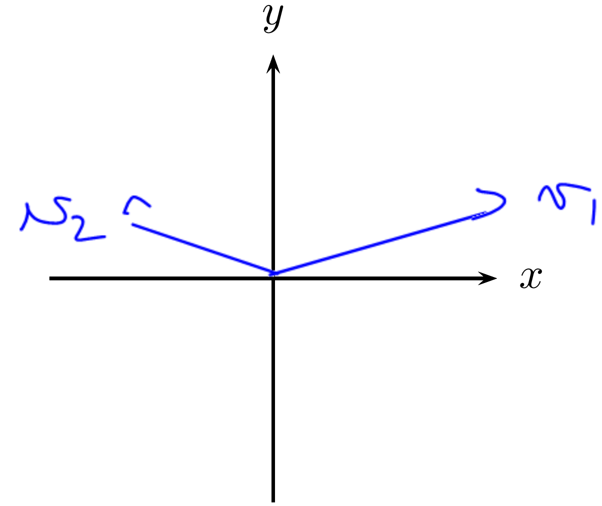
Linearly Independent

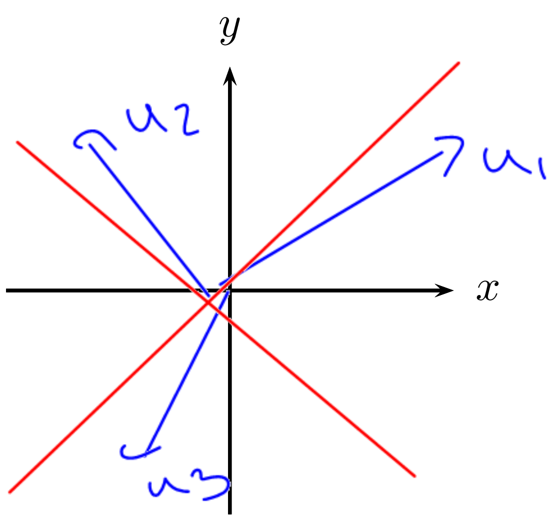
Generating Set



$a = 2$

$b = 2$

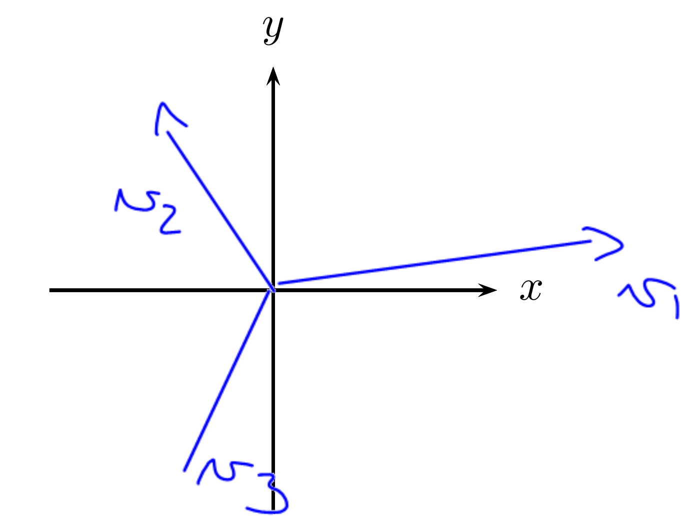




not lin'ly indep

at 3 vectors

$b = 3$



The Key Lemma can be proven in several ways. On request, there is a PDF with a proof based on systems of linear equations, and another proof based on induction. They are quite long though, so we won't cover them in these lectures.

However, with the use of the Key Lemma, it is easy to prove the result we need to uniquely define dimensionality.

Theorem 14: Let $\mathbf{V}$ be a vector space and $\{v_1, v_2, \ldots v_p\}$ and $\{w_1, w_2, \ldots, w_q\}$ be two bases for $\mathbf{V}$. Then $p = q$.

p elements

q elements

Paraphrase this theorem.

each of our bases for V

must have the same number of elements

=> every basis for V is the same size

=> dim(V) is well-defined

Theorem 14: Let **V** be a vector space and $\overbrace{\{v_1, v_2, \ldots v_p\}}^{\text{Set } A}$ and $\overbrace{\{w_1, w_2, \ldots, w_q\}}^{\text{Set B}}$ be two bases for **V**. Then the sets must be the same size, i.e. $p = q$.

