Week #10: Invertible Matrices and Determinants

Text Section 14 - Invertible Matrices and Determinants

Quick Review of Linear Functions

input from $\mathbb{R}^2$

$$L(x,y) = (\underbrace{2x-3y}, \underbrace{x+y}, \underbrace{4x+5y})$$

output in $\mathbb{R}^3$

$$L(x,y) = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} x + \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} y$$

$\mathrm{Im}(L) = \mathrm{span}\left(\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}\right)$

standard matrix for L.

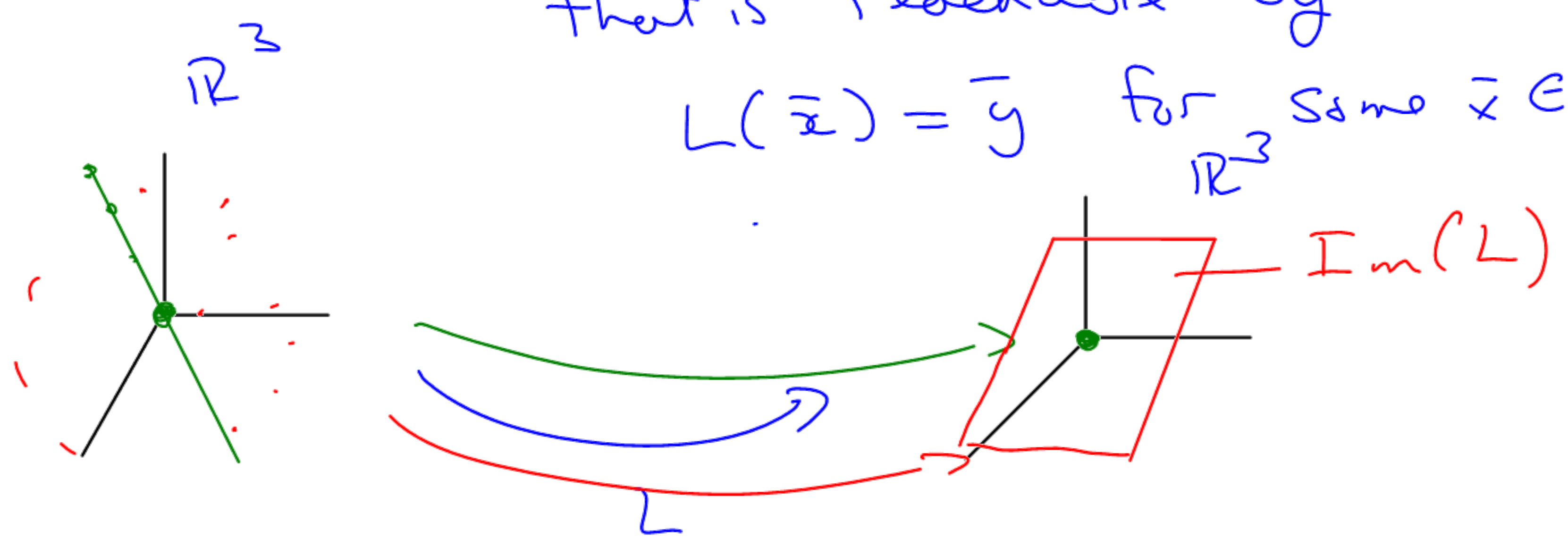
$$A_L = \begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 4 & 5 \end{bmatrix}$$

$$\text{and } L(x,y) = \begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Also recall:

Definition: for a linear mapping $L : \mathbf{V} \to \mathbf{W}$, the set of all **input** vectors that are mapped to $\mathbf{0_W}$ is called the Kernel of L, or the **null space** of L. Formally:

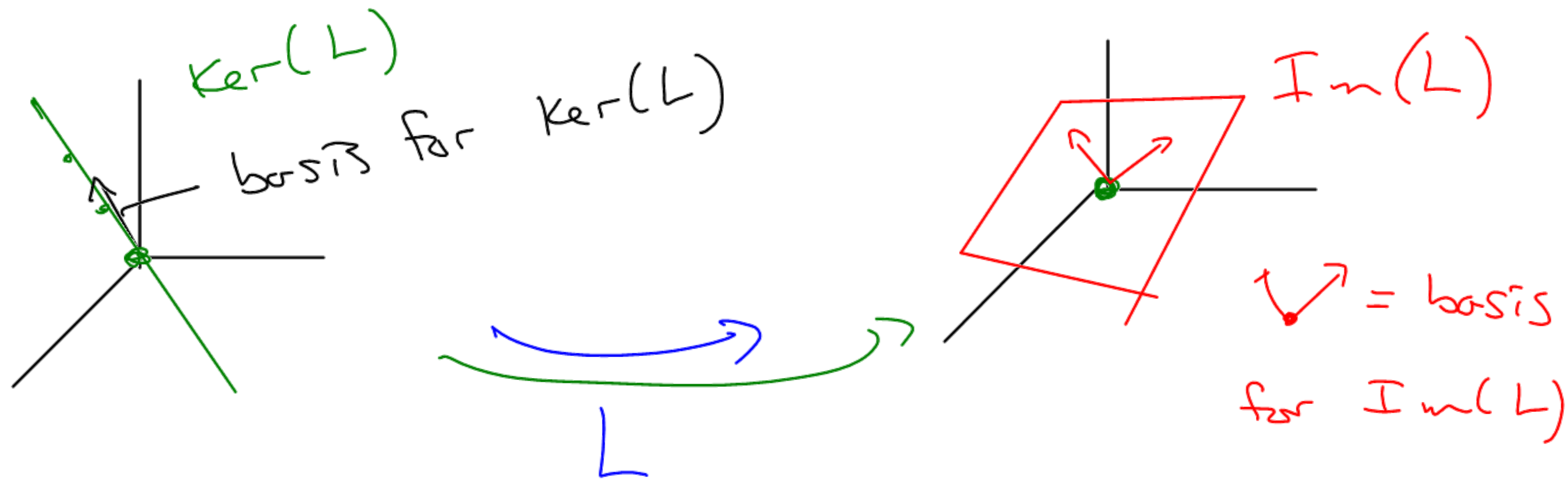
Definition: Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation. We define the **image** of L, $\mathrm{Im}(L)$ by: the set $\bar{y} \in W$ (output) that is reachable by $L(\bar{x}) = \bar{y}$ for some $\bar{x} \in V$



Both the kernel and the image of $L : V \to W$ are subspaces:

- kernel is a subspace of input space
- image is a " " output space.

Since they are subspaces, each of them must have **bases** and **dimension**. Sketch:



Note: every linear transformation L has a kernel and an image.
We will also used the same nomenclature for every matrix A, where we infer that A is the standard matrix for a linear transform L.

$$\begin{tikzcd} V \arrow[r, bend left, "L"] & W \arrow[l, bend left, "L"] \end{tikzcd}$$

Invertibility

We will further explore the idea of our matrices-as-functions, and specifically whether they are **invertible**. But first, we review the invertibility in the general case, including non-linear functions.

