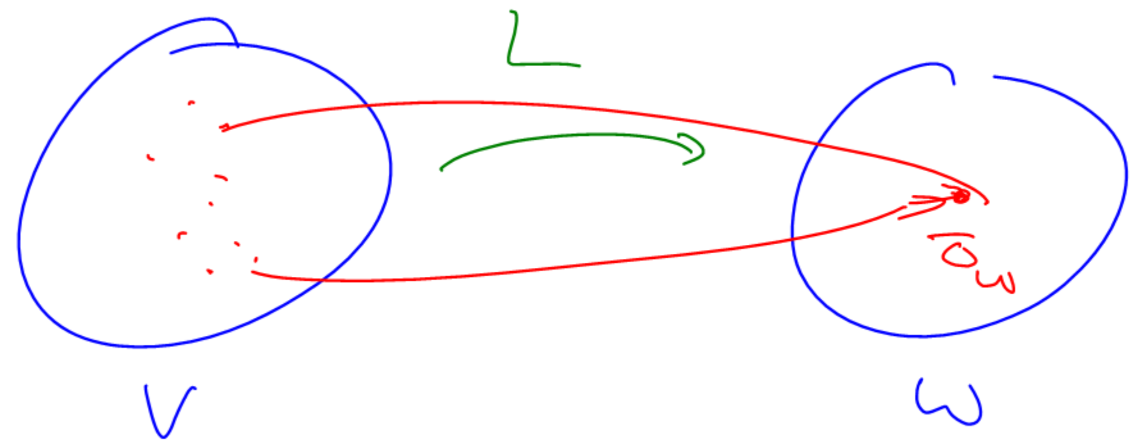
Week #8: Null Space, Image, Matrix Forms

Review:

Definition: for a linear mapping $L : \mathbf{V} \to \mathbf{W}$, the set of all **input** vectors that are mapped to $\mathbf{0_W}$ is called the:



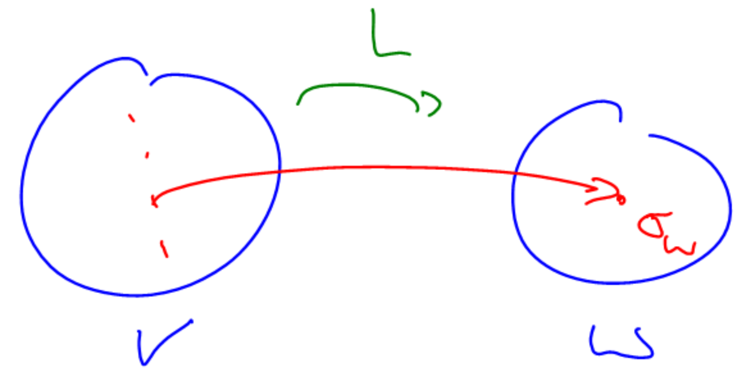
(A) Function of L.

(B) Image of L.

(C) Kernel of L.   or nullspace of L

(D) Zeros of L.

The kernel of a linear mapping $L : \mathbf{V} \to \mathbf{W}$ is:



(A) some subset of the input space $\mathbf{V}$.

~~(B) some subset of the output space $\mathbf{W}$.~~

(C) a vector subspace of the input space $\mathbf{V}$.

~~(D) a vector subspace of the output space $\mathbf{W}$.~~

Ker(L)

Kernel and Image of Linear Transformations

Where we are heading: for a linear mapping $L : \mathbf{V} \to \mathbf{W}$, we will be able to divide up the dimensions of the **input** space, $\mathbf{V}$.

n dim

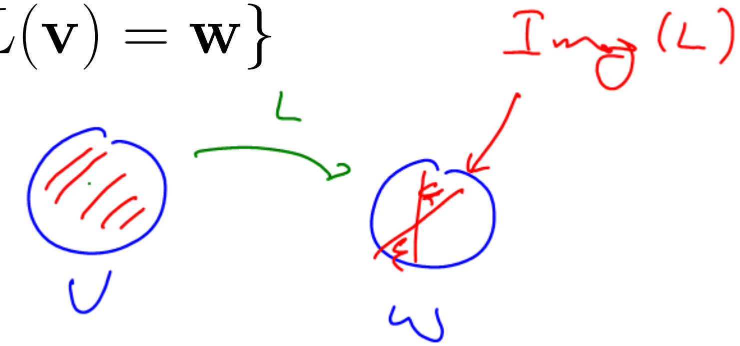
Let $\dim(\mathbf{V}) = n$ be the starting dimension.

- The **kernel** of L, with $L(\mathbf{v}) = \mathbf{0_W}$, will have some dimension k, and

- The **image** of L, all the output $\mathbf{w}$'s we can get from some $L(\mathbf{v}) = \mathbf{w}$, will have dimension $n - k$.

Definition: Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation. We define the **image** of L, $\mathrm{Im}(L)$, by

$$\mathrm{Im}(L) = \{\mathbf{w} \in \mathbf{W} : \text{There is some } \mathbf{v} \in \mathbf{V} \text{ such that } L(\mathbf{v}) = \mathbf{w}\}$$



Note: the whole output space $\mathbf{W}$ is called the **co-domain** or **target set**.

or output set

Note that for a linear transformation $L : \mathbf{V} \to \mathbf{W}$,

- the kernel $\mathrm{Ker}(L)$ is a subset of the **input** space $\mathbf{V}$, while

  $\hookrightarrow = \{ v \in V : L(v) = \bar{0}_w \}$

- the image $\mathrm{Im}(L)$ is a subset of the **output** space $\mathbf{W}$.

  $\hookrightarrow = \{ w \in W : \exists\, v \text{ s.t. } L(v) = w \}$

- the image **may or may not** be all of $\mathbf{W}$.

