Week #9: Kernel and Image Bases, Matrix Multiplication

Review For a linear transformation $L : \mathbf{V} \to \mathbf{W}$,

(A) some subset of the input space $\mathbf{V}$.

(B) some subset of the output space $\mathbf{W}$.

(C) a vector subspace of the input space $\mathbf{V}$.

(D) a vector subspace of the output space $\mathbf{W}$.

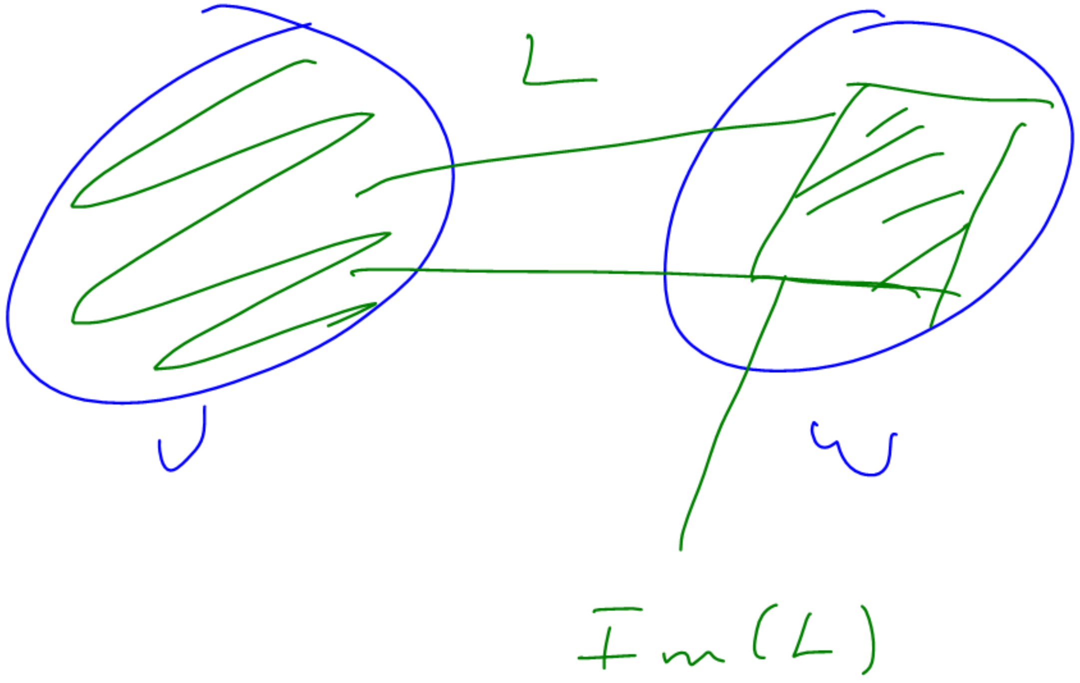
For a linear transformation $L : \mathbf{V} \to \mathbf{W}$, the image $\mathrm{Im}(L)$ is:

(A) some subset of the input space $\mathbf{V}$.

(B) some subset of the output space $\mathbf{W}$.

(C) a vector subspace of the input space $\mathbf{V}$.

[✓ (D) a vector subspace of the output space $\mathbf{W}$.]



points in output that can be reached by L

$\mathrm{Im}(L) = \{ w \in W : \text{there exists some } v \in V \text{ s.t. } L(v) = w \}.$

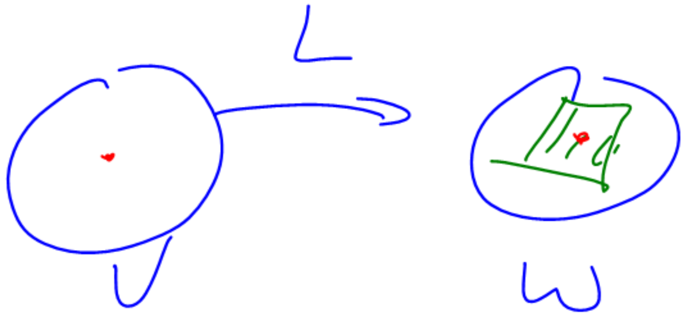
input

For a linear transformation $L : \mathbf{V} \to \mathbf{W}$, the image Im($L$):

(A) will be all of **W**.

[✓ (B) will be all or some of **W**.]

(C) ~~will be just $\mathbf{0}_{\mathbf{W}}$ if Ker(L) is non-empty.~~



$\mathrm{Ker}(L) = \overset{\text{just}}{\{0_V\}}$

$\Leftrightarrow$ only 0_V input maps to 0_W output

$\Downarrow$

L is injective/ one-to-one

174 PSAe / TA strike

- sec 103–105 has no instructor
- tutorials run as normal

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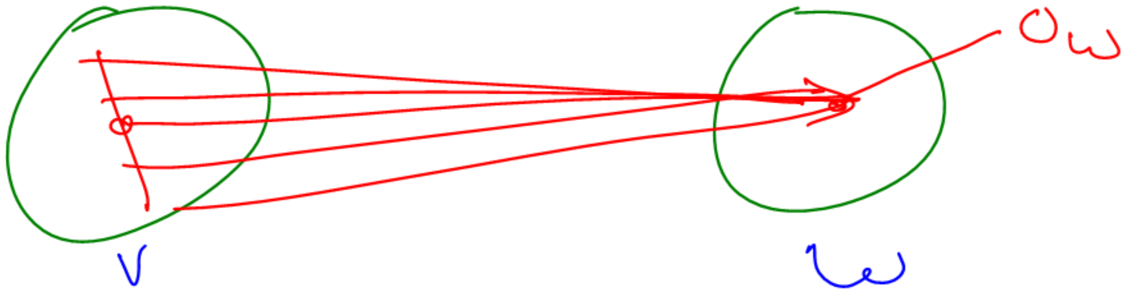
- tutorials run as usual.

Two major sets L → Kernel
→ Image

Definition: For a linear mapping $L : \mathbf{V} \to \mathbf{W}$, the set of all **input** vectors that are mapped to $\underline{\mathbf{0_W}}$ is called the **kernel** of L, or the **null space** of L.