Proof. Use the Key Lemma to show that $p = q$.

Set A is a basis for V,
$\Rightarrow$ Set A is lin'ly indep

bases are sets which are
1) generating set for V
2) lin'ly indep.

Set B is a basis for V
$\Rightarrow$ set B is a generating set for V

$\Rightarrow \underbrace{p}_{\text{size of Set A}} \leq \underbrace{q}_{\text{size of Set B}}$ by the Key Lemma

But also:

Set A is a generating set

and Set B is lin'ly indep

by Key Lemma

$$\underbrace{q}_{\text{size of set B}} \leq \underbrace{p}_{\text{size of set A}}$$

$\Rightarrow \quad p = q$ is the only option

$\Rightarrow$ size of any two bases must be the same.

Example: Consider the vector subspace

$$\mathbf{V} = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\} \subset \mathbb{R}^3.$$

Show that the pair of vectors $\{v_1, v_2\} = \{(1, 0, -1), (0, 1, -1)\}$ is a basis for $\mathbf{V}$.

To show $\{v_1, v_2\}$ is a basis for V, must show

1) $\{v_1, v_2\}$ is lin'ly indep, and

2) $\{v_1, v_2\}$ is a spanning set for V.

1) $\{v_1, v_2\} = \{(1,0,-1), (0,1,-1)\}$ is only 2 vectors, which are not multiples of each other

$\Rightarrow$ $\{v_1, v_2\}$ is lin'ly indep.

2) Claim: $\{v_1, v_2\}$ spans V

$\Rightarrow$ any element of V is a lin comb'n of $\{v_1, v_2\}$.

$$\mathbf{V} = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\} \subset \mathbb{R}^3.$$

To be in V, a point (x, y, z) satisfies

$$x + y + z = 0$$

or $z = -x - y$

$\Rightarrow$ every element in V has form $(x, y, -x-y)$

Can we solve $a \cdot v_1 + b\, v_2 = (x, y, -x-y)$ for all x, y?

def'n v_1, v_2 $\quad a(1, 0, -1) + b(0, 1, -1) = (x, y, -x-y)$

Solve by components: $a = x$ ✓

$b = y$ ✓

check 3^{rd} component:

$a(-1) + b(-1) = x(-1) + y(-1) = -x-y$ ✓

$\Rightarrow$ For $(x, y, -x-y) \in V$, we can find $a, b \in \mathbb{R}$ s.t.

$a v_1 + b v_2 = (x, y, -x-y) \Rightarrow \{v_1, v_2\}$ spans ✓

plane in $\mathbb{R}^3$

$$\mathbf{V} = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\} \subset \mathbb{R}^3.$$

Use that result to determine the dimension of **V**.

" # of elements in a basis for V.

$\{v_1, v_2\} = \{(1,0,-1), (0,1,-1)\}$ is a basis for V.

$\Rightarrow$ V is 2 dim'l

Example: Make a case for the dimensionality of $C^\infty(\mathbb{R})$ being infinite.

Taylor Series

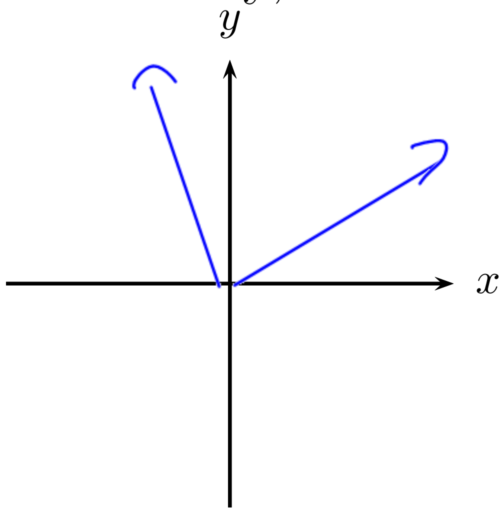
$\in C^\infty \rightarrow e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

basis for $C^\infty = \{1, x, x^2, x^3, x^4 \ldots\}$ (?)