Invertible functions: Given a function $f : \mathbb{R} \to \mathbb{R}$, the notation f^{-1} indicates the *inverse function* of f (if the inverse function exists).

$$\begin{tikzcd} \mathbb{R} \arrow[r, bend left, "f"] & \mathbb{R} \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Examples

(a) Let $f(x) = x^3$.

$f^{-1}(x) = \sqrt[3]{x}$

(b) Let $g(x) = 2x \boxed{- 7}$. not linear for linear algebra

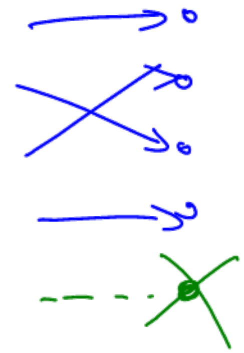
$g^{-1}(x) = \frac{x+7}{2}$

The f^{-1} function, where it exists, **un-does** the effect of f:

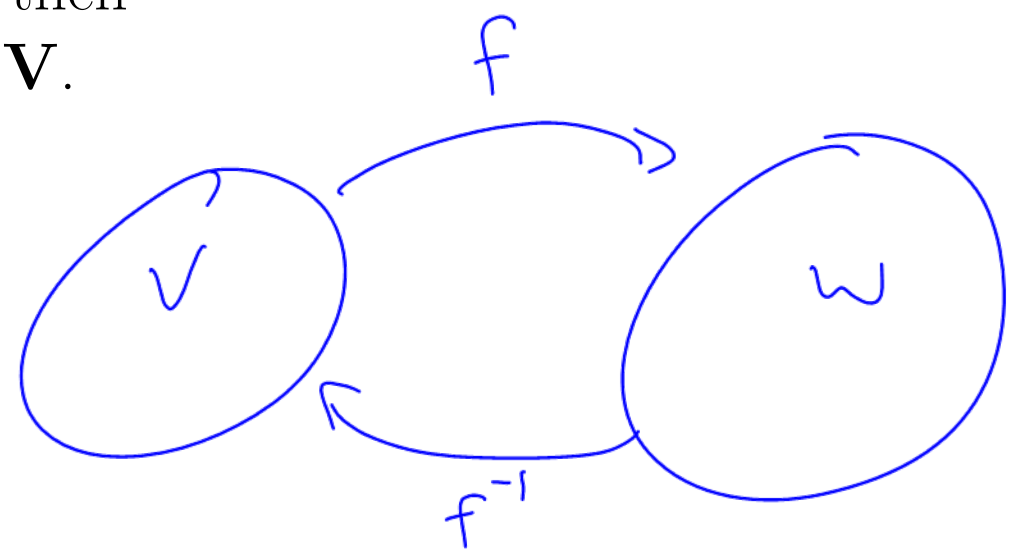
$$f^{-1}(\ f(x)\ ) = x, \text{ and } f(\ f^{-1}(x)\ ) = x$$

When does a function have an inverse?

A function $f : V \to W$ has an inverse function f^{-1} if and only if f is **injective (one-to-one)** and **surjective (onto)**.

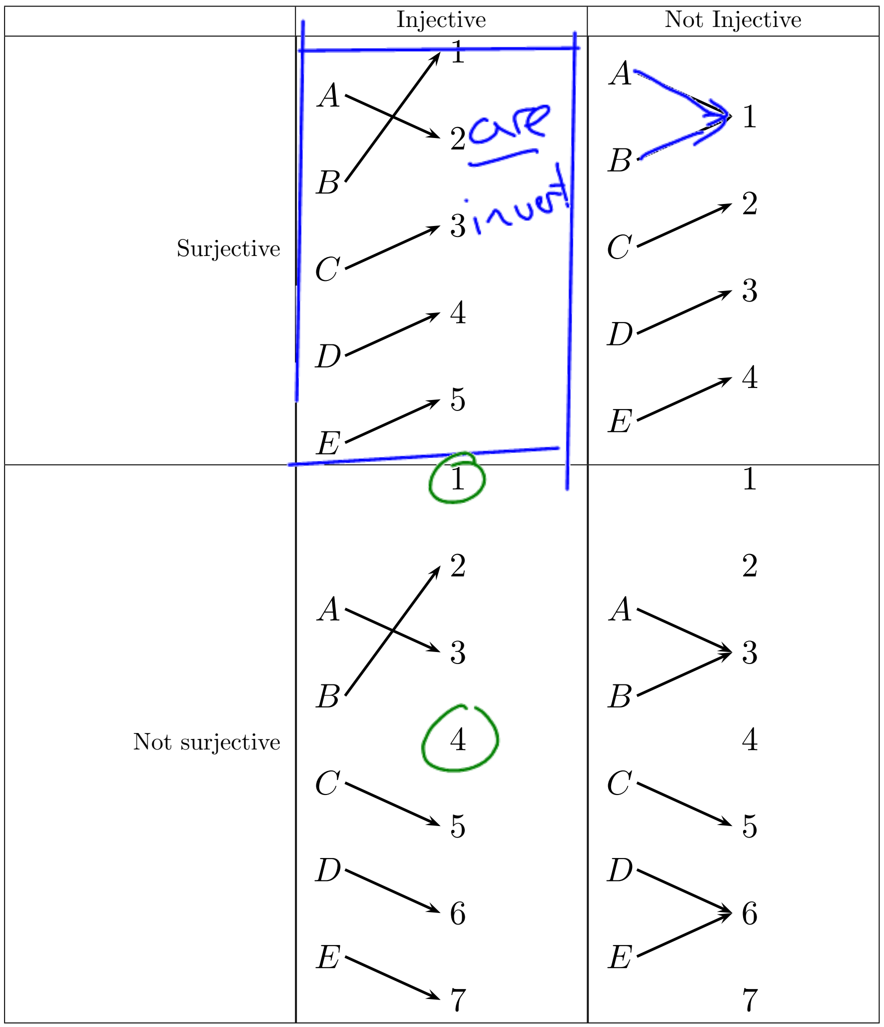


Note that if an inverse function exists and
$f : \mathbf{V} \to \mathbf{W}$ then
$f^{-1} : \mathbf{W} \to \mathbf{V}$.
Diagram:



A function $f : V \to W$ has an inverse function f^{-1} if and only if f is **injective (one-to-one)** and **surjective (onto)**.

Question: Use the diagram below to identify problems with the inverse function when a function is *not* surjective, or *not* injective.



$f^{-1}(1) = A$ or B

not invertible b/c not injective

$f^{-1}(1) = ???$

or

$f^{-1}(4) = ???$

not invertible b/c not surjective

Invertibility of $\boxed{\text{Linear}}$ Functions or Mappings

Suppose that $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

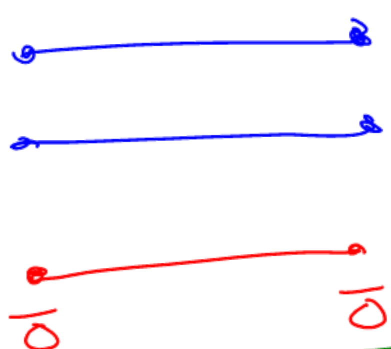
Under what conditions is L invertible, or alternatively *injective and surjective*?