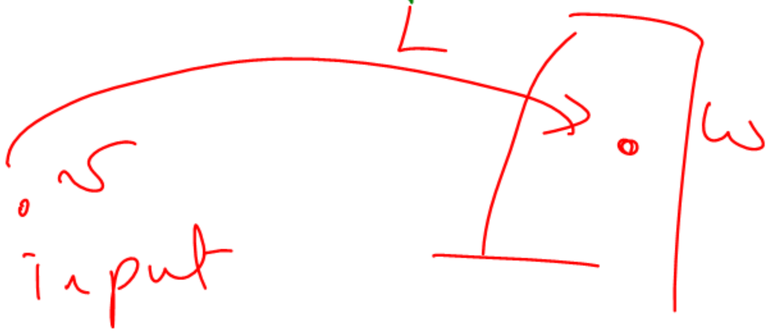
Formally:

$$\ker(L) = \{ v \in V : L(v) = 0_W \}$$



Definition: Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation. We define the **image** of L, $\mathrm{Im}(L)$ by:

$$\mathrm{Im}(L) = \{ w \in W : \exists\, v \in V \text{ s.t. } L(v) = w \}$$



Recall: We have seen different forms of a linear transformation.

$$L(x,y) = (2x-3y,\ x+y,\ 4x+5y)$$

calculus

$$L(x,y) = \begin{bmatrix}2\\1\\4\end{bmatrix}\boxed{x} + \begin{bmatrix}-3\\1\\5\end{bmatrix}y$$

standard matrix

$$A_L = \begin{array}{c}\\ \\ \\ \end{array}\begin{array}{cc} L(1,0) & L(0,1) \\ \left[\begin{matrix}2\\1\\4\end{matrix}\right. & \left.\begin{matrix}-3\\1\\5\end{matrix}\right]\end{array}$$

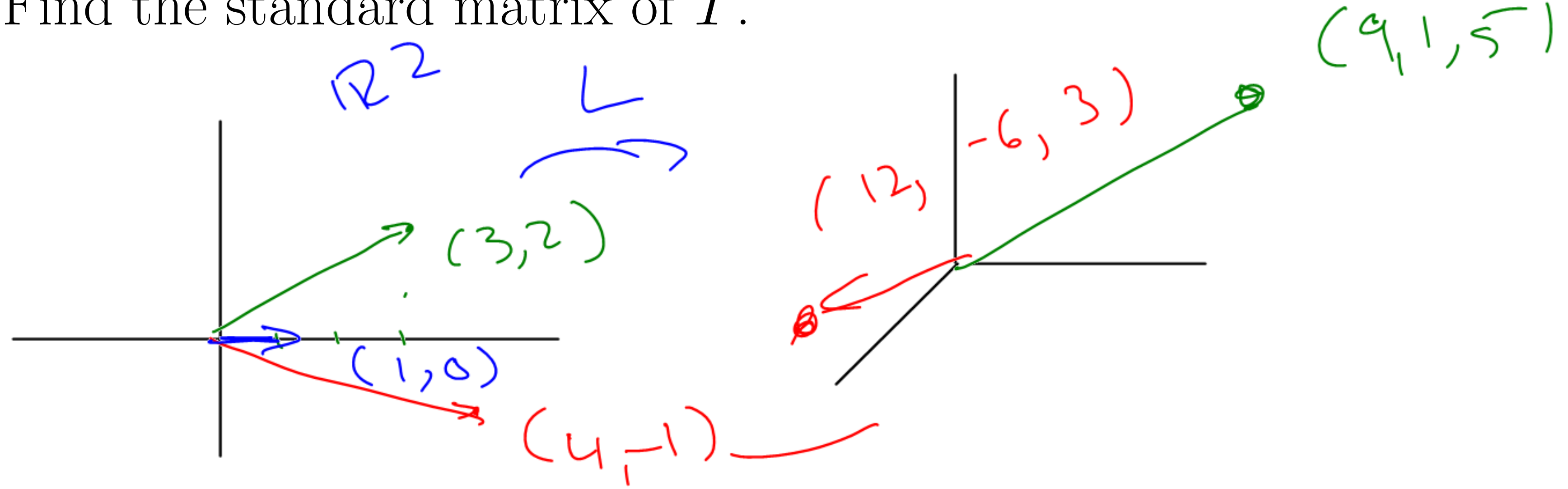
$$\text{and } L(x,y) = \begin{bmatrix}2 & -3\\1 & 1\\4 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

But what if we don't know the transform of the simple basis vectors $(1, 0, 0\ldots)$, $(0, 1, 0, \ldots)$?

Example: Consider $T : \mathbb{R}^2 \to \mathbb{R}^3$ as a linear transformation such that

$$T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}9\\1\\5\end{bmatrix} \text{ and } T\left(\begin{bmatrix}4\\-1\end{bmatrix}\right) = \begin{bmatrix}12\\-6\\3\end{bmatrix}$$

Find the standard matrix of T.



$\left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}\right\}$ are a basis for the input space, $\mathbb{R}^2$

$\Rightarrow \begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, needed for standard matrix, are in the span of $\left\{\begin{bmatrix}3\\2\end{bmatrix} \begin{bmatrix}4\\-1\end{bmatrix}\right\}$

$$T\left(\begin{bmatrix}3\\2\end{bmatrix}\right)=\begin{bmatrix}9\\1\\5\end{bmatrix} \text{ and } T\left(\begin{bmatrix}4\\-1\end{bmatrix}\right)=\begin{bmatrix}12\\-6\\3\end{bmatrix}$$

Solve $\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}a+\begin{bmatrix}4\\-1\end{bmatrix}b$

components $1 = 3a + 4b$ ①

$0 = 2a - b$ ②

$b = 2a \rightarrow$ ①

$1 = 3a + 4(2a)$

$1 = 11a$

$a = 1/11$, $b = 2a = \frac{2}{11}$

so $\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{11}\begin{bmatrix}3\\2\end{bmatrix}+\frac{2}{11}\begin{bmatrix}4\\-1\end{bmatrix}$

$\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}c+\begin{bmatrix}4\\-1\end{bmatrix}d$

components:

$0 = 3c + 4d$ ③

$1 = 2c - d$ ④ $\rightarrow d = 2c-1$

$\downarrow$

$0 = 3c + 4(2c-1)$ ③

$4 = 11c \qquad c = \frac{4}{11}$

$d = 2c - 1 = 2\left(\frac{4}{11}\right) - 1 = -3/11$

So $\begin{bmatrix}0\\1\end{bmatrix}=\frac{4}{11}\begin{bmatrix}3\\2\end{bmatrix}-\frac{3}{11}\begin{bmatrix}4\\-1\end{bmatrix}.$

$$T\left(\begin{bmatrix}3\\2\end{bmatrix}\right)=\begin{bmatrix}9\\1\\5\end{bmatrix} \text{ and } T\left(\begin{bmatrix}4\\-1\end{bmatrix}\right)=\begin{bmatrix}12\\-6\\3\end{bmatrix}$$

Need $L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = L\left(\underbrace{\frac{1}{11}\begin{bmatrix}3\\2\end{bmatrix} + \frac{2}{11}\begin{bmatrix}4\\-1\end{bmatrix}}_{=}\right)$

Know these!

$$= \frac{1}{11} L\left(\begin{bmatrix}3\\2\end{bmatrix}\right) + \frac{2}{11} L\left(\begin{bmatrix}4\\-1\end{bmatrix}\right)$$

$$= \frac{1}{11}\begin{bmatrix}9\\1\\5\end{bmatrix} + \frac{2}{11}\begin{bmatrix}12\\-6\\3\end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix}33\\-11\\11\end{bmatrix} = \begin{bmatrix}3\\-1\\1\end{bmatrix}$$

$\Leftarrow$ <u>first</u> column of our standard matrix for L.