Problem: Sketch possible the following configurations for linear mappings from $\mathbb{R}^2 \to \mathbb{R}^3$:

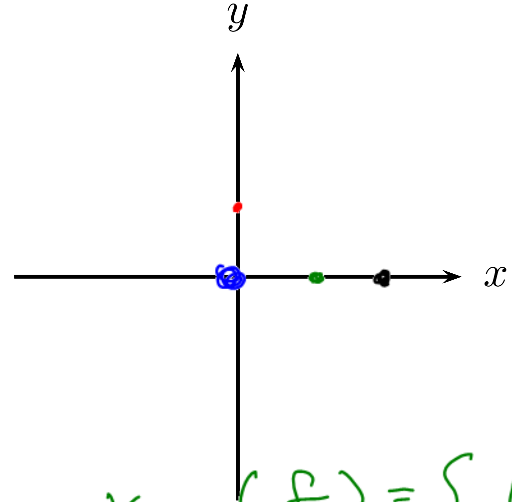
$f(0,0) = (0,0,0) = \bar{0}_{\mathbb{R}^3}$

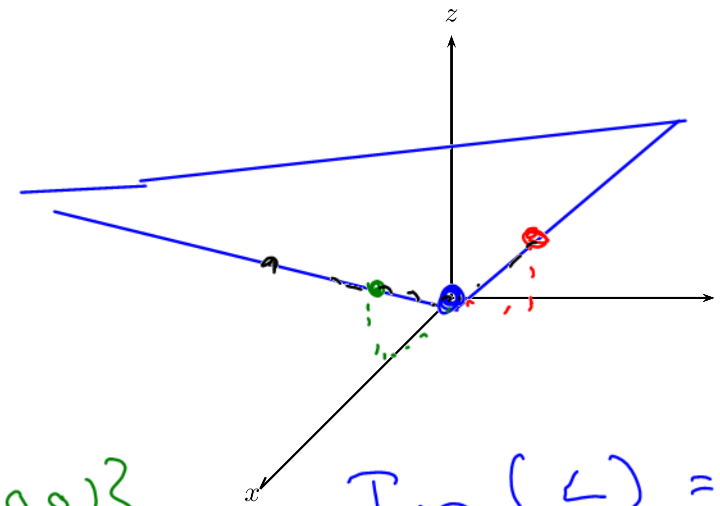
$f(1,0) = (1,0,1)$

$f(0,1) = (0,1,1)$

$f(2,0) = (2,0,2)$

$f(x, y) = [x, y, x + y]$





Ker(f) = $\{(0,0)\}$
single point/
0 dim'l

Im(L) = plane (2D)
through origin.

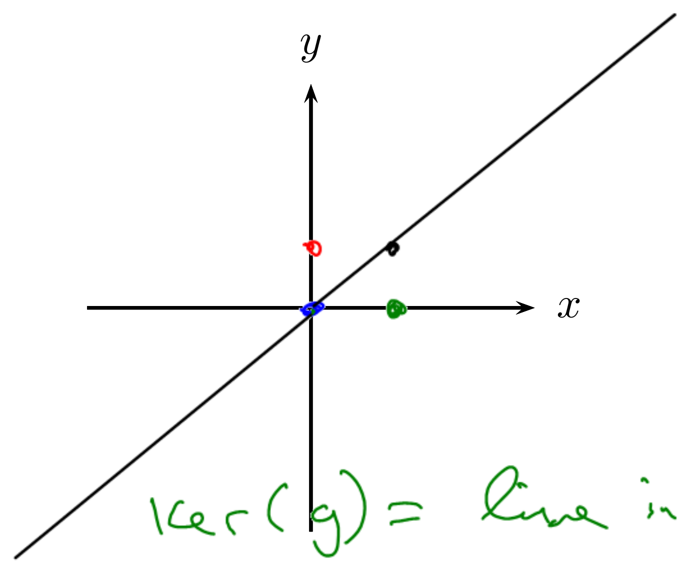
$g(0,0) = (0,0,0)$

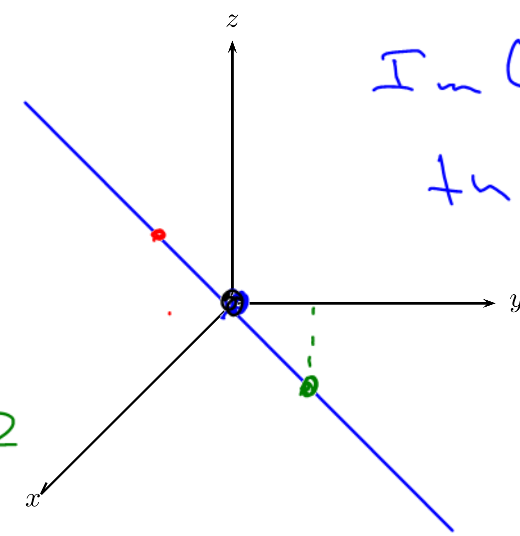
$g(1,0) = (0,1,-1)$

$g(0,1) = (0,-1,1)$

$g(1,1) = (0,0,0)$

$g(x, y) = [0, x - y, y - x]$





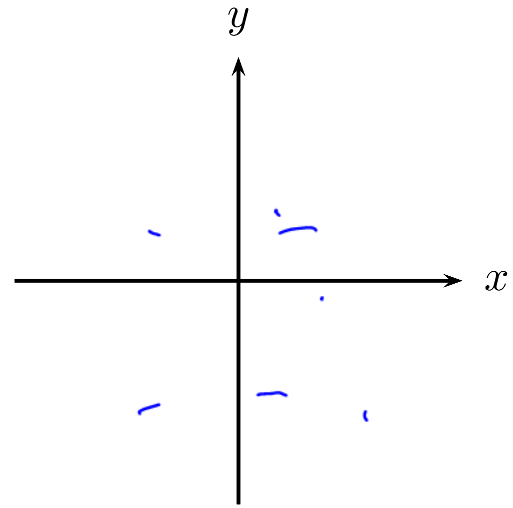
Ker(g) = line in $\mathbb{R}^2$
(1D)