a finite set cannot span C^∞

Corollaries Related to Dimension

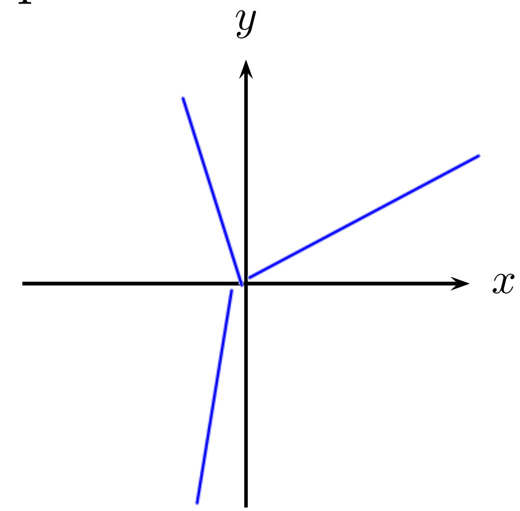
Intuitively, can we have 3 or more vectors in $\mathbb{R}^2$ that are linearly **in**dependent?



2 → # vectors in a basis

vectors if set: lin'ly indep

and a spanning set



set of 3 : spanning set contr

+ lin'~~ly~~ indep

Prove in general than if $\dim(V) = N$ then any set of $N+1$ or more vectors in V must be linearly **de**pendent.

$\dim(V) = N \Rightarrow$ any basis for V contains N elements

$\rightarrow$ is a spanning set

Let $\{u_1, u_2 \ldots u_{N+1}\}$ is a given set.

Claim " is lin'ly indep.

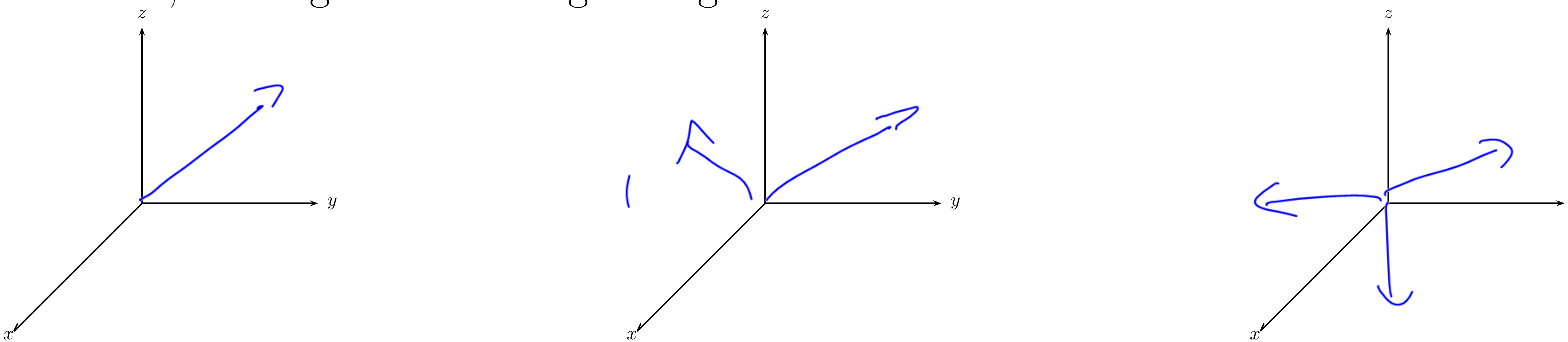
Key Lemma: $\left|\begin{matrix}\text{set w/} \\ \text{lin'ly indep}\end{matrix}\right| \leq \left|\begin{matrix}\text{spanning} \\ \text{set}\end{matrix}\right|$

$N+1 \leq N$

impossible

$\Rightarrow$ claim that $\{u_1, u_2 \ldots u_{N+1}\}$ is lin'ly indep is false $\Rightarrow$ a set w/ $N+1$ vectors must be lin'ly dep.

And looking at the reverse of this, if we have a linearly **in**dependent set, or are trying to build one, how big can that set get? E.g. in $\mathbb{R}^3$.



Prove in general than if $\dim(V) = N$, and we have a set $\{\underbrace{v_1, \ldots, v_p}_{p \text{ vectors}}\}$ of linearly **in**dependent vectors, then $p \leq N$.

Key Lemma:

A basis for V has N vectors and it is a spanning set

$\Longrightarrow$ $p \leq N$.

These results give us **alternative definitions** of the dimension of a vector space $\mathbf{V}$:

- $\dim(\mathbf{V})$ is the minimum number of vectors in any generating set for $\mathbf{V}$.

- $\dim(\mathbf{V})$ is the maximum number of vectors in any linearly independent set for $\mathbf{V}$.