Also need $L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = L\left(\underbrace{\frac{\overset{4}{\cancel{-3}}}{11}\begin{bmatrix}3\\2\end{bmatrix} + \left(\frac{\overset{-3}{\cancel{4}}}{11}\right)\begin{bmatrix}4\\-1\end{bmatrix}}\right)$

known!

linearity $= \frac{\overset{4}{\cancel{-3}}}{11} L\left(\begin{bmatrix}3\\2\end{bmatrix}\right) + \frac{\overset{-3}{\cancel{4}}}{11} L\left(\begin{bmatrix}4\\-1\end{bmatrix}\right)$

$= \frac{\overset{4}{\cancel{-3}}}{11}\begin{bmatrix}9\\1\\5\end{bmatrix} + \frac{\overset{-3}{\cancel{4}}}{11}\begin{bmatrix}12\\-6\\3\end{bmatrix}$

$= \begin{bmatrix}0\\2\\1\end{bmatrix}$ ← 2nd col of our standard matrix

$\Rightarrow A_L = \begin{bmatrix}3 & 0\\-1 & 2\\1 & 1\end{bmatrix}$

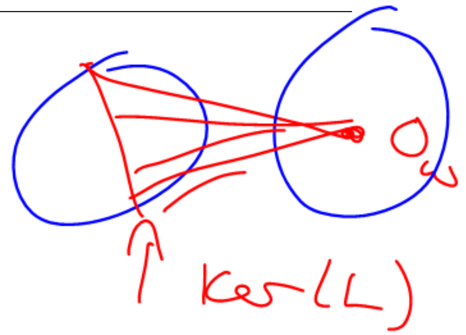
standard matrix for L

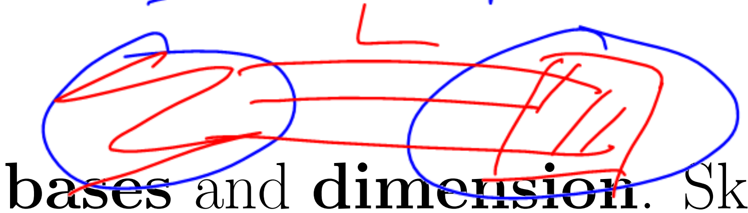
$L(x,y)$
$= (3x+0y,$
$-x+2y,$
$x+y)$

Back to kernel and image!

Both the kernel and the image are subspaces:

- Ker is a subspace of the input space
- Im is a subspace of the output space





Since they are subspaces, each of them must have **bases** and **dimension**. Sketch:

Note: every *linear transformation* L has a kernel and an image.

We will also used the same nomenclature for every *matrix* A, where we infer that A is the standard matrix for a linear transform L.

E.g. "The kernel of the matrix A is...", or "The image of the matrix A is..."

Theorem 23: Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and A its standard matrix. Then the **columns of** A are a **generating set** for $\mathrm{Im}(L)$.

or spanning set

Illustration:

$$L(x,y) = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} x + \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} y$$

$$A_L = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 4 & 5 \end{bmatrix}$$

$$\mathrm{Im}(L) = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : L(x,y) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} x + \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} y \right\}$$

$\Rightarrow \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} \right\}$ is a spanning set for the $\mathrm{Im}(L)$.

We see that the columns of A are a **generating set** for $\mathrm{Im}(L)$. Is that enough for that set to be a **basis** for $\mathrm{Im}(L)$?

No : also need $\underset{\text{the set to be}}{\wedge}$ linearly indep.

$\text{Im}(L) = \text{Image}$

Example: compare the images of the two matrices below, and find a basis each for their images.

$$L = \begin{bmatrix}2\\1\\4\end{bmatrix}x + \begin{bmatrix}-3\\1\\5\end{bmatrix}y$$

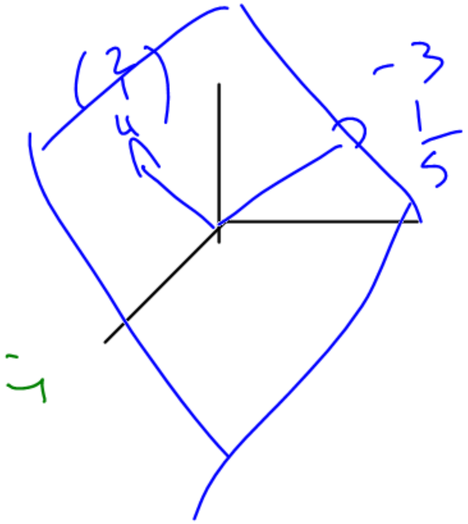
$$A = \begin{bmatrix}2 & -3\\ 1 & 1\\ 4 & 5\end{bmatrix}$$

\# rows = output dim

$L: \mathbb{R}^2 \to \mathbb{R}^3$

↓

2 col's, each col is $L(1,0)$, $L(0,1)$ ...



$$\Rightarrow \left\{ \begin{bmatrix}2\\1\\4\end{bmatrix}, \begin{bmatrix}-3\\1\\5\end{bmatrix} \right\}$$ is a spanning set for $\text{Im}(L)$

2 vectors, not multiples

$\Rightarrow$ also linearly indep

$\Rightarrow$ this set is a basis for $\text{Im}(A)$

$\Rightarrow$ $\text{Im}(A)$ is $\underline{2}$ dimensional.

$L: \mathbb{R}^3 \to \mathbb{R}^3$

$\text{Im}(B)$ is 2 dim'l

$$B = \begin{bmatrix}2 & 4 & -3\\ 1 & 2 & 1\\ 4 & 8 & 5\end{bmatrix}$$

3 col's,

$$\left\{ \begin{bmatrix}2\\1\\4\end{bmatrix} \begin{bmatrix}4\\2\\8\end{bmatrix} \begin{bmatrix}-3\\1\\5\end{bmatrix} \right\}$$ is a spanning set for $\text{Im}(B)$.

$\underline{\text{Not}}$ linearly indep

Remove $\begin{bmatrix}4\\2\\8\end{bmatrix}$,

$$\left\{ \begin{bmatrix}2\\1\\4\end{bmatrix} \begin{bmatrix}-3\\1\\5\end{bmatrix} \right\}$$ is lin $\underline{\text{indep}}$

$\Rightarrow$ basis.

Now compare the RREF for each matrix and look for patterns.