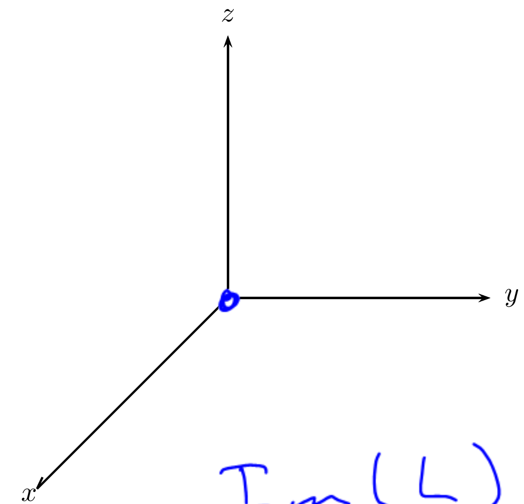
Im(L) = line
through the
origin.
(1D)

$\mathbb{R}^2 \to \mathbb{R}^3$ examples.

$h(0,0) = (0,0,0)$

$h(1,0) = (0,0,0)$





$$h(x, y) = [0, 0, 0]$$

$\ker(h) = \{(x,y) \in \mathbb{R}^2\}$

(2D)

$\mathrm{Im}(L) = \{(0,0)\}$

0D

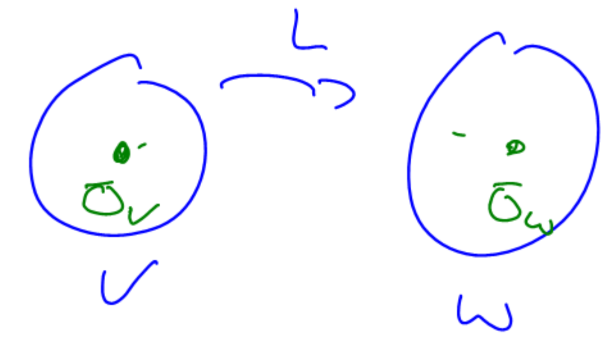
Theorem 19 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation, and

output set   some input

$$\mathrm{Im}(L) = \{\mathbf{w} \in \mathbf{W} : \text{There is some } \mathbf{v} \in \mathbf{V} \text{ such that } L(\mathbf{v}) = \mathbf{w}\}$$

Then $\mathrm{Im}(L)$ is a **subspace** of $\mathbf{W}$ (not just a subset).

Proof



1) $\mathrm{Im}(L)$ contains the $\bar{0}_W$

Proved earlier that for $\bar{0}_V$,

$L(\bar{0}_V) = \bar{0}_W$ for any linear transform

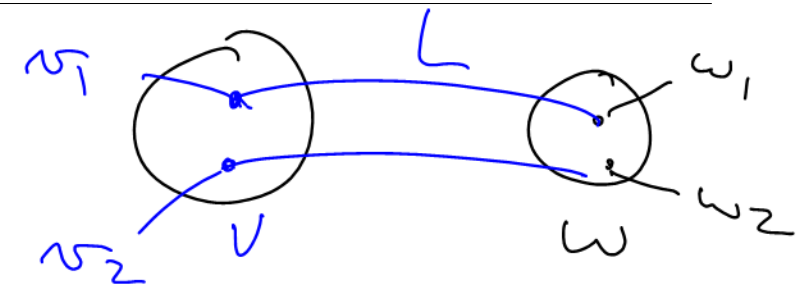
$\Rightarrow \bar{0}_W \in \mathrm{Im}(L)$

Theorem 19 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
Then $\mathrm{Im}(L)$ is a subspace of $\mathbf{W}$.

$$\mathrm{Im}(L) = \{\mathbf{w} \in \mathbf{W} : \text{There is some } \mathbf{v} \in \mathbf{V} \text{ such that } L(\mathbf{v}) = \mathbf{w}\}$$

---

2) Im(L) closed under addition



Consider $w_1 \in \mathrm{Im}(L)$, $w_2 \in \mathrm{Im}(L)$

$\Rightarrow$ Know v_1 and v_2 exist s.t

$v_1, v_2 \in V$ and $L(v_1) = w_1$ and

$L(v_2) = w_2$

Q: is $w_1 + w_2$ in $\mathrm{Im}(L)$?

Consider $L(v_1 + v_2)$

$= L(v_1) + L(v_2)$ by linearity of L

$= w_1 + w_2$. Since $v_1 + v_2 \in V$ and $L(v_1 + v_2) = w_1 + w_2$

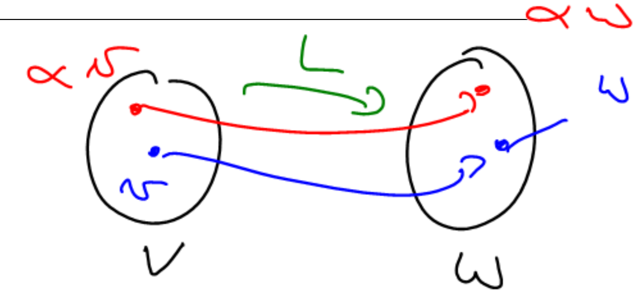
$w_1 + w_2 \in \mathrm{Im}(L)$

Theorem 19 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
Then $\mathrm{Im}(L)$ is a subspace of $\mathbf{W}$.

$$\mathrm{Im}(L) = \{\mathbf{w} \in \mathbf{W} : \text{There is some } \mathbf{v} \in \mathbf{V} \text{ such that } L(\mathbf{v}) = \mathbf{w}\}$$

---

3) Im(L) closed under scalar mult'n.



Consider $w \in \mathrm{Im}(L)$ and $\alpha \in \mathbb{R}$

Know for some $v \in V$, $L(v) = w$

Q: is αw in $\mathrm{Im}(L)$?