For the work below, assume A is the standard matrix for L.

Required: L is injective (one-to-one)

want

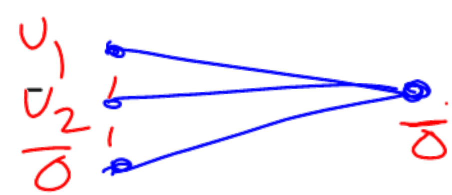
$\mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$



$\boxed{\ker(L) = \{\bar{0}\}}$

Bad

$\mathbb{R}^n \longrightarrow \mathbb{R}^m$



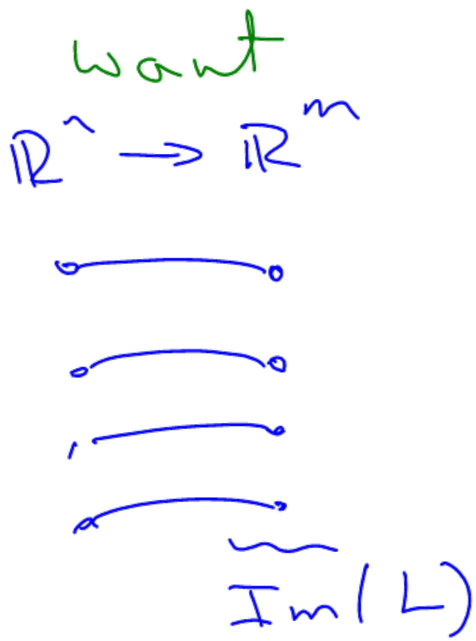
bad

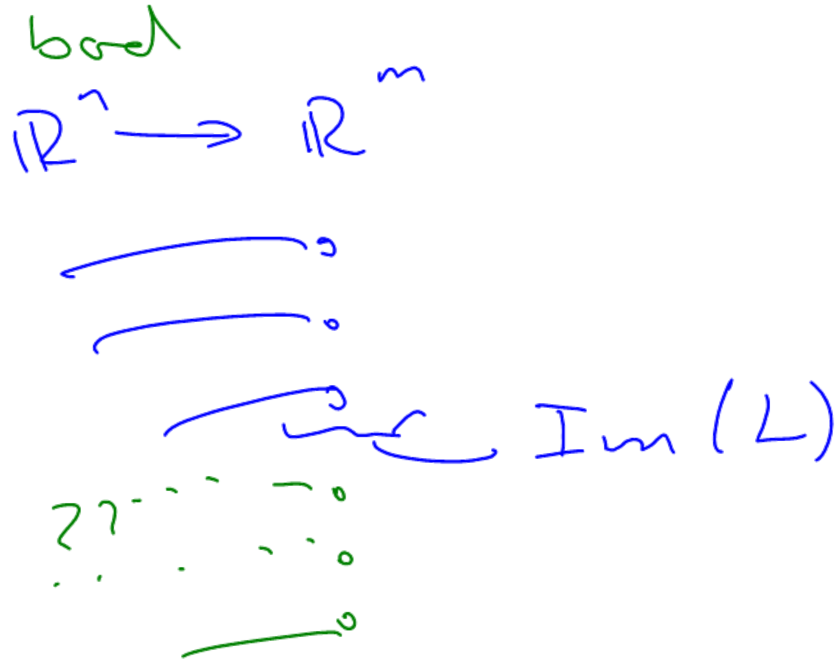
Kernel larger than $\{\bar{0}\}$

week 7:

Thm: L linear, L is injective if and only if $\ker(L) = \{\bar{0}\}$

Required: L is surjective (onto)





$$\boxed{\mathrm{Im}(L) = \text{all of } \mathbb{R}^m}$$

matrix A

Conclusion: $L : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if

- $\mathrm{Ker}(L) = \{\bar{0}\}$ → columns of A are linearly indep.

  +

- $\mathrm{Im}(L) =$ all of $\mathbb{R}^m$ → columns of A span $\mathbb{R}^m$.

  $\Downarrow$

  columns of A are a basis for $\mathbb{R}^m$

$$A = \underbrace{\left.\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\right\}}_{n \text{ colums.}} \begin{bmatrix} | \\ n \\ | \end{bmatrix}$$

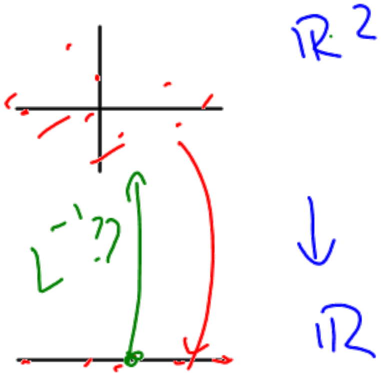
m rows

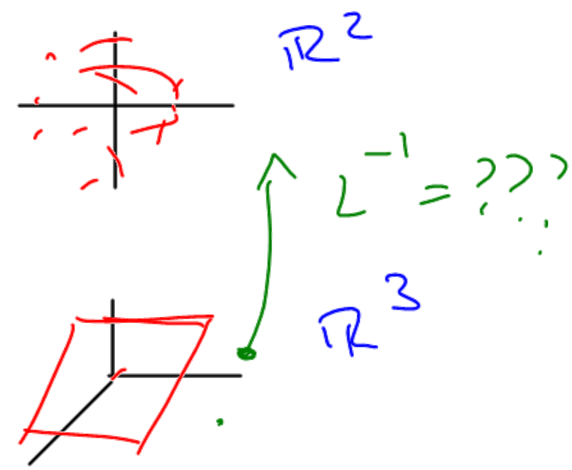
$L : \mathbb{R}^n \to \mathbb{R}^m$ is invertible.

Consequence: if L is invertible, what are the dimensions of its standard matrix A?

A must be a square matrix
- same number of rows and cols.

$\Rightarrow$ Linear mappings can be invertible only if input & output spaces have the same dimension.





Nomenclature: since there is a such a close tie between linear functions and their matrix representations, we can talk equally about

(1) The linear function L being invertible;

standard

(2) The matrix for that function, L_A, being invertible; or

(3) A generic matrix A being invertible.

Rank-nullity and invertibility

not square $\Rightarrow$ not invertible

We showed earlier that to be invertible, a matrix must first be square $\boxed{(n \times n)}$. However, not even all $n \times n$ matrices will be invertible. Here we connect important properties of matrices to invertibility. Assume the matrix A is the standard matrix for a linear transformation L.

A has ___n___ columns.

injective

To be invertible, L must be one-to-one and onto. This means that the kernel of A is: ___$\{\vec{0}\}$ – 0 dimen'l___

\# of lin'ly indep col's

So the rank of A must be: ___n ($\underset{n}{\text{rank}} + \underset{0}{\text{null}} = \underset{n}{\#\ \text{cols}}$)___

So the columns of A must be: ___linearly indep___

Any of these conditions can be used to test for a matrix's invertibility.

Examples: Which of the following matrices are invertible?

$$A = \begin{bmatrix} 8 & 1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \qquad \mathbb{R}^3 \to \mathbb{R}^2$$

not square $\Rightarrow$ not invertible.

$$B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

square. ✓

but cols of B are not linearly indep

$\Rightarrow$ B is not invertible.

$$C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

square ✓

and cols of C are lin'ly indep.