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 2 & 1 \\ 4 & 8 & 5 \end{bmatrix}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{RREF}(B) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

lin'ly indep

redundant

multiple/comb'n of the other cols

lin'ly indep

Algorithm for finding a basis of Im(L)

→ remove any redundant columns.

Let A be the standard matrix for L.

- Put A into RREF.
- Look at the columns of A with leading ones. The corresponding columns in **the original** A are a basis for Im(L).

Example: $L : \mathbb{R}^4 \to \mathbb{R}^3$, with its standard matrix

$$A = \underbrace{\begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix}}_{\text{4 cols} \Rightarrow \mathbb{R}^4 \text{ is input dimension}}$$

Find a basis for Im(L).

Reminders

$\mathbb{R}^3$ is our output space $\rightarrow \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ -7 \\ 7 \end{bmatrix} \right\}$ are a spanning set for Im(L).

$$\begin{array}{c} R_3 \\ R_1+2R_2 \\ R_2 \end{array} \begin{bmatrix} 1 & -3 & 1 & 7 \\ 0 & 0 & -3 & -12 \\ -1 & 3 & -1 & -7 \end{bmatrix} \rightarrow \begin{array}{c} R_1 \\ R_2/_{-3} \\ R_1+R_3 \end{array} \begin{bmatrix} \boxed{1} & -3 & \boxed{1} & 7 \\ 0 & 0 & \boxed{1} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix}$$

$$\begin{array}{c} R_1 - R_2 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>is</u> now in RREF.

Im(L) has a basis of just 2 vectors,

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix}$$

What is the dimension of Im(L)?

$$\text{b/c a } \underline{\text{basis}} \text{ for } \mathrm{Im}(L) = \left\{ \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}_{\text{2 vectors}} \right\}$$

$\Rightarrow$ Im(L) is

2 dimensional.

Why does this work?

Row operations **preserve linear relationships between the columns**.

When we get to RREF, columns that are linearly independent there (leading ones) where also linearly independent in the original A.

Illustration: Let

$$A = \begin{array}{c} \begin{bmatrix} 2 & 1 & \cancel{-4} \\ -1 & -1 & \cancel{1} \\ 1 & 1 & \cancel{-1} \end{bmatrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} \end{array}$$

Confirm that $c_3 = -3c_1 + 2c_2$.

Check $c_3 = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$

$$-3c_1 + 2c_2 = -3\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6+2 \\ 3-2 \\ -3+2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$$

equal.

Below are some steps in the row reduction of this matrix:

$$\begin{bmatrix} 2 & 1 & -4 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Show that the same column relationship $c_3 = -3c_1 + 2c_2$ exists at each stage.

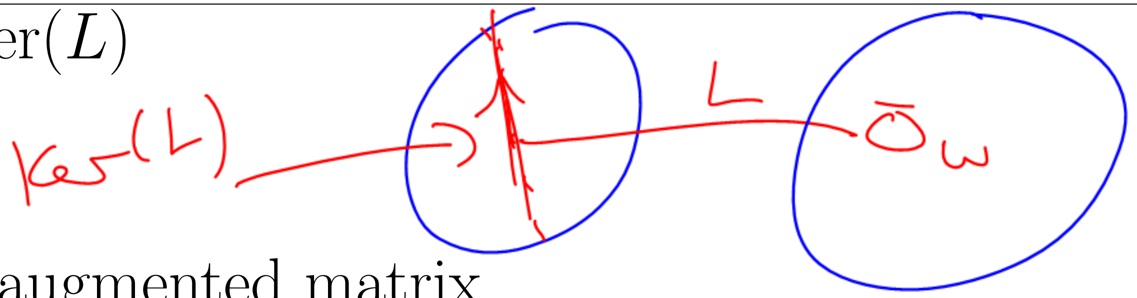
$$-3\begin{bmatrix}1\\0\\-1\end{bmatrix} + 2\begin{bmatrix}1\\-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-2\\1\end{bmatrix} \qquad c_3 = \begin{bmatrix}-1\\-2\\1\end{bmatrix}$$

$$-3\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} = c_3 = \begin{bmatrix}-3\\2\\0\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Follow-up: why are the columns with leading 1's in RREF guaranteed to be a linearly independent set of vectors?

Algorithm for finding a basis of Ker(L)



Let A be the standard matrix for L.

- Put A into RREF.
- Add an extra column of 0's to get an augmented matrix.
- Write down the general solution to the corresponding system in vector form.
- The vectors associated with the free variables are a basis for the kernel.

Example: $L : \mathbb{R}^4 \to \mathbb{R}^3$, with its standard matrix

$$\mathrm{Ker}(L) = \{ v : L(v) = \bar{0}_w \}$$

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix}$$

Find a basis for Ker(L). (Start on next page)

Ker : want

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} z + \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} w = 0$$

Find a basis for Ker(L).

Solve

$$\begin{array}{cccc|c} x & y & z & w & \\ \textcircled{1} & -3 & 0 & 3 & 0 \\ 0 & 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

leading one col's / variables locked in as

other columns represent "free" variables

Let $y = s$

and $w = t$

Row 3 → nothing

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \to \left[\begin{array}{cccc|c} 1 & -3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Row 2 → $z + 4\underset{t}{\cancel{w}} = 0$

$$\boxed{z = -4t}$$

$$\boxed{\begin{aligned} y &= s \\ w &= t \end{aligned}}$$

Row 1 → $x - 3\underset{s}{\cancel{y}} + 3\underset{t}{\cancel{w}} = 0$

$x = +3s - 3t$

vector form

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{bmatrix} 3s - 3t \\ s \\ -4t \\ t \end{bmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix} t$$

Solved for

$L(x\,y\,z\,w) = \bar{0}$.

$\Rightarrow (x, y, z, w) \in \mathrm{Ker}(L)$

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Confirm the kernel membership of a few points.

row · column
dot product.

eg $L\left(\begin{bmatrix}3\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2&-6&-1&2\\-1&3&-1&-7\\1&-3&1&7\end{bmatrix}\begin{bmatrix}3\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ ✓ $=\bar{0}_W$

so $\begin{bmatrix}3\\1\\0\\0\end{bmatrix}$ is in $\ker(L)$

$L\begin{pmatrix}-3\\0\\-4\\1\end{pmatrix} = \begin{bmatrix}2&-6&-1&2\\-1&3&-1&-7\\1&-3&1&7\end{bmatrix}\begin{bmatrix}-3\\0\\-4\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ ✓

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

How many dimensions are there in Ker(L)?

Basis for Ker(L)

$$\text{is } \left\{ \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}}_{\text{2 vectors}} \right\}$$

$\Rightarrow$ Ker(L) is 2 dimensional.

or Dim(Ker(L)) = 2.