Consider $L(\alpha v)$

$= \alpha L(v)$ by linearity of L

$= \alpha w$

$\Rightarrow$ since $L(\alpha v) = \alpha w$ and $\alpha v \in V$ $\Rightarrow$ $\alpha w \in \mathrm{Im}(L)$

so $\mathrm{Im}(L)$ is a vector subspace of W.

Theorem 19 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
Then $\mathrm{Im}(L)$ is a subspace of $\mathbf{W}$.

$$\mathrm{Im}(L) = \{\mathbf{w} \in \mathbf{W} : \text{There is some } \mathbf{v} \in \mathbf{V} \text{ such that } L(\mathbf{v}) = \mathbf{w}\}$$

Here is a theorem that will be helpful when we study dimensions related to transforms. It ties into our idea that $\text{Ker}(L)$ measures how much L "collapses" the input space $\mathbf{V}$, under its linear transformation.

Theorem 18 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
L is **injective** or one-to-one if and only if $\text{Ker}(L) = \mathbf{0}_{\mathbf{V}}$.

Concept Question: *injective* functions are which of these?

(a) Let $f : S \to T$. $\forall x, y \in S$: $x \neq y \Rightarrow f(x) \neq f(y)$.

only one

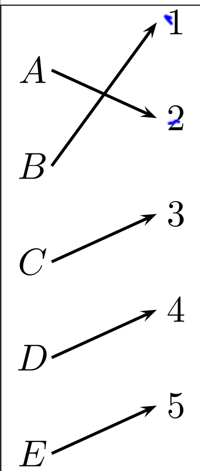
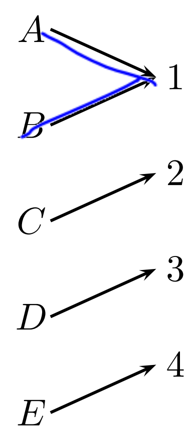
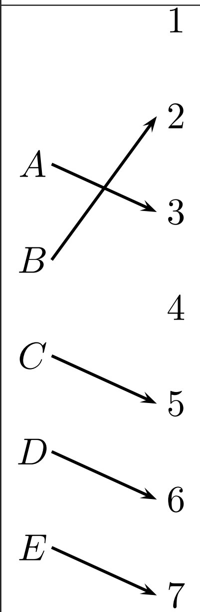
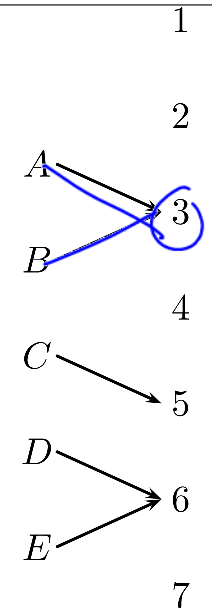
$v \xrightarrow{f} w$

$v_2 \longrightarrow w_2$

(b) Let $f : S \to T$. $\forall t \in T, \exists x \in S : f(x) = t$.

surjective / onto / covering the output space

Diagrams

	Injective	Not Injective
Surjective		
Not surjective		

Theorem 18 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
L is **injective** if and only if $\mathrm{Ker}(L) = \{\mathbf{0_V}\}$ single point

Proof: $\mathrm{Ker}(L) = \{\mathbf{0_V}\} \implies L$ is injective.

iff requires

$\mathrm{Ker}(L) = \{0_V\} \Rightarrow L$ is injective

and

L is injective $\Rightarrow \mathrm{Ker}(L) = \{0_V\}$

Know only $L(\bar{0}_V) = \bar{0}_W$

no other $v \in V$ satisfies

$L(v) = \bar{0}_W$

To get to injectivity, consider $v_1, v_2 \in V$ (input space)

s.t. $L(v_1) = L(v_2)$

Goal: show that v_1 must $= v_2$

Consider $L(v_1) = L(v_2)$

Subtract $L(v_2)$ from both sides

Theorem 18 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
L is **injective** if and only if $\mathrm{Ker}(L) = \mathbf{0}_{\mathbf{V}}$.

$$L(v_1) - L(v_2) = L(v_2) - L(v_2)$$

$$L(v_1) - L(v_2) = \bar{0}_w$$

$$L(v_1 - v_2) = \bar{0}_w \qquad \text{b/c } L \text{ is linear}$$

so $v_1 - v_2 \in \mathrm{Ker}(L)$

but $\bar{0}_v$ is only vector in $\mathrm{Ker}(L)$

so $v_1 - v_2 = \bar{0}_v \Rightarrow v_1 = v_2$

$\Rightarrow$ L is injective if $\mathrm{Ker}(L) = \{\bar{0}_v\}$

Theorem 18 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
L is **injective** if and only if $\text{Ker}(L) = \mathbf{0_V}$.

Proof: L is injective $\implies$ $\text{Ker}(L) = \{\mathbf{0_V}\}$.

If L is injective, then only one vector $v \in V$ satisfies $L(v) = \bar{0}_w$

and b/c L is linear

know $L(\bar{0}_v) = \bar{0}_w$

$\Rightarrow$ only $L(\bar{0}_v) = \bar{0}_w$

$\Rightarrow$ $\text{Ker}(L) = \{\bar{0}_v\}$ if L is injective.

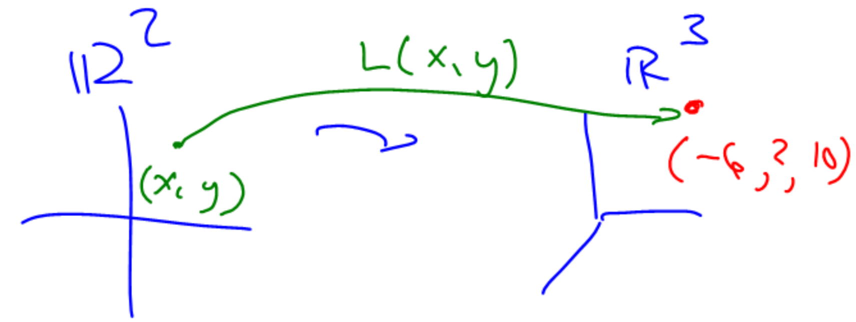
Theorem 18 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
L is **injective** if and only if $\mathrm{Ker}(L) = \mathbf{0}_{\mathbf{V}}$.

