Week #11: Determinants, Invertible Matrices, Eigenvalues and Eigenvectors

Section 14 - Invertible Matrices and Determinants - Continued

Recall: Algorithm for Computing the Inverse Matrix

If A is an invertible $\underline{n \times n}$ matrix, write the following double-width matrix

$$[\ \underset{\text{original}}{A} \mid \underset{\text{Identity}}{I_n}\ ]$$

Put this combined matrix into RREF. The resulting matrix will be

$$[\ I_n \mid \underbrace{A^{-1}}_{\text{inverse of } A}\ ]$$

where A^{-1} is the inverse of A.

Example: Given the invertible 2×2 matrix

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix},$$

compute the matrix A^{-1}.

$$\left[\begin{array}{cc|cc} \underbrace{\begin{matrix} 3 & 5 \\ 2 & 4 \end{matrix}}_{A} & & \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \end{array}\right] \longrightarrow \begin{matrix} R_1 \\ 2R_1 - 3R_2 \end{matrix} \left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 0 & -2 & 2 & -3 \end{array}\right]$$

$$\longrightarrow \begin{matrix} R_1 \\ -\frac{1}{2}R_2 \end{matrix} \left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 0 & 1 & -1 & 1.5 \end{array}\right] \xrightarrow[R_2]{R_1 - 5R_2} \left[\begin{array}{cc|cc} 3 & 0 & 6 & -7.5 \\ 0 & 1 & -1 & -1.5 \end{array}\right]$$

$$\longrightarrow \begin{matrix} 1/3\,R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 0 & \underbrace{\begin{matrix} 2 & -2.5 \\ -1 & -1.5 \end{matrix}}_{A^{-1}} \\ 0 & 1 & \end{array}\right]$$

$$\text{so } A^{-1} = \begin{bmatrix} 2 & -2.5 \\ -1 & -1.5 \end{bmatrix}$$

Alternative determinant-based method for computing a matrix inverse

So we note for $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, that $A^{-1} = \begin{bmatrix} 2 & -2.5 \\ -1 & 1.5 \end{bmatrix}$,

Compute the determinant of A:

$$\det(A) = 12 - 10 = 2$$

Recall

$$\Rightarrow \det(A^{-1}) = \frac{1}{2}$$

Look for patterns in the inverse in this 2×2 case.

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$$

flip sign

swapped diag.

Hmm...

Inverse Matrices using Adjoint Matrices

Adjoint

$$\boxed{A^{-1}} = \overset{\text{know}}{\boxed{\frac{1}{\det(A)}}}\underbrace{\text{Adj}(A)}_{\text{new}}$$

$\bar{v} = \begin{bmatrix} 1 & 5 \end{bmatrix}$

$\bar{v}^T = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

where Adj(A) is a new matrix, based on determinants. Bear with it, this will take a few steps:

eg $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, then $A^T = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$

Adj$(A) = C^T$, where:

- The ^{T} indicates a *transpose*, or a reflection of values along the diagonal.
- The C matrix is the **cofactor** matrix of A, which is defined element-by-element as element of $\rightarrow C_{ij} = (-1)^{i+j} \det(M_{ij})$, and
- M_{ij} is the sub-matrix of A with row i and column j removed.

Thankfully, the co-factor matrix's elements are something you've seen already through the recursive determinant definition.

Checkboard signs

$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ . . .

Example: find the determinant of $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, using the adjoint method.

so $A^{-1} = \frac{1}{\det(A)} \underline{\text{Adj}(A)}$ need

know $= 12 - 10 = 2$

well $\text{Adj}(A) = C^{T}$, with $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$, check board $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$c_{11} = (+)\det([4]) = 4$

$c_{12} = (-)\det([2]) = -2$

$c_{21} = (-)\det([5]) = -5$

$c_{22} = (+)\det([3]) = 3$

so $C = \begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix}$

$\Rightarrow C^{T} = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \text{Adj}(A)$

$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -2,5 \\ -1 & 1.5 \end{bmatrix}$

Example: find the determinant for a general matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, using the adjoint method.