Dimensional Analysis

Example:

$\overset{\text{rows}}{3} \times 4$ col's

$$A = \underset{\xrightarrow{\text{4 cols}}}{\begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix}} \to \text{RREF} \begin{bmatrix} \boxed{1} & -3 & 0 & 3 \\ 0 & 0 & \boxed{1} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$L : \underline{\mathbb{R}^4} \to \underline{\mathbb{R}^3}$

$\text{Dim}(\text{Im}(L)) = \underline{\#}$ cols in RREF w/ leading ones

$= 2$

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \to \text{RREF} \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Dim}(\text{Ker}(L)) =$ __\#__ cols in RREF <u>without</u> leading ones
(free variables)

$= 2$

General Case:

rows

$n \times m$ cols

m dimensional input

↓

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ a_{31} & a_{32} & \dots \\ \dots & & \end{bmatrix} \to \text{RREF} \begin{bmatrix} \textcircled{1} & c & \dots & \\ 0 & 0 & \textcircled{1} & \dots \\ 0 & 0 & 0 & \textcircled{1} \\ \ddots & & & \end{bmatrix}$$

$L : \underline{\mathbb{R}^m} \to \underline{\mathbb{R}^n}$

Dim(Im(L)) = # of cols with leading ones in RREF

+ = m = # cols

Dim(Ker(L)) = # of cols without leading ones in RREF.

Rank - Nullity Theorem

If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$n = \text{Dim}(\text{Im}(L)) + \text{Dim}(\text{Ker}(L))$$

n dimensioned input

input dim

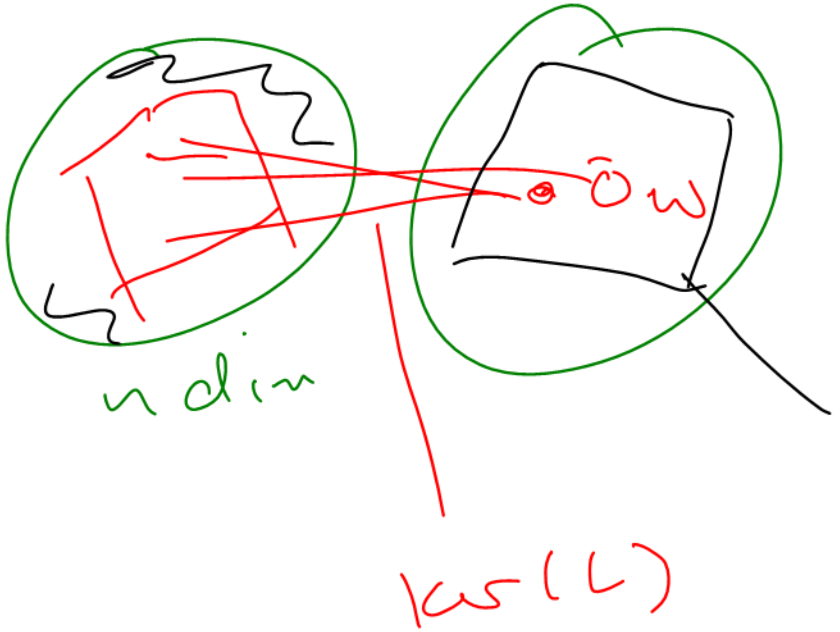
rank = dim of Im(L)

= # of leading ones in RREF

||

$$n = \text{Rank}(A) + \text{Nullity}(A)$$

↳ dimension of Ker(L)



Consequences of Rank and Nullity for Function Properties

What properties of a function are equivalent to the function being **injective** or **one-to-one**?

L is injective (one to one) if and only if $\mathrm{Ker}(L) \overset{\text{only}}{=} \{\overline{0}_V\}$.

$\Updownarrow$

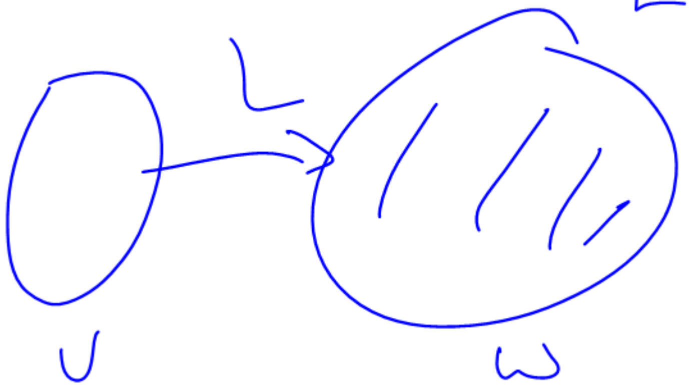
Ker(L) is 0 dim'l

$\Updownarrow$

all cols of L have leading 1's in RREF $\iff$ L's matrix has no cols w/out leading 1's RREF

What properties of a function are equivalent to the function being **surjective** or **onto**?

L covers W



$\iff$ $\mathrm{Im}(L) =$ all of W

$\dim(\mathrm{Im}(L)) = \dim(W)$. = # rows in matrix

$\|$

\# leading 1's in RREF. → Every row in RREF has a leading 1.

$\Rightarrow$ If we want surjective + injective
(= bijective)

then → leading 1's in RREF in every col + every row

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & - \\ 0 & 0 & 1 & \\ - & - & & \ddots \end{bmatrix} \Leftrightarrow \text{matrix must be square.}$$

$\Downarrow$

input dim = output dim.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Section 12 - Applications to Solutions of Linear Systems

We have seen augmented matrices before when we introduced RREF as a way to solve systems of equations.

Example: Consider the system of equations

$$\left(\begin{array}{c}1\\0\\2\end{array}\right|\begin{array}{rrrcr} x+ & y & +2z & = & -3 \\ x & -2y & +z & = & 2 \\ 2x & & +3z & = & -4 \end{array}$$

Big question: How many solutions does this system have?

Interim: Write this system using its matrix form.

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -3 \\ 0 & -2 & 1 & 2 \\ 2 & 0 & 3 & -4 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -3 \\ 0 & -2 & -1 & 2 \\ 2 & 0 & 3 & -4 \end{array}\right]$$

Interpret this system in terms of the span of a set of vectors.

$$\begin{bmatrix}1\\0\\2\end{bmatrix}x + \begin{bmatrix}1\\-2\\0\end{bmatrix}y + \begin{bmatrix}2\\-1\\3\end{bmatrix}z = \begin{bmatrix}-3\\2\\-4\end{bmatrix}$$

Is $\begin{bmatrix}-3\\2\\-4\end{bmatrix}$ in the span of $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\-2\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\3\end{bmatrix}\right\}$?

$$\Updownarrow$$

Does the system of equations have a solution?

$$A_L \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & -3 \\ 0 & -2 & -1 & 2 \\ 2 & 0 & 3 & -4 \end{array}\right]$$

Interpret this system in terms of the Im(L) for some transform L.