Example: Determining if a point is in $\mathrm{Im}(L)$.

Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear mapping defined by

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

Show that the point $(-6, 2, 10)$ **is** in the $\mathrm{Im}(L)$.



Goal: find (x,y) input s.t. $L(x,y) = (-6,2,10)$

i.e. $$L(x,y) = \begin{bmatrix} 2x-3y \\ x+y \\ 4x+5y \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 10 \end{bmatrix} \to \left[\begin{array}{cc|c} 2 & -3 & -6 \\ 1 & 1 & 2 \\ 4 & 5 & 10 \end{array}\right]$$

$$\begin{array}{c} R_2 \\ R_1-2R_2 \\ 2R_1-R_3 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -5 & -10 \\ 0 & -11 & -22 \end{array}\right] \to \begin{array}{c} R_1 \\ R_2/-5 \\ \end{array} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \to \begin{array}{c} R_1-R_2 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$x = 0, \ y = 2$

Continued

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

Check $L(0,2) = (-6, 2, 10)$ ✓

so $(-6, 2, 10) \in \mathrm{Im}(L)$

$L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear mapping defined by

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

Example: Show that the point $(1, 2, 3)$ is **not** in the $\mathrm{Im}(L)$.

Goal: Find (if) x, y exists. s.t. $L(x, y) = (1, 2, 3)$

i.e. $L(x,y) = \begin{bmatrix} 2x - 3y \\ x + y \\ 4x + 5y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & -3 & 1 \\ 1 & 1 & 2 \\ 4 & 5 & 3 \end{array}\right]$

$$\begin{array}{l} R_2 \\ R_1 - 2R_2 \\ 2R_1 - R_3 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -5 & -3 \\ 0 & -11 & -1 \end{array}\right] \to \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 3/5 \\ 0 & 1 & 1/11 \end{array}\right)$$

$y = 3/5$ and $y = 1/11$ impossible

$\Rightarrow$ no solution.

Continued

no (x, y) exists s.t. $L(x,y) = (1,2,3)$

$\Rightarrow$ $(1,2,3)$ is <u>not</u> in the image of L

Tie these ideas about linear transformations back to our earlier ideas about span and linear systems of equations.

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

is $(a, b, c) \in \mathrm{Im}(L)$?

$$\begin{aligned} 2x - 3y &= a \\ x + y &= b \\ 4x + 5y &= c \end{aligned} \rightsquigarrow x \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is (a, b, c) in the span of

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} \right\} ?$$

Bases in transformations and Im(L)

$L: \mathbb{R}^2 \to \mathbb{R}^3$

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

Earlier we used a bit of trial and error and equation solving to see if a vector was or wasn't in the image of L. Can we use some insight instead to develop a **basis for the image**?

$$L(x,y) = \begin{bmatrix} 2x-3y \\ x+y \\ 4x+5y \end{bmatrix} = x\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + y\begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}$$

Q: what elements of $\mathbb{R}^3$ can be made by L by choosing diffent x, y input values?

A: Any linear comb'n of $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} \right\}$

$\Rightarrow \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} \right\}$ is a spanning set for $\mathrm{Im}(L)$

$$L(x,y) = (2x-3y, x+y, 4x+5y)$$

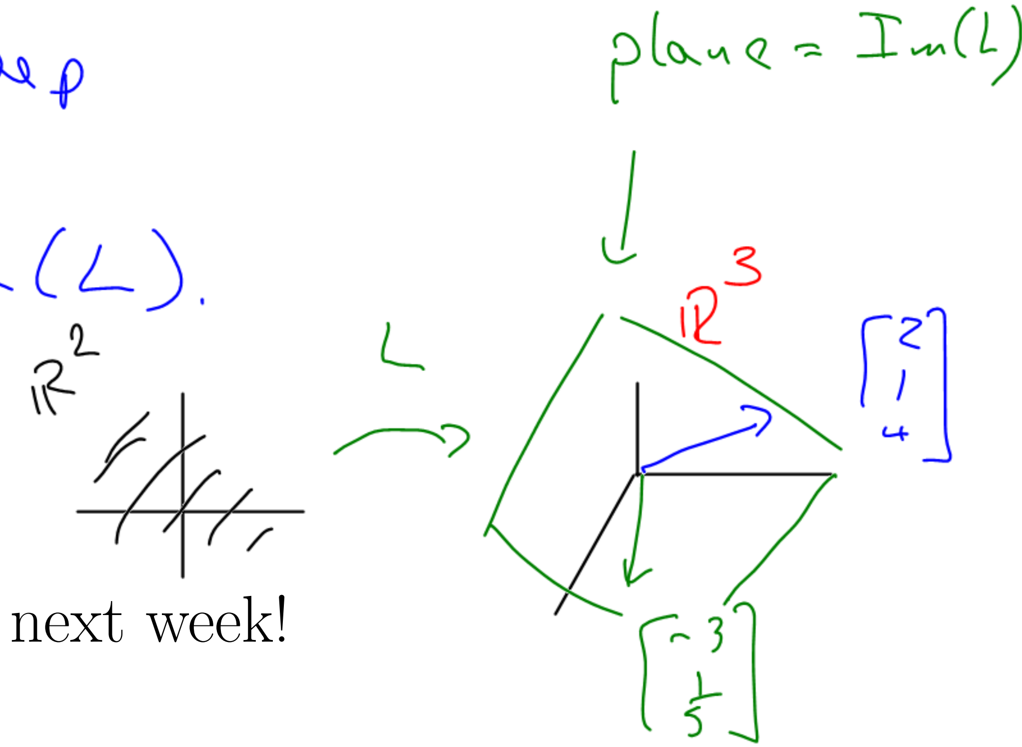
Recall: basis = (span set) + (linearly indep)

Our set $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} \right\}$ contains 2 non-multiple vectors

$\Rightarrow$ set is linearly indep.