- $\dim(\mathbf{V})$ is the lowest number N such that all sets of $N+1$ vectors must be linearly **de**pendent.

Linear Transformations

Back from Week 1: **Mappings and Functions.**

Definition: Let S and T be two sets. A **mapping** or **function** from S to T is a rule which assigns to each element of S **one and only one** element of T.

function

Notation:

$$f : S \to T$$

$$x \to f(x)$$

actual transformation

S is called the input set or domain

T is called the output set or co-domain

not Image

New special kind of functions: *linear* functions.

Definition: Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces.

A function $L : \mathbf{V} \to \mathbf{W}$ is called a **linear function** if it passes two tests.

1. Addition Test: For all $v_1, v_2 \in \mathbf{V}$,

$$L(v_1 + v_2) = L(v_1) + L(v_2)$$

→ safe to swap
- addition &
- transform

2. Scalar Multiplication Test: For all $\alpha \in \mathbb{R}$, and $v \in \mathbf{V}$,

$$L(\alpha v) = \alpha L(v)$$

swap order of
- scalar multiplication &
- transform

Equivalent terms: linear **function**, **transformation**, **map** and **mapping**.

Example: the **identity map** $L : \mathbf{V} \to \mathbf{V}$ is defined by $L(v) = v$. Show that this is a linear map.

1) Test whether $\underbrace{L(v_1 + v_2)}_{LHS} = \underbrace{L(v_1) + L(v_2)}_{RHS}$ for some $v_1, v_2 \in V$

Consider $LHS = L(v_1 + v_2)$

$= v_1 + v_2$ by def'n of L

$RHS = L(v_1) + L(v_2)$

$= v_1 + v_2$ by def'n of L

Since $LHS = RHS$, $L(v) = v$ passes 1^{st} criterion for a linear function.

$$L(v) = v$$

Test whether $\underbrace{L(\alpha v)}_{LHS} = \underbrace{\alpha\,(L(v))}_{RHS}$ $\quad \alpha \in \mathbb{R},\ v \in V$

$LHS = L(\alpha v)$

$= \alpha v$ by def'n of L

and $RHS = \alpha L(v)$

$= \alpha v$ by def'n of L

Since $LHS = RHS$, $L(v) = v$ passes 2^{nd} criterion

$\Rightarrow L(v) = v$ <u>is</u> a linear transform.

Example: Show that the function $L : \mathbb{R} \to \mathbb{R}$ given by $L(x) = \sqrt{|x|}$ is **not** a linear transformation.

We expect this <u>not</u> to be linear.

Try to build a counter-example to

$$L(v_1 + v_2) = L(v_1) + L(v_2)$$

eg. let $v_1 = 4$,

$v_2 = 9$

$$\text{LHS} = L(v_1 + v_2) = L(4+9) = L(13) = \sqrt{13}$$

$$\text{while RHS} = L(v_1) + L(v_2) = L(4) + L(9) = \sqrt{4} + \sqrt{9} = 2+3 = 5$$

but $\sqrt{13} \neq 5$ so $L(x) = \sqrt{|x|}$ is <u>not</u> a linear function.

Example: Determine whether the function $L : \mathbb{R} \to \mathbb{R}$ given by $L(x) = 3x + 1$ is a linear transformation or not.

affine function

Let $x_1 = 1$, $x_2 = 3$

Requirement $L(x_1 + x_2) = L(x_1) + L(x_2)$

$LHS = L(1+3) = L(4) = 3 \cdot 4 + 1 = 13$

while $RHS = L(1) + L(3) = [3(1)+1] + [3(3)+1]$

$= 4 + 10$

$= 14$

Since $13 \neq 14$, this L fails this linearity criterion

$\Rightarrow$ $L(x) = 3x+1$ is not a linear function in linear algebra

Quick fact: helpful for non-linearity check:
In any linear function $L : \mathbf{V} \to \mathbf{W}$, $L(\mathbf{0_V}) = \mathbf{0_W}$.
Why is this the case?

Linearity requires $L(\alpha v) = \alpha L(v)$ $\alpha \in \mathbb{R}$, $v \in V$

but if $\alpha = 0$

$L(0_v) = 0 \cdot L(v)$

$L(0_v) = 0_w$

$L : \mathbb{R}^3 \to \mathbb{R}^2$

Example: Use this to show that $L(x, y, z) = (x + 1, y + z)$ is not a linear transformation.