Is $\begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}$ in the image of L?

in the image of A_L?

$\Updownarrow$

Does the sys of eq's have a solution?

original

2 columns are a basis for image of L

$$\left[\begin{array}{ccc|c}1&1&2&-3\\0&-2&-1&2\\2&0&3&-4\end{array}\right] \to \text{RREF} \left[\begin{array}{ccc|c}1&0&\frac{3}{2}&-2\\0&1&\frac{1}{2}&-1\\0&0&0&0\end{array}\right]$$

Red flag
0 0 0
=> no sol'n
[illegible]

2 cols w/ leading 1's
lin'ly indep
1 col w/out a leading 1.

What are the dimension of Ker(L) and Im(L)?

1 dimensional

2 dimensional

Does original system of equations have ~~no solution~~, ~~a single solution~~, or an infinite number of solutions?

$$\begin{bmatrix}1&1&2\\0&-2&-1\\2&0&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{pmatrix}-3\\2\\-4\end{pmatrix}$$

L is injective

⇕

Ker(L) is only $=\{\overline{0}\}$ ✓ 0 dimen'l

our Ker(L) is 1 dime'l

More generally: let A be thought of as the standard matrix for a transformation, and the system of equations being written as $[A|b]$ for some vector b. Work through the cases for A and b.

$$\left[\begin{array}{c|c} A & b \end{array}\right]$$

↳ std matrix for some transform L

Case 1: $b \notin \text{Im}(A)$.

⟹ system of eq'ns has no solution

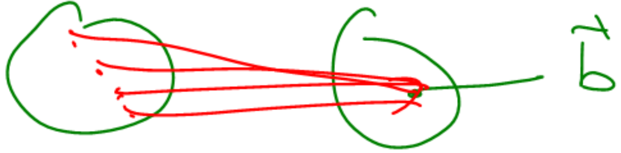
Case 2a: $b \in \text{Im}(A)$, and $\text{Ker}(A) \overset{\text{just}}{=} \{\mathbf{0}\}$.

there is at least one sol'n + L is injective ⟹ system of eq'ns has exactly 1 sol'n.

Case 2b: $b \in \text{Im}(A)$, and $\text{Ker}(A) \neq \{\mathbf{0}\}$.

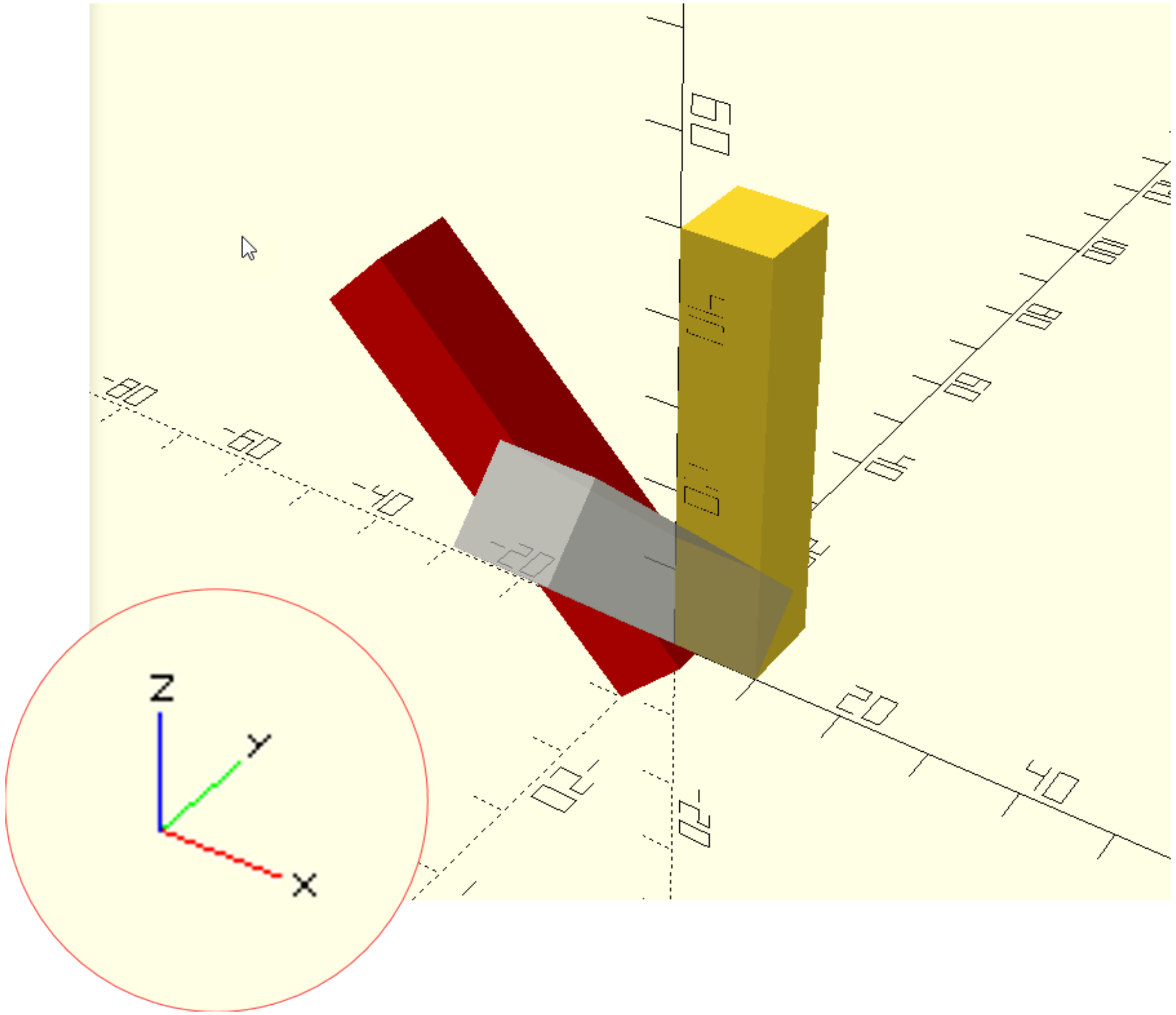
Kernel is larger than just {0}

there is at least one sol'n + L is not injective ⟹ system of eq'ns has an infinite number of sol'ns.



Section 13 - Matrix Multiplication

A very handy property of linear transforms is that the **composition of two linear transforms is also linear**. For example, in graphics, we frequently want to compose rotations of an object.



Start: Yellow block.