Since $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} \right\}$ is
- a spanning set for Im(L) and
- linearly indep

$\Rightarrow$ This set is a basis for Im(L).



We will study this basis-of-image question in more detail next week!

Function Composition

Let $\mathbf{V}$ be a vector space with elements $u, v \in \mathbf{V}$. We then add transforms $T_1 : \mathbf{V} \to \mathbf{V}$ and $T_2 : \mathbf{V} \to \mathbf{V}$ which are linear transforms such that:

$$V \xrightarrow{T_1} V \xrightarrow{T_2} V$$

u, v

or

$$V \xrightarrow{L_1} W \xrightarrow{L_2} X$$

$T_1(u) = 3u - 6v$ and $T_1(v) = -u + 3v$, and

$T_2(u) = -4u + 7v$ and $T_2(v) = 2u + 2v$.

Find the images of u and v under the composition:

$T_2(T_1(u)) =$ $T_2(3u - 6v) = 3T_2(u) - 6T_2(v)$

linearity of T_2

$= 3(-4u + 7v) - 6(2u + 2v)$

$= -24u + 9v$

$T_2(T_1(v)) =$ $T_2(-u + 3v)$

linearity of T_2 $= -T_2(u) + 3T_2(v) = -(-4u + 7v) + 3(2u + 2v)$

$= 10u - v.$

Matrices Defined by Linear Transformations

Any linear transform from $\mathbb{R}^n \to \mathbb{R}^m$ can be represented by an $m \times n$ matrix.

Bonus: the matrix form of a linear transform, after reducing to RREF, will let us

- find the dimension of $\mathrm{Ker}(L)$, and

- find a basis for $\mathrm{Ker}(L)$;

- find the dimension of $\mathrm{Im}(L)$, and

- find a basis for $\mathrm{Im}(L)$;

Sizes of matrices

Example: identify the size of each of the matrices below. Make sure the dimensions are presented in the standard order.

(rows) by (columns)

2 x 3
row col

$$\begin{bmatrix} 2 & 5 & 8 \\ 0 & 1 & 6 \end{bmatrix}$$

3 x 3
row col

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we fix m and $n \geq 1$, we define the set of all $m \times n$ matrices as $M_{m,n}(\mathbb{R})$.

Fun fact: this set of matrices is a vector space! e.g. vect space of 2x3 matrices

- We define $M_1 + M_2$ addition as element-by-element addition, and
- We define scalar multiplication αM as multiplying every element of the matrix M by α.

Note: we only have a vector space $M_{m,n}(\mathbb{R})$ if we keep the dimensions of the matrices constant.

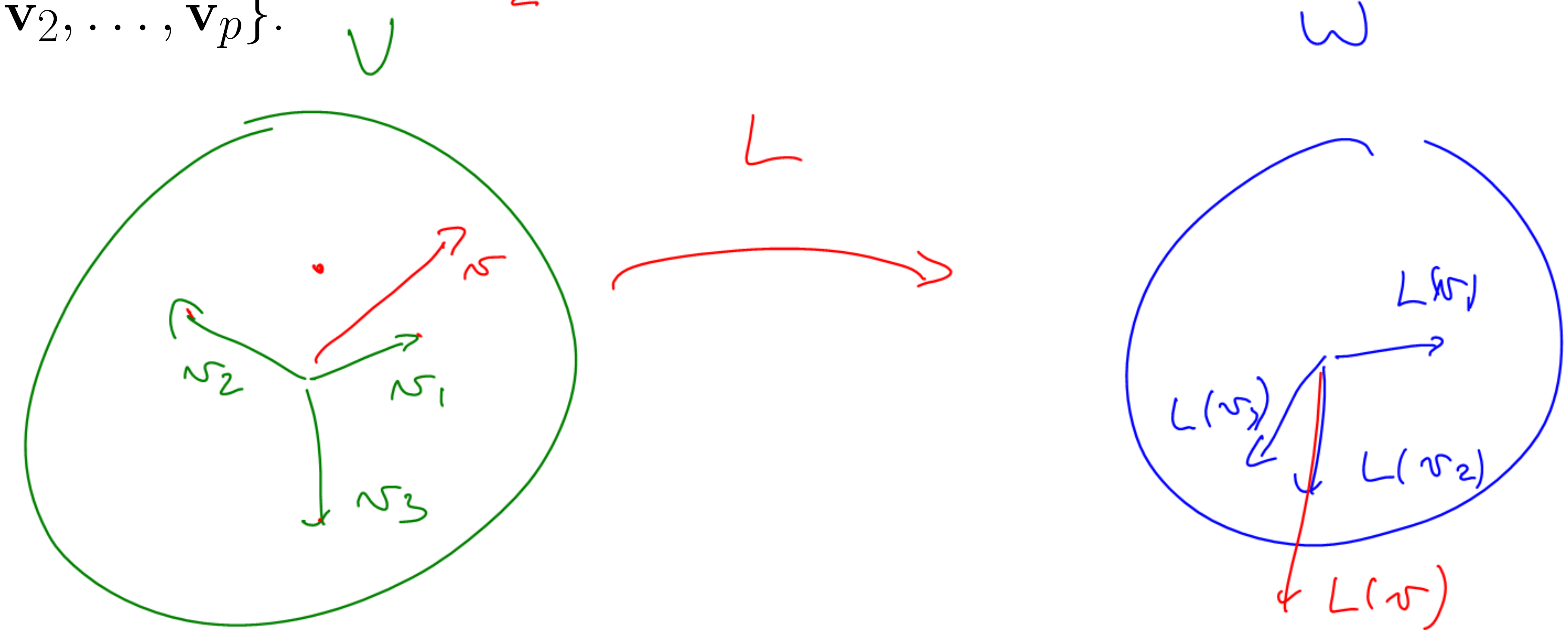
Example: try adding two $\underline{2 \times 3}$ matrices, then separately a 2×2 and a 3×1 matrix.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \quad ???$$