Check: $L(0,0,0)$ b/c $(0,0,0) \in \mathbb{R}^3$ is $\bar{0}$ in $\mathbb{R}^3$

$= (1,0) \neq \bar{0}$ in $\mathbb{R}^2$

$\Rightarrow L(x, y, z)$ is not a linear function.

Example: show this test isn't always sufficient by giving examples of non-linear transformations that still satisfy $L(\mathbf{0_V}) = \mathbf{0_W}$.

$L(x)$: $\mathbb{R} \to \mathbb{R}$

$x \to \sqrt{|x|}$

then $L(0) = 0$

but still L is not linear

Summary of the $L(\mathbf{0_V}) = \mathbf{0_W}$ test:

- If $L(\mathbf{0_V}) = \mathbf{0_W}$ then L might be linear; need to use both linearity tests

- If $L(\mathbf{0_V}) \neq \mathbf{0_W}$ then L definitely not linear

Example: Determine whether the definite integral over a fixed interval $[a, b]$ is a linear transformation.

$L : \underline{C^\infty} \to \underline{\mathbb{R}}$

$L(f) = \int_a^b f(x)\,dx$

Check 1) For linearity, $L(f_1 + f_2) = L(f_1) + L(f_2)$ for $f_1, f_2 \in C^\infty$

Consider LHS

$$= \int_a^b (f_1 + f_2)(x)\,dx = \int_a^b f_1(x) + f_2(x)\,dx$$

$$= \left(\int_a^b f_1(x)\,dx\right) + \left(\int_a^b f_2(x)\,dx\right)$$

$$= L(f_1) + L(f_2) \quad \checkmark$$

2) for linearity, $L(\alpha f) = \alpha L(f)$ $\quad \alpha \in \mathbb{R}$, $f \in C^{\infty}$

$$LHS = L(\alpha f) = \int_a^b (\alpha f)(x)\,dx$$

$$= \int_a^b \alpha \cdot f(x)\,dx$$

$$= \alpha \int_a^b f(x)\,dx$$

$$= \alpha L(f) \quad \checkmark$$

$\Rightarrow \quad L(f) = \int_a^b f(x)\,dx$ is a linear transform.

The Kernel or Null Space of Linear Transformations

Example: consider the transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ with $L(x, y, z) = (x - y, y - z)$.
What is the input set or *domain* of L, and what is its dimension?

$\mathbb{R}^3$, a 3 dim'l space

What is the output set or *co-domain* of L, and what is its dimension?

$\mathbb{R}^2$, a 2 dim'l space

$$L(x, y, z) = (x - y, y + z)$$

Compute $L(1, 1, -1) =$ $(1-1, \ 1+(-1)) = (0,0)$

scalar multiples

Compute $L(-4, -4, 4) =$ $(-4-(-4), \ -4+4) = (0,0)$

What do you think all the points in the input space $(x, y, z) \in \mathbb{R}^3$ satisfying $L(x, y, z) = \mathbf{0}$ in $\mathbb{R}^2$ will have in common?

scalar multiples of $(1,1,-1)$

What is another name for that set of points for which $L(x, y, z) = \mathbf{0}$?

<u>span</u> of $\{(1,1,-1)\}$

vector subspace of $\mathbb{R}^3$ defined by the span of $\{(1,1,-1)\}$

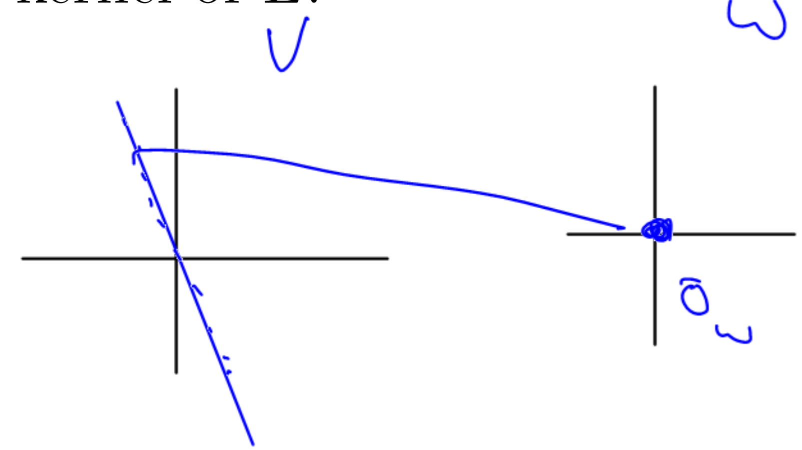
Definition: for a linear mapping $L : \mathbf{V} \to \mathbf{W}$, the set of all **input** vectors that are mapped to $\mathbf{0_W}$ is called the **kernel** of L, or the **null space** of L. Formally:

$$\mathrm{Ker}(L) = \{\mathbf{v} \in V : L(\mathbf{v}) = \mathbf{0_W}\}$$

Example: for $L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $L(f) = \frac{d}{dx} f(x)$, find the kernel of L.

$\mathrm{Ker}(L) = \{$ all functions f s.t. $L(f) = \bar{0}$ in $C^\infty \}$

zero function



$= \{$ any constant function $\}$

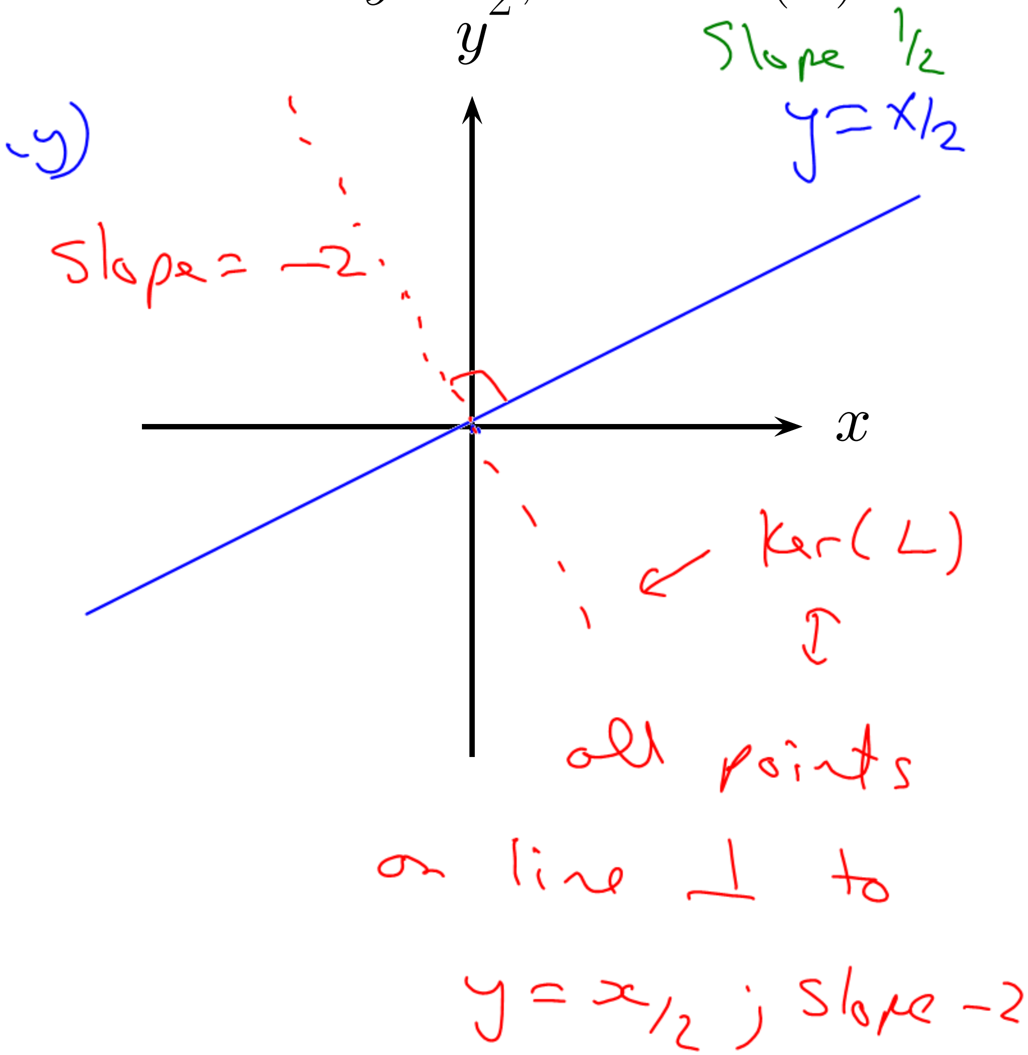
$= \{ f \in C^\infty \mid f(x) = c$ for all x's, $c \in \mathbb{R} \}$.