$\Rightarrow$ C is invertible

Foreshadowing: how would we tell if a larger, e.g. 5x5, matrix were invertible?

Defining the Matrix of an Inverse Linear Function

Suppose that $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, **and** that it is invertible.

We can define the inverse function $L^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by the rule:

$$L^{-1} : \underset{①}{\mathbb{R}^n} \to \underset{②}{\mathbb{R}^n}$$

for each $\bar{b} \in \underset{②}{\mathbb{R}^n}$,

$\bar{x} = L^{-1}(\bar{b})$ is the unique input $\bar{x} \in \underset{①}{\mathbb{R}^n}$ such that

$$L(\bar{x}) = \bar{b}$$

or $$L^{-1}(\underbrace{L(\bar{x})}_{\bar{b}}) = \bar{x}$$

Theorem: If $L : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation, then $L^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is also a linear transformation.
Note: though this theorem is true, we will not prove it in this class.

Consequence: since L^{-1} is also a linear transformation, it too must have a standard matrix, A^{-1}.

One obvious question is how to compute the matrix A^{-1}, based on the matrix A.

"A inverse"

Algorithm for Computing the Inverse Matrix

If A is an invertible $n \times n$ matrix, write the following double-width matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & & & 0 \\ 0 & 0 & \ddots & & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\ A \ \mid \ I_n\ ].$$

orig'l

$n \times n$ identity matrix

Put this combined matrix into RREF. The resulting matrix will be

$$[\ I_n \ \mid \ A^{-1}\ ]$$

where A^{-1} is the inverse of A.

Example: Given the invertible 2×2 matrix

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix},$$

compute the matrix A^{-1}.

Goal: get LHS into I_2

$$\left[\begin{array}{cc|cc} \underbrace{\begin{matrix} 3 & 5 \\ 4 & 7 \end{matrix}}_{A} & & \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{2\times 2\ I} \end{array}\right] \xrightarrow{\substack{R_1 \\ 4R_1 - 3R_2}} \left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 0 & -1 & 4 & -3 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 + 5R_2 \\ -R_2}} \left[\begin{array}{cc|cc} 3 & 0 & 21 & -15 \\ 0 & 1 & -4 & 3 \end{array}\right] \xrightarrow{\substack{\frac{1}{3}R_1 \\ R_2}} \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{2\times 2\ I} & \underbrace{\begin{matrix} 7 & -5 \\ -4 & 3 \end{matrix}}_{A^{-1}} \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$ ☒

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$$

Example: Given the invertible 3×3 matrix

$$B = \begin{bmatrix} 3 & 1 & 3 \\ 0 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix},$$

compute the matrix B^{-1}.

Goal: get I_3 on LHS.

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ \underbrace{2 \quad 1 \quad 2}_{B} & & & \underbrace{0 \quad 0 \quad 1}_{3 \times 3\ I} & & \end{array}\right] \longrightarrow \begin{array}{c} R_1 \\ R_2 \\ 2R_1 - 3R_2 \end{array} \left[\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 & 0 & -3 \end{array}\right]$$

$$\begin{array}{c} R_1 \\ -R_3 \\ R_2 + 3R_3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 6 & 1 & -9 \end{array}\right] \longrightarrow \begin{array}{c} R_1 - R_2 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 3 & 0 & -3 \\ 0 & 1 & 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 6 & 1 & -9 \end{array}\right]$$

$-3+27=24$

$$B = \begin{bmatrix} 3 & 1 & 3 \\ 0 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix},$$

$$\begin{array}{c} R_1 - 3R_3 \\ \to R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & -15 & -3 & 24 \\ 0 & 1 & 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 6 & 1 & -9 \end{array}\right]$$

$$\begin{array}{c} R_1/3 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -1 & 8 \\ 0 & 1 & 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 6 & 1 & -9 \end{array}\right]$$

(3×3 I)

so $$B^{-1} = \begin{bmatrix} -5 & -1 & 8 \\ -2 & 0 & 3 \\ 6 & 1 & -9 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 & 3 \\ 0 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix},$$

Some Properties of the Inverse Matrix

- $A^{-1}\ A = I_n$

Proof: $A^{-1}(A\ \bar{x}) = \bar{x}$ $\Rightarrow A^{-1}A = I_n$

$I_n \bar{x} = \bar{x}$

- $A\ A^{-1} = I_n$

Proof: $\bar{y} = A\bar{x}$

$\downarrow$

$A^{-1}\bar{y} = A^{-1}A\bar{x}$

$A^{-1}\bar{y} = I_n\bar{x}$ $\Rightarrow$ $I_n\bar{x} = \bar{x}$, so

$A^{-1}\bar{y} = \bar{x}$

$A\ A^{-1}\bar{y} = A\bar{x} = \bar{y}$

and $I_n\bar{y} = \bar{y}$

so $A\ A^{-1} = I_n$

1. - $A\bar{x} = \bar{b} \implies \bar{x} = A^{-1}\bar{b}$

$\uparrow$ system of equation, n eq'ns, n unknowns.

want to use RREF instead

Not the most efficient/ Not the most accurate.

Goal: find $\bar{x}$

$A^{-1}\cdot$ 1 $\Rightarrow A^{-1}(A\bar{x}) = A^{-1}b \Rightarrow \bar{x} = A^{-1}b$, sol'n to our system of equations.

Alternative Test for Matrix Invertibility

Reminder: not all square matrices will be invertible.

require all columns be linly indep..

If our rank/column-linear-independence checks aren't immediately obvious, is there another test we can use to see if the matrix is invertible, before we try the full RREF inverting process?

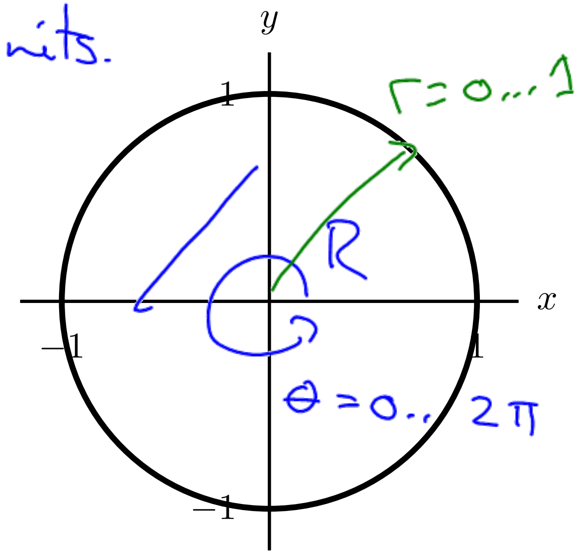
We will see that the an important quantity called the *determinant* of a matrix can be used for this. Even better, the matrix determinant can also give other information about an implied linear transformation.

Introduction to Determinants

Easy question: what is the area of the unit circle?