Key Property of Linear Transformations

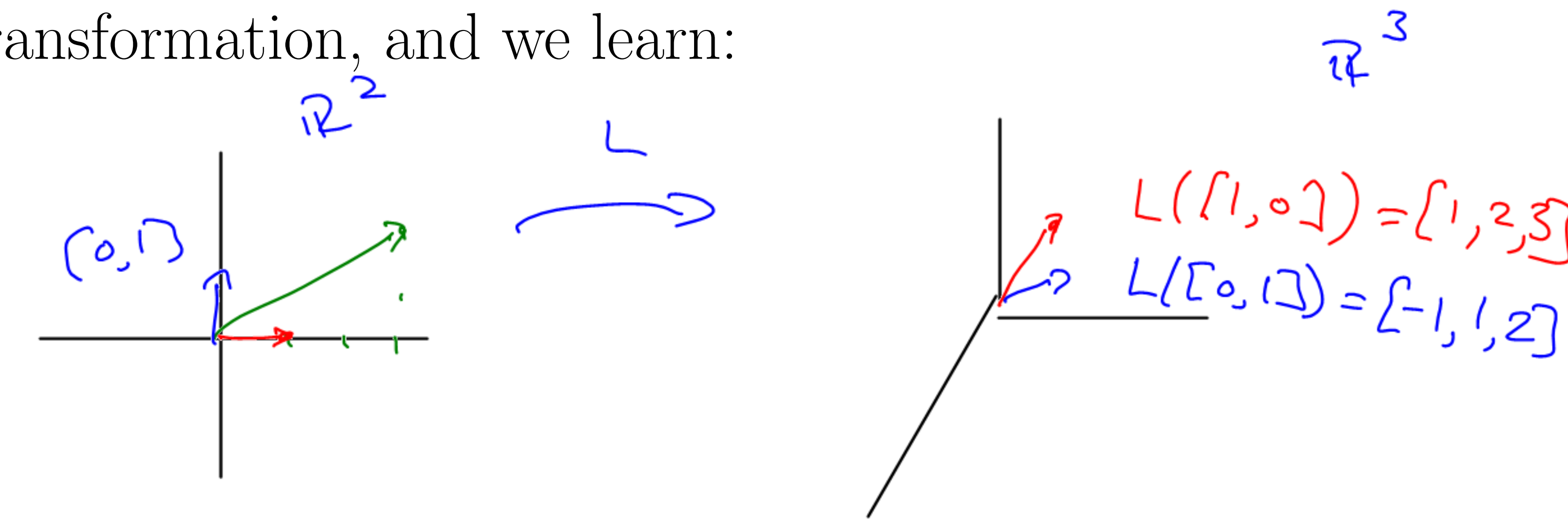
Suppose that $L : \mathbf{V} \to \mathbf{W}$ is a linear transformation, and that we have a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in \mathbf{V}$ for which we know $L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_p)$.
Then, for any new vector $\mathbf{v}$ we can easily deduce $L(\mathbf{v})$, so long as $\mathbf{v}$ is in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.



Example: $L : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation, and we learn:

- $L([1,0]) = [1,2,3]$, and
- $L([0,1]) = [-1,1,2]$.

Find $L([3,2])$.



$$L([3,2]) = L(3[1,0] + 2[0,1])$$
$$= 3L([1,0]) + 2L([0,1])$$
$$= 3[1,2,3] + 2[-1,1,2]$$
$$= [1,8,13]$$

$L([1,0]) = [1,2,3]$, and $L([0,1]) = [-1,1,2]$.

Find in general, $L([x,y])$ for any x, y.

$$L([x,y]) = L(x \cdot [1,0] + y \cdot [0,1])$$
$$= x \; L([1,0]) + y \cdot L([0,1])$$
$$= x[1,2,3] + y[-1,1,2] \quad \text{vector output form}$$
$$= [x-y, \; 2x+y, \; 3x+2y] \quad \text{function form}$$

$L([1,0]) = [1,2,3]$, and $L([0,1]) = [-1,1,2]$.

This can give us the idea to encode a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ using an $m \times n$ matrix.

Find the **standard matrix** associated with L.

matrix w/ columns representing $L([0 \ldots 1, 0 \ldots])$ w/ the 1 in the entry corresponding to the col

$$A_L = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 2 \\ \vdots & \vdots \end{bmatrix}$$

y vect multipliers.

col = $L([0,1])$

col = $L([1,0])$

x vect multiplier

Definition For a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, we associate to L the $m \times n$ **standard matrix** M for which where:

- 1st column of M is $L(1, 0, \ldots, 0)$,
- 2nd column of M is $L(0, 1, \ldots, 0)$,
- ...
- nth column of M is $L(0, 0, \ldots, 1)$,

rows = output dim

cols = input dim

3 output dim ⇒ 3 rows

Example: Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by

$$L(x, y, z, w) = (5x - 2y,\ \ 3z - w,\ \ y - 7z + 8w).$$

Find the standard matrix for L. 4 cols in M = 4 input dim's

x=1, all others 0

Need $L([1,0,0,0]) = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$

$L([0,1,0,0]) = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

y=1

z=1

$L([0,0,1,0]) = \begin{bmatrix} 0 \\ 3 \\ -7 \end{bmatrix}$

$L([0,0,0,1]) = \begin{bmatrix} 0 \\ -1 \\ 8 \end{bmatrix}$

w=1

$$L(x, y, z, w) = (5x - 2y,\ \ 3z - w,\ \ y - 7z + 8w)$$

$$M = \begin{array}{c} \begin{bmatrix} 5 & -2 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 1 & -7 & 8 \end{bmatrix} \\ \begin{matrix} x, & y, & z, & w \end{matrix} \end{array}$$