Example: for $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(x, y) =$ projection of (x, y) onto the line $y = \frac{x}{2}$, find $\mathrm{Ker}(L)$.

$\mathrm{Ker}(L) = \{$ a set of input values (x,y) s.t. $L(x,y) = (0,0) \}$

$= \{$ a set of input values (x,y) s.t. projection of (x,y) onto line $y = x/2$ is $(0,0) \}$

$= \{ (x,y) \mid y = -2x \}$



Kernel as a Vector Subspace

$\{\} \xrightarrow{L} \bar{0}_W$

From these examples, we get an idea that the kernel is not just some subset of the input space $\mathbf{V}$, but rather a highly structured subset...

Theorem 17. Let $\mathbf{V}$ and $\mathbf{W}$ be real vector spaces, and let $L : \mathbf{V} \to \mathbf{W}$ be a **linear** mapping. Then the kernel of L, $\mathrm{Ker}(L)$, is a **vector subspace** of $\mathbf{V}$, the input space

Recall: What are the requirements for a subset $S \subset \mathbf{V}$ to be a vector subspace?

1) $\bar{0}_V \in S$

input $\bar{0}_V$

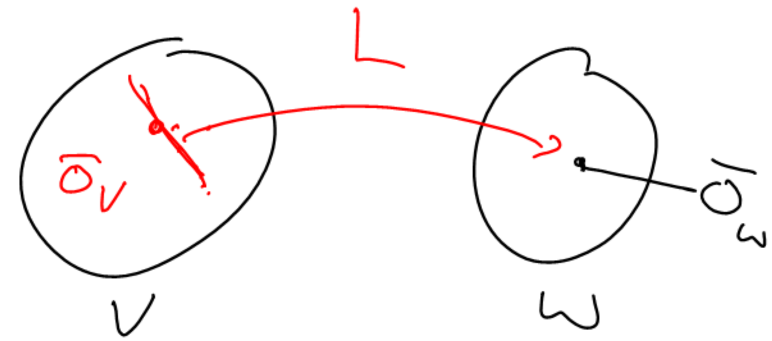
2) S closed under addition

3) S " " scalar multiplication

Theorem 17. Let **V** and **W** be real vector spaces, and let $L : \mathbf{V} \to \mathbf{W}$ be a **linear** mapping. Then the kernel of L, $\mathrm{Ker}(L)$, is a vector subspace of **V**.

Proof

Recall: $\mathrm{Ker}(L) = \{ v \in V : L(v) = \bar{0}_W \}$



1) Q: is $L(\bar{0}_V) = \bar{0}_W$?

Consider linearity property

$$L(\alpha v) = \alpha L(v), \quad \text{then } w/\alpha = 0$$

$$L(\bar{0}_V) = L(0 \cdot v) = 0 \cdot \underbrace{L(v)}_{\text{some } w \in W} = \bar{0}_W$$

so $\bar{0}_V \in \mathrm{Ker}(L)$

Theorem 17. Let $\mathbf{V}$ and $\mathbf{W}$ be real vector spaces, and let $L : \mathbf{V} \to \mathbf{W}$ be a **linear** mapping. Then the kernel of L, $\mathrm{Ker}(L)$, is a vector subspace of $\mathbf{V}$.

2) Closed under addition

Consider $v_1, v_2 \in \mathrm{Ker}(L)$ $\to$ know $L(v_1) = \bar{0}_W$

and $L(v_2) = \bar{0}_W$

Q: is $v_1 + v_2$ also in $\mathrm{Ker}(L)$?

i.e is $L(v_1 + v_2) = \bar{0}_W$?

$L(v_1 + v_2)$

$= L(v_1) + L(v_2)$ by linearity of L $\Rightarrow v_1 + v_2 \in \mathrm{Ker}(L)$

$= \bar{0}_W + \bar{0}_W$

$= \bar{0}_W$

Theorem 17. Let $\mathbf{V}$ and $\mathbf{W}$ be real vector spaces, and let $L : \mathbf{V} \to \mathbf{W}$ be a **linear** mapping. Then the kernel of L, $\text{Ker}(L)$, is a vector subspace of $\mathbf{V}$.