$r = 1$

$$\text{Area} = \pi r^2 = \pi 1^2 = \pi \text{ sq. units.}$$



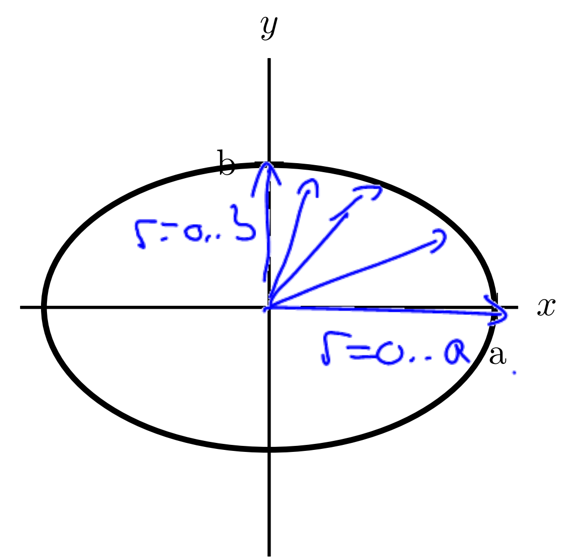
Area = from 17.2

$$\iint_R 1\, dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 1\, r\, dr\, d\theta$$

$$= \pi \text{ sq units.}$$

Harder question: what is the area of an ellipse with axis lengths a and b?

more challenging.



We will find a non-calculus way to compute the ellipse area using matrices, specifically using a new property of transformations and matrices called the **determinant**. $A = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & & - \\ \cdot & \cdot & \end{bmatrix} \rightarrow D = \#$

Associated with any linear transformation L (and its associated standard matrix A), is a single scalar value called the **determinant** D.

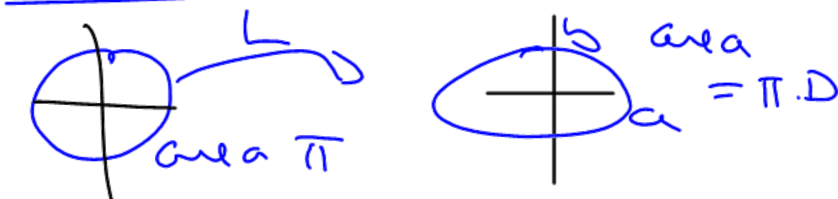
(1) The **magnitude** of D represents the **overall scaling** done by the transformation.

(2) The **sign** of D is related the **directional or orientation flips** done by the transformation.

Determinant in 2D

More specifically for maps from a plane to a plane, $L : \mathbb{R}^2 \to \mathbb{R}^2$:

(1) The **magnitude** of D represents the **scaling of the area** for a closed region by the transformation.



(2) The **sign** of D is related the relative direction/mirroring of vector pairs after the transformation.

Computing the Determinant - 2×2 case

det(A) matrix determinant

$$\text{Definition: } D(A) = |A| = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ab - cd$$

Example: compute the determinant of $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$.

$\det(A) = (2)(-3) - (0)(0) = -6$

APSC 172 cross-over: compute the determinant of the matrix of second derivatives: $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.

$D = f_{xx} f_{yy} - (f_{xy})^2$

Hessian matrix

$\det(H) = f_{xx} f_{yy} - f_{xy} f_{yx}$

$\quad\quad\quad\quad = f_{xx} f_{yy} - (f_{xy})^2$

$$D(A) = |A| = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ab - cd$$

Example: compute the determinant of $B = \begin{bmatrix} -1 & 1 \\ -4 & 2 \end{bmatrix}$.

$$\det(B) = (-1)(2) - (-4)(1)$$
$$= -2 + 4$$
$$= +2$$

Note: we will re-use these same A and B matrices in the following examples as well.

Example: Compute the transformation or image of the triangle shown after applying the linear transformation defined by $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$.