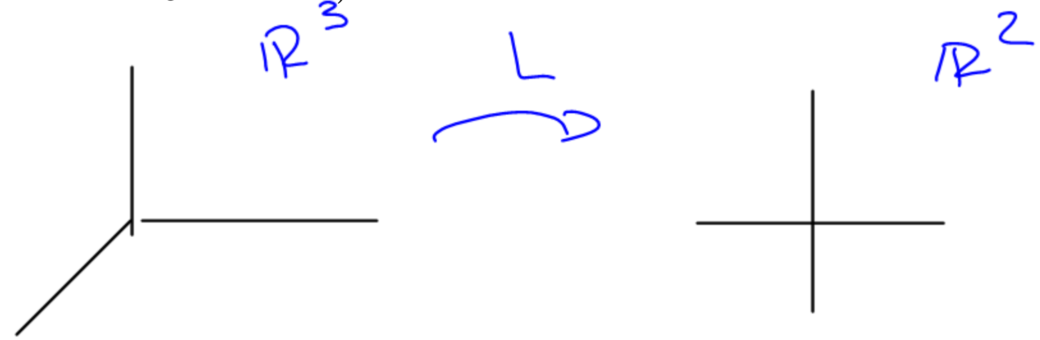
3×4

\# out \# inputs

Example: Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$L(x, y, z) = (-2x - 4y + 3z,\ 11x + 15y + 8z).$$

Find the standard matrix for L.



$$M = \begin{array}{ccc} \left[\begin{array}{ccc} -2 & -4 & 3 \\ 11 & 15 & 8 \end{array}\right] \\ \begin{array}{ccc} x & \quad y & \quad z \end{array} \end{array}$$

rows = # outputs = 2

cols = # inputs = 3

$L([0,1,0]) = [-4, 15]$ ✓

Concept Check: What is the size of the standard matrix for a linear transformation $L : \mathbb{R}^7 \to \mathbb{R}^5$?

rows = # outputs

1 col for each input

M is $\underline{\ 5\ }$ $\times$ $\underline{\ 7\ }$

rows cols

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} \cdots$$

each col = an output vector

Concept Check: If the standard matrix for L is 6 rows $\times$ 8 columns,

then L: $\underline{\mathbb{R}^8}$ $\to$ $\underline{\mathbb{R}^6}$

6

\# output dim

8

Moving between standard matrix and the matching linear transform

each col ∈ output space, $\mathbb{R}^4$

Example: $A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix}$

each col is for 1 input dim'n

If A is the standard matrix for a transform L_A, what are the dimensions of the input and output spaces of L?

$L : \mathbb{R}^3 \to \mathbb{R}^4$

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix}$$

Interpret A as the transform of $(\underline{1,0,0})$, $(\underline{0,1,0})$ and $(\underline{0,0,1})$.

Each col of A = L(one simple unitary inputs)

$L([1,0,0]) = 1^{st}$ col of $A = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix}$

1st element

$L([0,1,0]) = 2^{nd}$ col of $A = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 7 \end{bmatrix}$

2nd el.

$L([0,0,1]) = 3^{rd}$ col of $A = \begin{bmatrix} 3 \\ 8 \\ 6 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix}$$

Compute $L_A(x, y, z)$.

$$= L_A(\, x[1,0,0] + y[0,1,0] + z[0,0,1])$$

$$= L_A([1,0,0]) \cdot x + L_A([0,1,0]) \cdot y + L([0,0,1]) \cdot z$$

$$= \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix} x + \begin{bmatrix} 5 \\ 0 \\ 1 \\ 7 \end{bmatrix} y + \begin{bmatrix} 3 \\ 8 \\ 6 \\ 0 \end{bmatrix} z = \begin{bmatrix} 1x + 5y + 3z \\ 2x + 0y + 8z \\ 4x + 1y + 6z \\ 0x + 7y + 0z \end{bmatrix}$$

like dot product!

Note any patterns in the resulting vector.

$$[1, 5, 3] \cdot [x, y, z]$$

Since we often start with a matrix representation for a linear transform, we would like a simpler process for defining the output of $L_A(\mathbf{v})$.

We define the (**non-scalar**) matrix product $A\mathbf{v}$, with $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and

matrix   input vector

$\mathbf{w}_i$ as the ith column of A as:

$$A\mathbf{v} = \mathbf{w}_1 x_1 + \mathbf{w}_2 x_2 + \dots \mathbf{w}_n x_n$$

What kind of product (from calculus class) does this look like?

Cols of A · v's coef's or components

Dot product

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 19 \\ -4 \\ 6 \\ 28 \end{bmatrix}$$

Evaluate $L_A(\underline{2, 4, -1})$.

$L_A : \mathbb{R}^3 \to \mathbb{R}^4$

from prev page

$$L_A(2,4,-1) = \begin{bmatrix} \Big| & \Big| & \Big| \\ w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix} \cdot 2 + \begin{bmatrix} 5 \\ 0 \\ 1 \\ 7 \end{bmatrix} 4 + \begin{bmatrix} 3 \\ 8 \\ 6 \\ 0 \end{bmatrix} (-1)$$

Av

<u>matrix</u> product.

$$= \begin{bmatrix} 19 \\ -4 \\ 6 \\ 28 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix}$$

Evaluate $L_A(0, -1, 1)$.

$$L_A(0,-1,1) = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 5 \\ -7 \end{bmatrix}$$

finger walking method

- in A, left to right
  eg. $[1, 5, 3]$
- in v, top to bottom
  $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Take dot product

Next week: matrix multiplication, and the dimensions/bases for the kernel and image!