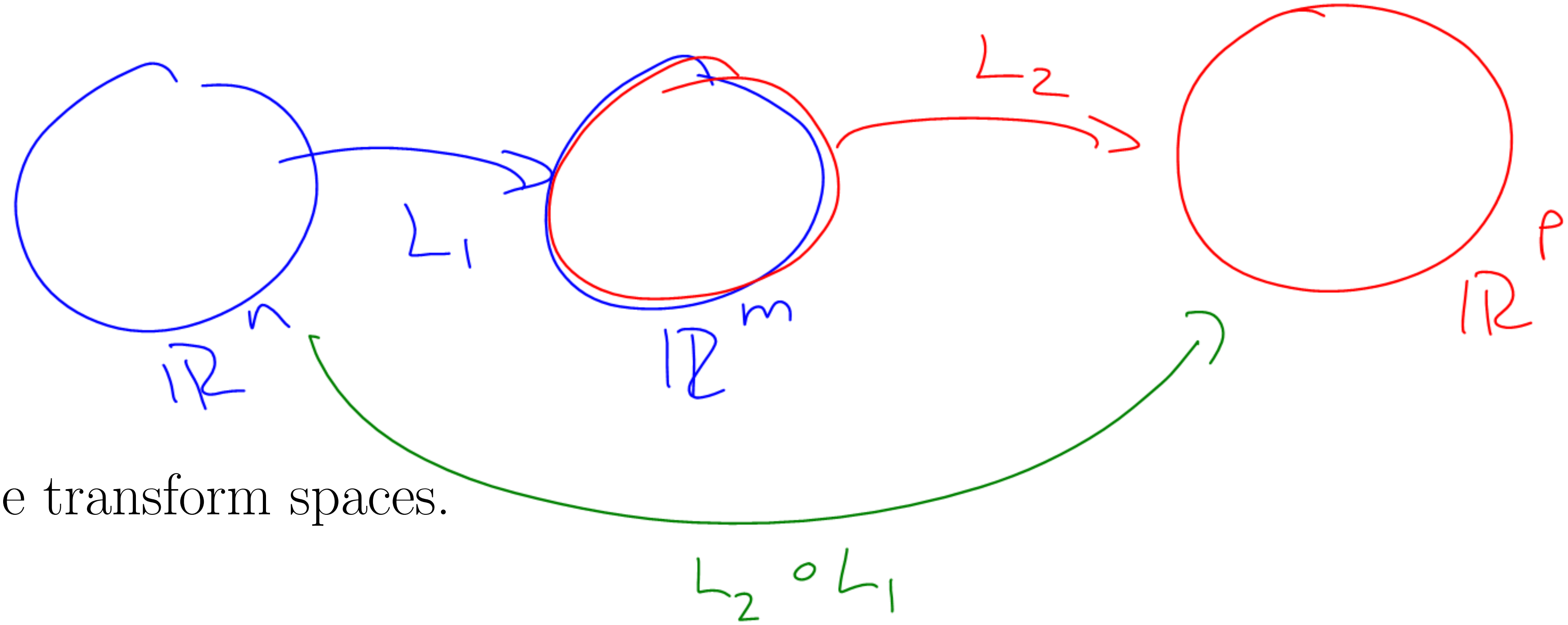
Grey block: Rotated by 45 degrees around the x axis.

Red block: Then rotated by 90 degrees around the z axis.

L_1 output = L_2 input

Theorem: Suppose that $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^p$ are both linear transformations. Then $L = L_2 \circ L_1$ is also a linear transformation.

Reminder: interpret $L_2 \circ L_1$ in words. "L_2 follows L_1" or $L_2(L_1(v))$



Sketch the transform spaces.

If $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^p$ are linear then so is their composition, $L_2 \circ L_1$.

Proof

Need criteria for a linear transform:

1) $L(v_1 + v_2) = L(v_1) + L(v_2)$

2) $L(\alpha v) = \alpha L(v)$

Check for $L_2 \circ L_1 \Rightarrow L_2(L_1(v))$ for any $v \in \mathbb{R}^n$

1) Consider $v_1, v_2 \in \mathbb{R}^n$ (inputs for L_1)

Goal

Look at $L_2(L_1(v_1 + v_2)) = L_2(L_1(v_1)) + L_2(L_1(v_2))$

$= L_2(\underbrace{L_1(v_1)} + \underbrace{L_1(v_2)})$ b/c L_1 is linear.

$\in \mathbb{R}^m \to$ inputs to L_2

$= L_2(L_1(v_1)) + L_2(L_1(v_2))$ ✓ b/c L_2 is linear

If $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^p$ are linear then so is their composition, $L_2 \circ Ł_1$.

2) consider $\alpha \in \mathbb{R}$, $v \in \mathbb{R}^n$

Look at $L_2(L_1(\alpha v))$ = Good $\alpha L_2(L_1(v))$

$= L_2[\alpha L_1(v)]$ b/c L_1 is linear

✓ $= \alpha L_2(L_1(v))$ b/c L_2 is linear.

So if L_1, L_2 are linear transforms, so too is $L_2 \circ L_1$

If $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^p$ are linear then so is their composition, $L_2 \circ \text{Ł}_1$.

Matrix Representation of Linear Transformations

Suppose that L_A and L_B have standard matrices A and B. Since $L_B \circ L_A$ is also a linear transformation, it will have a standard matrix, say C.

Question: how is C related to the original A and B?

Example: Let L_A and L_B be defined by the standard matrices below

cols = # input dimensions

L(1,0) and L(0,1)

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 4 & 1 & 3 \\ -3 & 0 & 1 & 1 \end{bmatrix}$$

rows = output dim's.

$L_B : \mathbb{R}^4 \to \mathbb{R}^3$

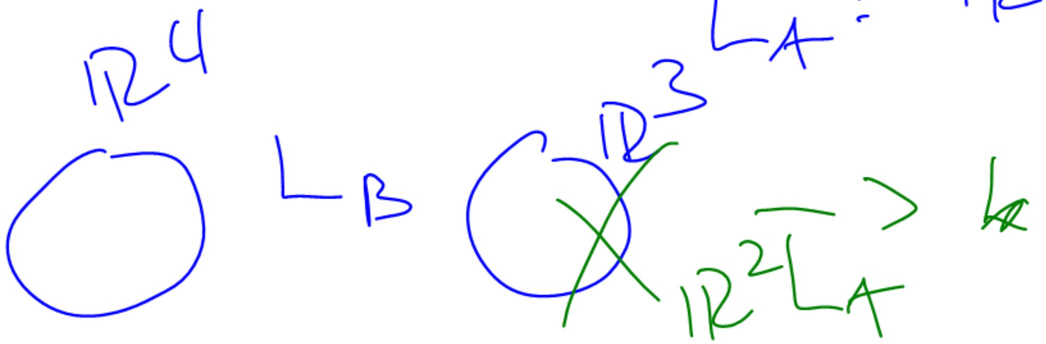
The C matrix must be $\underline{\ 3\ } \times \underline{\ 2\ }$.