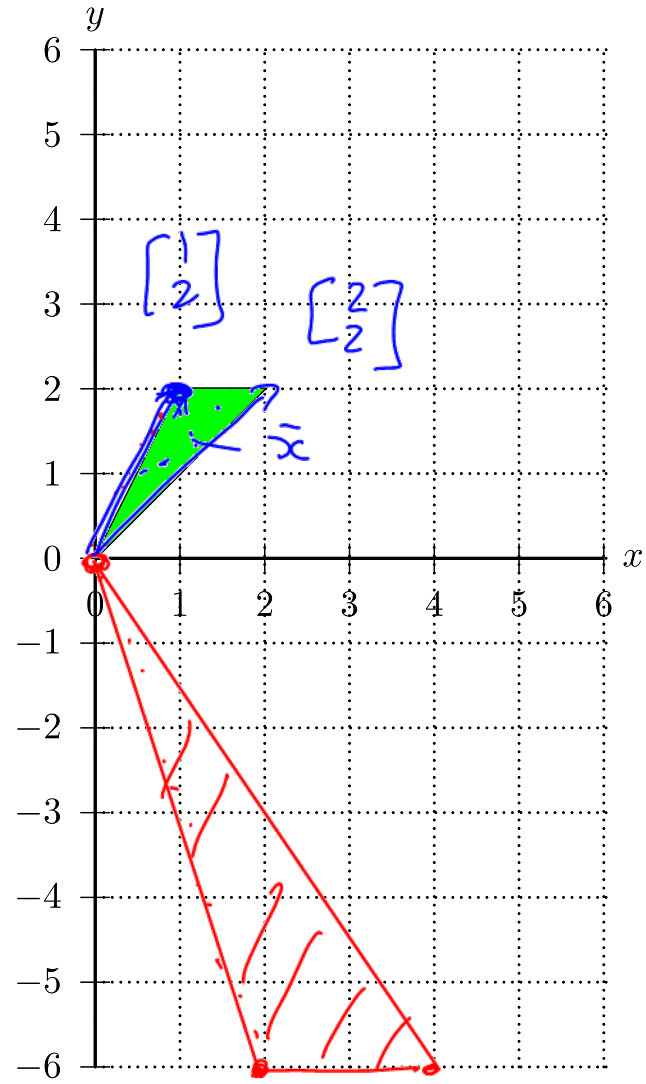
Recall: $|A| = -6$

Compute $A\bar{x}$

$$A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

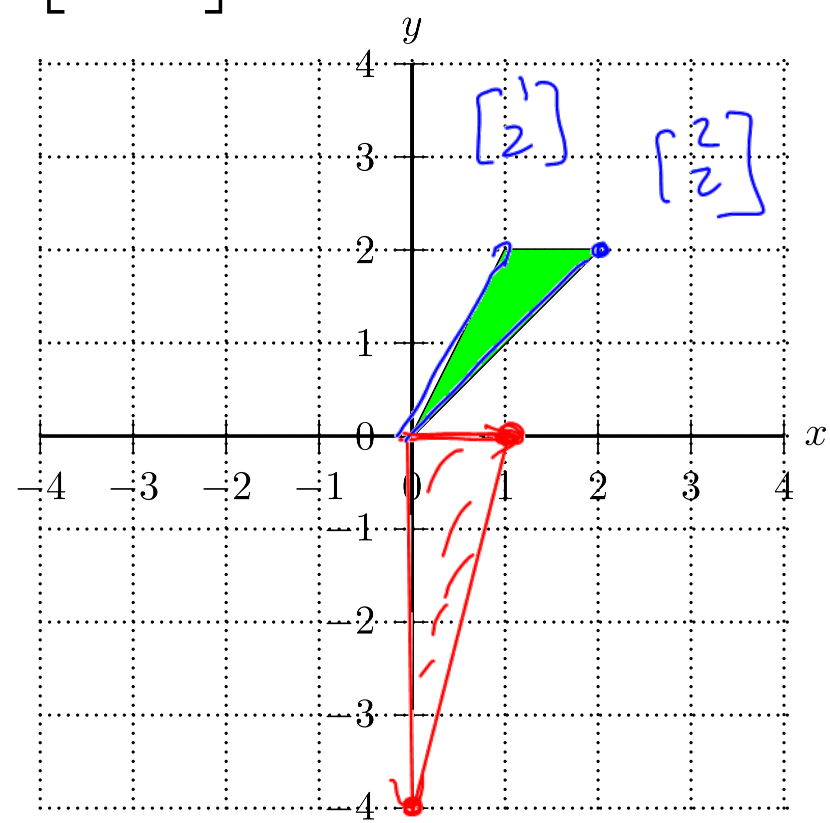
$$A\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$



Example: Compute the transformation of the triangle shown by the linear transformation defined by $B = \begin{bmatrix} -1 & 1 \\ -4 & 2 \end{bmatrix}$. Recall that $|B| = 2$.

$$B\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-1 & 1\\-4 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$B\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}-1 & 1\\-4 & 2\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}0\\-4\end{bmatrix}$$

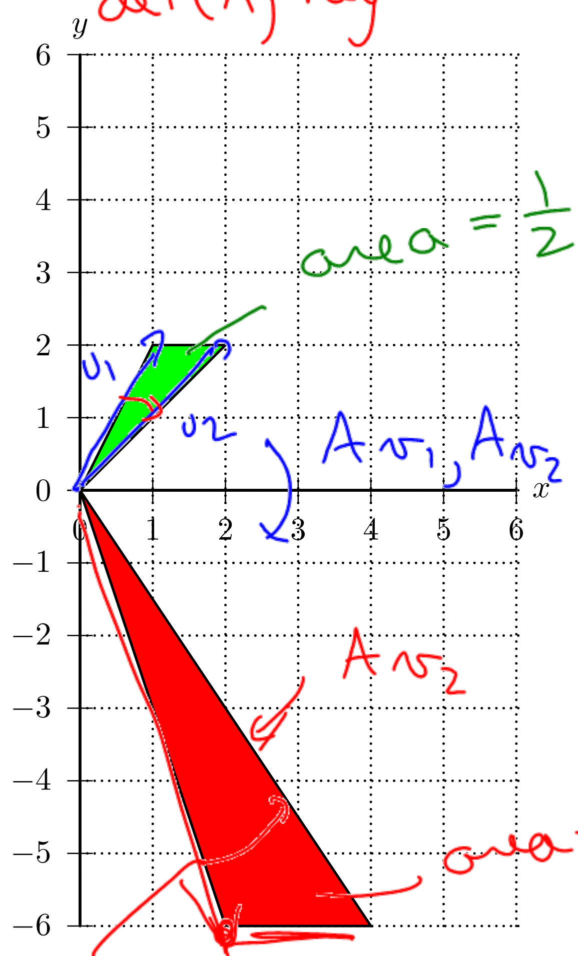


Recall:

(1) The **magnitude** of D represents the **scaling of the area** for a closed region by the transformation.

(2) The **sign** of D is related the relative direction/mirroring of vector pairs after the transformation.