3) Closed under scalar multiplication

Consider $v \in \text{Ker}(L)$ and $\alpha \in \mathbb{R}$ $\to$ know $L(v) = \bar{0}_W$

Q: is $L(\alpha v) \in \text{Ker}(L)$, i.e. is $L(\alpha v) = \bar{0}_W$?

Look at $L(\alpha v)$

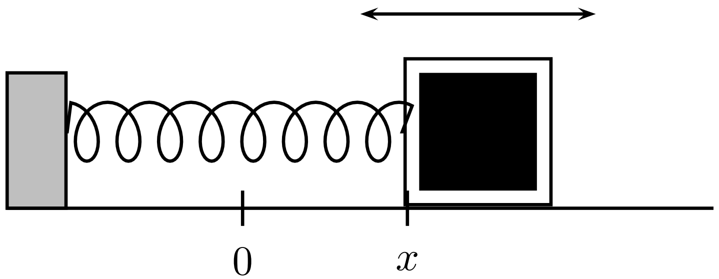
$= \alpha L(v)$ by linearity of L

$= \alpha \bar{0}_W$ so $L(\alpha v) = \bar{0}_W$ too

$= \bar{0}_W$ so $L(\alpha v) \in \text{Ker}(L)$

B/c 1), 2), 3) were all satisfied, $\text{Ker}(L)$ is a subspace of the input space V.

The Kernel: Application to Differential Equations



Example: For the spring system shown above, what is the differential equation that governs the position of the mass over time, $x(t)$?

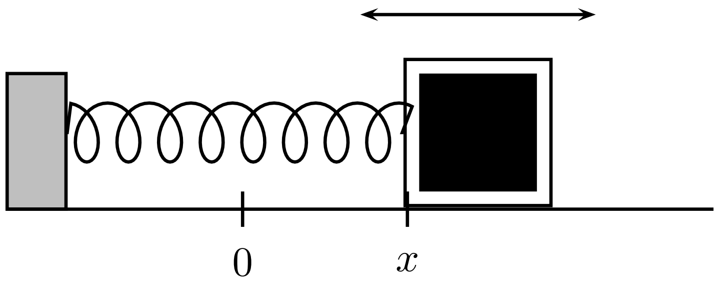
$$\sum F = ma$$

$$-kx = m \cdot x''(t)$$

$$x''(t) = -\frac{k}{m} x(t)$$

For the specific case where the mass is 1 kg, and the spring constant is 1 N/m, what is the differential equation?

$$x'' = -x$$



By inspecting $x''(t) = -x(t)$, we found that possible solutions were
$x_1(t) =$ cos(t) and
$x_2(t) =$ sin(t).

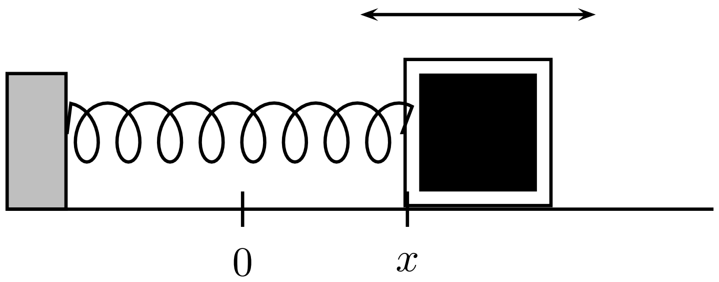
period 2π s

amplitude?

We showed earlier that the set of solutions to
$x''(t) = -x(t)$
were a vector subspace of $C^\infty(\mathbb{R})$, by showing the set satisfied the 3 subspace axioms.

1) $\bar{0}$ ∈ sol'ns
2) sol'ns closed add
3) " " sc. mult.

We could also then conclude that functions of the form
$x(t) =$ $c_1 \cos(t) + c_2 \sin(t)$
were solutions too. Provide a rationale for that statement.



Prove again that the set of solutions to $x''(t) = -x(t)$ is a vector subspace, but now using a linear transformation argument.

Ker (L) is a vector subspace for any linear L.

Rewrite DE as $\underbrace{x'' + x}_{\text{define } L} = 0$ ⇐ 0 function; $=0$ for all t

$$L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$$
$$x \to x'' + x$$

⇒ Ker(L) is a vector subspace of $C^\infty(\mathbb{R})$.

⇒ sol'ns to $x'' + x = 0$ are a vector subspace.