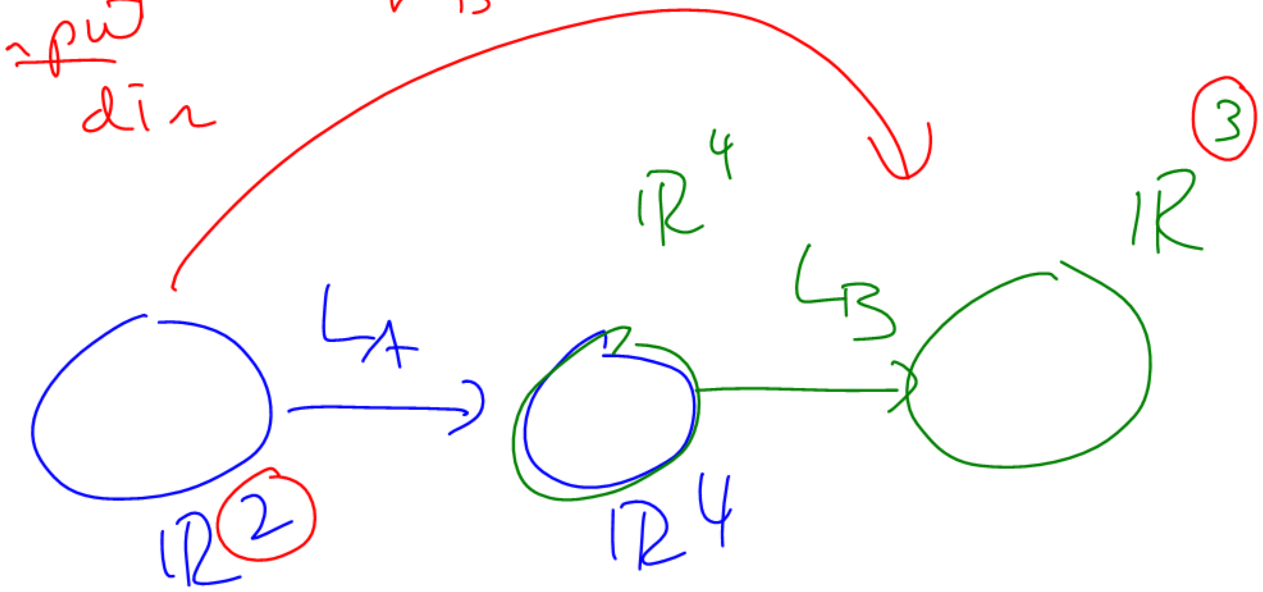
$L_B \circ L_A = L_C : \mathbb{R}^2 \to \mathbb{R}^3$

output dim. = rows

cols = # input dim

$L_A : \mathbb{R}^2 \to \mathbb{R}^4$





$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 4 & 1 & 3 \\ -3 & 0 & 1 & 1 \end{bmatrix}$$

Find the standard matrix C for $L_B \circ L_A$, using the definition of this matrix as the transform of the canonical basis vectors $(1, 0)$ and $(0, 1)$.

dot prod rows × cols.

$$L_B \circ L_A(1,0) = L_B\left(\begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = L_B\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 5 \end{bmatrix}\right)$$

copy 1st col.

$$= \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 4 & 1 & 3 \\ -3 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 4 & 1 & 3 \\ -3 & 0 & 1 & 1 \end{bmatrix}$$

$$L_B(L_A(0,\underset{\substack{\uparrow \\ \text{in } 2^{nd} \\ \text{pos'n}}}{1})) = L_B\left(\underset{\substack{\uparrow \\ 2^{nd} \text{ col of } A}}{\begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}}\right)$$

$$= \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 4 & 1 & 3 \\ -3 & 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{bmatrix} 12 \\ 11 \\ -6 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 4 & 1 & 3 \\ -3 & 0 & 1 & 1 \end{bmatrix}$$

---

full std matrix $C = L_B \circ L_A$

$$\begin{bmatrix} 8 & 12 \\ 19 & 11 \\ -1 & -6 \end{bmatrix}$$

3x2 matrix, as we desired.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "L_A"] \arrow[rr, bend left, "L_C"] & \mathbb{R}^4 \arrow[r, "L_B"] & \mathbb{R}^3 \end{tikzcd}$$

From this example, we can infer the general pattern in the matrices for composing linear transformations.

Proposition Suppose that $L_A : \mathbb{R}^n \to \mathbb{R}^m$ and $L_B : \mathbb{R}^m \to \mathbb{R}^p$ are both linear transformations with standard matrices A and B respectively. Then the standard matrix C for the composition $L = L_B \circ L_A$ is

$$C = \left[Bc_1 \mid Bc_2 \mid Bc_3 \mid \ldots \mid Bc_n \right]$$

where the c_i are the columns of A, and Bc_i is the output of $L_B(c_i)$.

This tool of combining matrices turns out to be surprisingly useful, so we define it in general for any pair of (appropriately dimensioned) matrices.

Definition If A is an $m \times n$ matrix, and B is an $p \times m$ matrix, the we define the product BA to be the $p \times n$ matrix

$$BA = \left[Bc_1 \mid Bc_2 \mid Bc_3 \mid \ldots \mid Bc_n \right]$$

where the c_i are the columns of A.

Note: we can also compute the matrix product BA using the "finger-following method" related to the dot product, as we will demonstrate in an example.

$L_B \circ L_A$ has matrix BA (B times A matrix prod)

Example: Find the matrix product of the matrices below, by the finger-following method across (rows in B) and (columns in A).

$$\underset{B}{\begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 2 & 3 \\ -1 & 2 \\ -3 & 4 \end{bmatrix}}$$

row · col

dot products

1st r, 1st c = dot of 1st row of B · 1st col of A

$$= \begin{bmatrix} [3,0,1]\cdot\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} & [3\ 0\ 1]\cdot\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \\ [-1\ 2\ 1]\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} & [-1\ 2\ 1]\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 3 & 13 \\ -7 & 5 \end{bmatrix}$$

need 2nd row, 1st col = 2nd row of B · 1st col of A

Check: $\begin{bmatrix} 3 & 13 \\ -7 & 5 \end{bmatrix}$

Example: Find the product of the two matrices below.

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & 2 \\ -1 & 2 & -1 & 4 \end{bmatrix}$$

$$3 \times 4 = \begin{bmatrix} 3 & 8 & 1 & 8 \\ -2 & -3 & -1 & -2 \\ -5 & -4 & -3 & 0 \end{bmatrix}$$

4

$\sqrt{\quad}$ $(1+x)$ $7 \; {}^{1}n$

$$\frac{1}{1-x} = 1 + x \cdots -$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!}$$

Check:
$$\begin{bmatrix} 3 & 8 & 1 & 8 \\ -2 & -3 & -1 & -2 \\ -5 & -4 & -3 & 0 \end{bmatrix}$$

Next week: some unexpected properties of matrix multiplication!