det(A) neg



area $= \frac{1}{2}(1)(2) = 1$

$A v_1, A v_2$

area $\times 6$

$= $ area $\cdot |\det(A)|$

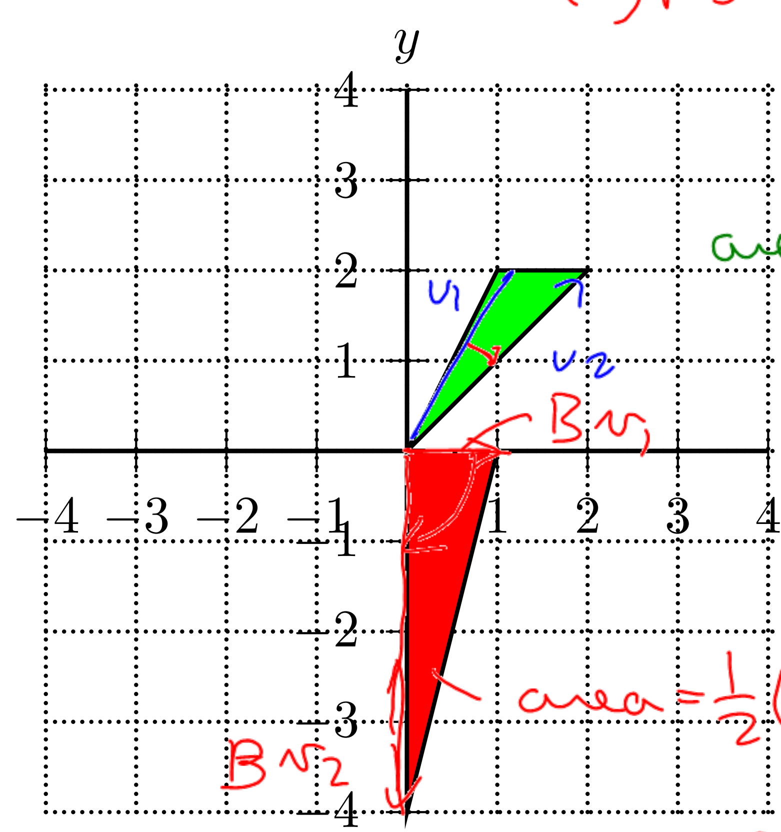
$A v_2$

area $= \frac{1}{2}(2)(6)$

$= 6$

$A v_1$

det(B) pos



area $= 1$

area $\times$ $|\det(B)|$

$B v_1$

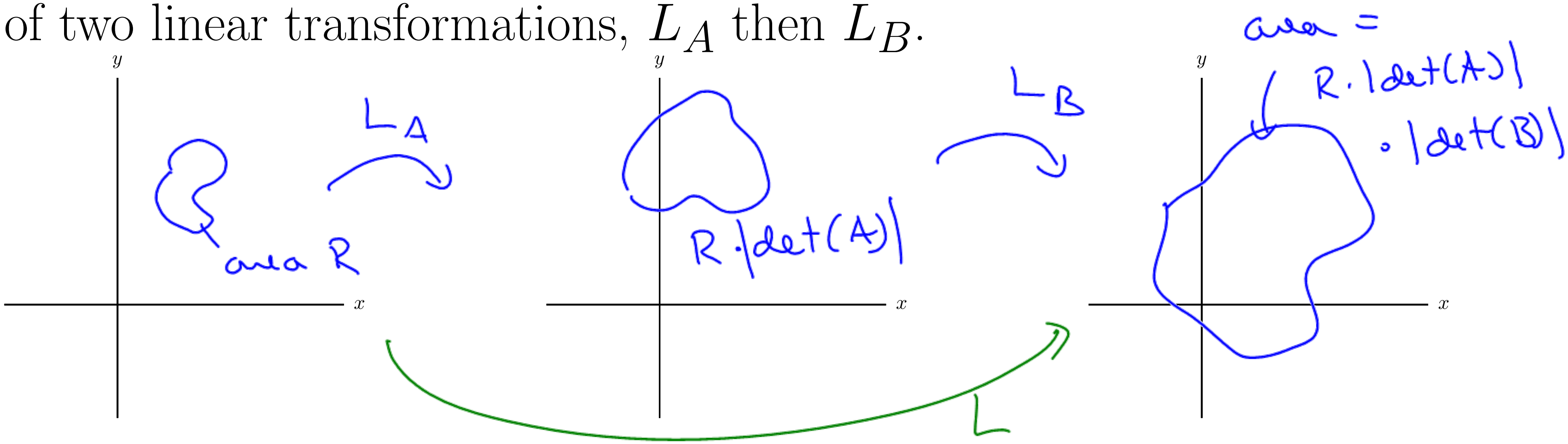
area $= \frac{1}{2}(1)(4)$

$= 2$

$B v_2$

The Determinant and Composition

Example: Sketch a region in $\mathbb{R}^2$ that is transformed by a composition of two linear transformations, L_A then L_B.



What is the effective change in area of the composition $L = L_B \circ L_A$?

mult by $|\det(A)||\det(B)|$

What does this say about the determinant of the standard matrix for $L_B \circ L_A$?

$\downarrow$ BA

$$|\det(BA)| = |\det(A)||\det(B)|$$

In general, for a matrix product (AB):

$$\boxed{\det(AB) = \det(BA) = \det(A)\det(B)}$$

Test for

The Determinant and Invertibility

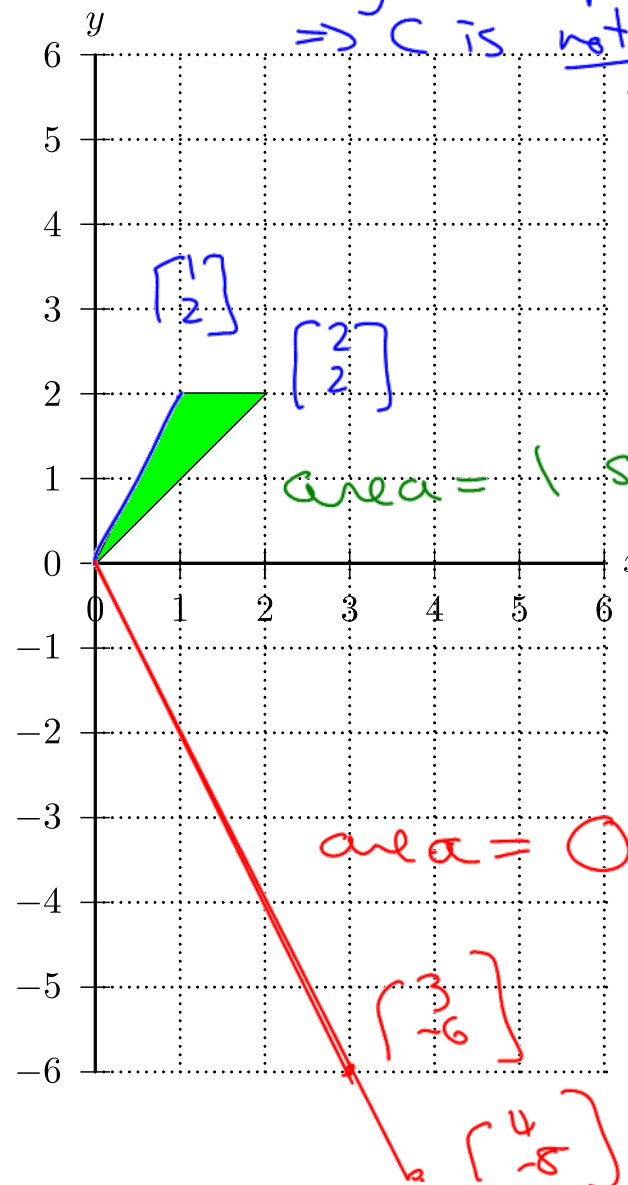
Example: Compute the transformation of the triangle shown by the linear transformation defined by $C = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$.

col's are not lin'ly indep $\Rightarrow$ C is not invertible

$$C\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

and

$$C\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$



area = 1 sq unit

area = 0

What do you notice about the resulting area?

$$C = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}.$$

What is the dimension of the image of C? $\Rightarrow$ 1D (line)

span of $\begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ = span $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

What is the dimension of the <u>kernel</u> of C?

Started in 2D

Rank/nullity $(\dim \ker) + (\overset{1}{\cancel{\dim \mathrm{Im}}}) = 2 \Rightarrow \dim(\ker) = 1$

What is the determinant of C?

From area, $\det(C) = 0$ | from 2x2 det formula

$\left| \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \right| = (-2) - (-2) = 0.$

Is C invertible?

<u>No!</u>

- not full rank
- Ker is not just $\{\vec{0}\}$
- columns are <u>not</u> lin'ly indep
- $\det(C) = 0$

In general, a square $n \times n$ matrix A **is** invertible if and only if:

seen already

- The columns of A are linearly independent.
- The rank of A is n.
- The kernel of A has dimension 0.

New!!

- The determinant of A is *not* zero.

Computing the determinant for 2×2 matrices

Recall: for a 2×2 matrix,

$$\det(A) = |A| = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ab - cd$$

Note: the vertical bars here are *not* absolute values. Remember the determinant can be negative!

Example: compute the determinant of $A = \begin{bmatrix} 2 & 7 \\ -1 & -2 \end{bmatrix}$

$\det(A) = (2)(-2) - (-1)(7)$

$= -4 + 7 = 3$ $\boxtimes$

row col

Computing the determinant for 3×3 matrices

$$\det(A) = |A| = \left| \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right|$$

(−) (−) (−)

$$= \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

+ + +

- Copy the 1st two columns beside the matrix, then
- Multiply down the diagonals, and up the diagonals as shown, then
- **Add** the down-right diagonal products, and **subtract** the up-right diagonals.

Note: the subscripting in matrices works like a_{ij} as the

- row i.
- col j.

Example: Compute the determinant of the 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \\ -1 & 1 & 2 \end{bmatrix} \begin{matrix} 1 & 2 \\ 4 & 0 \\ -1 & 1 \end{matrix}$$

copy

$$\det(A) = \left[0 + (-6) + (-4)\right] - \left[0 + 3 + 16\right]$$

tidy $= -10 - 19$

$= -29$

Computing the determinant for $n \times n$ matrices

We have seen techniques now for computing determinants of 2×2 and 3×3 matrices.

Sadly, there is no 'diagonal shortcut' for any matrices past 3×3 in size. Instead, we introduce a recursive process called **the Laplace Expansion Formula** or the **co-factor expansion**.

The co-factor expansion requires two building blocks:

(a) The checkerboard pattern of signs. This always starts as a $+$ in the top-left corner, and then alternates for every vertical or horizontal step taken.

E.g.

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}, \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

(b) The $(n-1)\times(n-1)$ sub-matrix associated with any entry.

For any entry a_{ij} in an $n \times n$ matrix, we make a smaller sub-matrix by crossing out the row i and column j that entry is in.

Example:

a_{23}

$$\begin{bmatrix} 1 & 5 & -1 & 6 \\ 4 & 0 & \boxed{3} & 7 \\ 0 & 0 & -2 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 6 \\ 0 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Example:

for a_{41}

$$\begin{bmatrix} 1 & 5 & -1 & 6 \\ 4 & 0 & 3 & 7 \\ 0 & 0 & -2 & 0 \\ \boxed{2} & 3 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 5 & -1 & 6 \\ 0 & 3 & 7 \\ 0 & -2 & 0 \end{bmatrix}$$

The Determinant - Laplace Expansion Formula

1. Pick any row or column. Single
2. For every entry a_{ij} in that row or column:
   compute (checkerboard sign) $\times(a_{ij})\times$ (det of sub-matrix for a_{ij})
   +, −
3. Take the sum of all the values from step 2.

Formally (Textbook, Page 186, 190): for A being a square $n \times n$ matrix, $\det(A)$ is the real number defined as follows:

(i) If $n = 1$, i.e. $A = \begin{bmatrix} k \end{bmatrix}$ for some real number k, then $\det(A) = k$.

(ii) If $n > 1$, then $\det(A)$ is recursively defined as follows:

pick row i →

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{i,j}\, \det([A]_{i,j})$$

checker . . . submatrix

pick col j

$$\text{or } = \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j}\, \det([A]_{i,j})$$

for a selection of row i or column j.

Example: Compute the determinant of the 4×4 matrix

$$A = \begin{bmatrix} 1 & 5 & -1 & 6 \\ 4 & 0 & 3 & 7 \\ 0 & 0 & -2 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

pick row or col

$\to$ pick row 1

$$\det(A) = \underbrace{(+1)}_{a_{11}} \det\left(\underbrace{\begin{bmatrix} 0 & 3 & 7 \\ 0 & -2 & 0 \\ 3 & 4 & 1 \end{bmatrix}}_{B}\right) + \underbrace{(-5)}_{a_{12}} \det\left(\underbrace{\begin{bmatrix} 4 & 3 & 7 \\ 0 & -2 & 0 \\ 2 & 4 & 1 \end{bmatrix}}_{C}\right)$$

$$+ \underbrace{(+(-1))}_{a_{13}} \underbrace{\det\left(\begin{bmatrix} 4 & 0 & 7 \\ 0 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}\right)}_{=0} + \underbrace{(-6)}_{a_{14}} \det\left(\underbrace{\begin{bmatrix} 4 & 0 & 3 \\ 0 & 0 & -2 \\ 2 & 3 & 4 \end{bmatrix}}_{D}\right)$$

$$= (1)(42) + (-5)(20) + 0 + (-6)(24)$$

$$= 42 - 100 - 144 = \boxed{-202}$$

(continued)

$$\det(B) \Rightarrow \begin{array}{ccc|cc} 0 & 3 & 7 & 0 & 3 \\ 0 & -2 & 0 & 0 & -2 \\ 3 & 4 & 1 & 3 & 4 \end{array} \rightarrow [0+0+0] - [-42+0+0]$$

$$= +42$$

$$\det(C) \Rightarrow \begin{array}{ccc|cc} 4 & 3 & 7 & 4 & 3 \\ 0 & -2 & 0 & 0 & -2 \\ 2 & 4 & 1 & 2 & 4 \end{array} \rightarrow [(-8)+0+0] - [-28+0+0]$$

$$= +20$$

$$\det(D) \Rightarrow \begin{array}{ccc|cc} 4 & 0 & 3 & 4 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 2 & 3 & 4 & 2 & 3 \end{array} \rightarrow [0+0+0] - [0-24+0]$$

$$= +24$$

Example: re-compute the determinant of the same matrix, using a different row or column.

$$A = \begin{bmatrix} 1 & 5 & -1 & 6 \\ 4 & 0 & 3 & 7 \\ 0 & 0 & -2 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

pick row or col

pick row 3

$$\det(A) = \underbrace{\cancel{(+0)\det\left(\begin{bmatrix} & \end{bmatrix}\right)}}_{=0} + \underbrace{\cancel{(-0)\det\left(\begin{bmatrix} & \end{bmatrix}\right)}}_{=0}$$

$$+ (-2)\det\left(\underbrace{\begin{bmatrix} 1 & 5 & 6 \\ 4 & 0 & 7 \\ 2 & 3 & 1 \end{bmatrix}}_{E}\right) + \underbrace{\cancel{(-0)\det\left(\begin{bmatrix} & \end{bmatrix}\right)}}_{=0}$$

$$= (-2)(101)$$

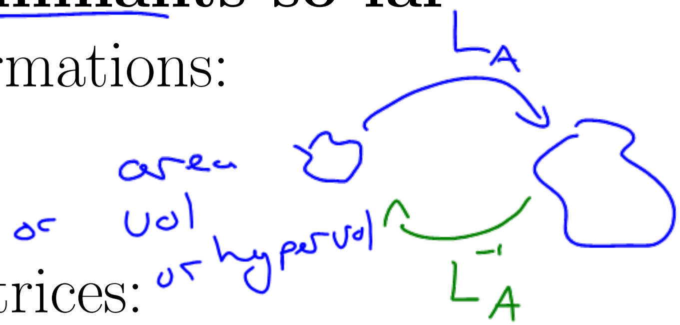
$$= -202$$

$$\det(E) \Rightarrow \begin{array}{ccc|cc} 1 & 5 & 6 & 1 & 5 \\ 4 & 0 & 7 & 4 & 0 \\ 2 & 3 & 1 & 2 & 3 \end{array}$$

$$= [0 + 70 + 72] - [0 + 21 + 20] = 142 - 41 = 101$$

Notes on determinants so far

Scaling by transformations:



scale area by $|\det(A)|$

Invertibility of matrices:

$$A \text{ invertible} \iff \det(A) \neq 0$$

Multiplication of matrices:

$$\det(B \cdot A) = \det(B)\det(A)$$

Determinants of inverse matrices:

$$A \cdot A^{-1} = I_n \qquad \det(A)\det(A^{-1}) = \det(I_n) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

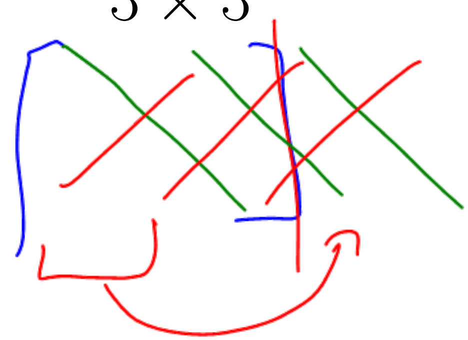
Computing determinants:

2×2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

3×3



$n \times n$

- check board
- sub matrices.