(only for 2×2 case)

$C_{11} = (+)\, d$

$C_{12} = (-)\, c \quad \Rightarrow \quad C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$C_{21} = (-)\, b$

$C_{22} = (+)\, a \quad \Rightarrow \quad C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \mathrm{Adj}(B)$

$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$\Rightarrow \quad B^{-1} = \dfrac{1}{\det(B)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

flip signs

swap places on diagonal

$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \dfrac{1}{2} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$ ☒

Optional: compute the matrix inverse for $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$, using the adjoint method.

could use RREF $[A \mid I_n]$ ✓

$$A^{-1} = \frac{1}{\det(A)} \text{Adj}(A) = \frac{1}{\det(A)} C^{T} \quad 3\times 3$$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}, \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Find all C's:

$$C_{11} = (+) \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2$$

$$C_{12} = (-) \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = -(-4) = 4$$

$$C_{13} = (+) \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$C_{21} = (-) \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} = 0$$

$$C_{22} = (+) \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$$

$$C_{23} = (-) \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$C_{31} = (+) \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} = -1$$

$$C_{32} = (-) \begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} = -12$$

$$C_{33} = (+) \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3$$

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix},$$

so $C = \begin{bmatrix} 2 & 4 & -1 \\ 0 & 5 & 0 \\ -1 & -12 & 3 \end{bmatrix}$

Need det(A):

$$\left.\begin{array}{ccc} 3 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 2 \end{array}\right| \begin{array}{cc} 3 & 0 \\ 0 & 1 \\ 1 & 0 \end{array}$$

$= [6+0+0]$

$-[1+0+0]$

$= 5$

$$\boxed{A^{-1} = \underbrace{\frac{1}{5}}_{1/\det(A)} \begin{bmatrix} 2 & 0 & -1 \\ 4 & 5 & -12 \\ -1 & 0 & 3 \end{bmatrix}}$$

✓

Check: is $A A^{-1} = I$?

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix} \left(\frac{1}{5}\right) \begin{bmatrix} 2 & 0 & -1 \\ 4 & 5 & -12 \\ -1 & 0 & 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Matrix Computation Summary

- For 2×2 matrices, the adjoint formula is the fastest by hand.
  - Though it's only 2×2, so both methods should be quick.
- For 3×3 matrices, both the adjoint and $[A|I_3]$ RREF methods will take roughly the same amount of time, depending on how much detail you write out.
- For 4×4 and larger, the $[A|I_n]$ RREF method is almost always faster.

German eigen = characteristic.

Section 15 - Eigenvalues and Eigenvectors

Recall: **determinants** summarize 'what this matrix does' as a transform, through the area/volume scaling and mirroring interpretation.

We can get more granular than that using the new concept of *eigenvalues* and *eigenvectors*, to be introduced shortly.

To set the stage, consider the question: out of all the square matrices representing transforms, which are the easiest to work with and understand their effects?

- Identity matrices I_n, e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.
  - Determinant: $\det(I_n) = 1$
  - Scaling effect: all vectors are scaled by 1.

- Next best: diagonal matrices, e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}}_{A}$, etc. $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

0's at all off-diagonal elements.

  - Determinant: product of diagonal elements.

  - Scaling effect:

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$ scaling of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ by 2

$$A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$ scaling of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ by -1

$$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$$ scaling of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ by $\frac{1}{2}$

What about non-diagonal matrices? Can some of them be distilled down to similar properties?

Example: Consider the transform implied by the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$.

Compute the transform of the following vectors by A.

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A scaled $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by 3

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

not just scaled

no simple pattern

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

not just scaled

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

A scaled $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ by 2.

For $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, summarize the effects of A as a linear transform.

- Determinant: $\det(A) = 6$ (= product of our scalings)

- Scaling effect:

$$\left.\begin{array}{l} A \text{ scaled } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ by } 3 \\ A \text{ scaled } \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ by } 2. \end{array}\right\} \text{ sufficient to define } A.$$

Is knowing those scaling effects for A enough to completely capture what A does, even it if is not a diagonal matrix? Yes!

In fact, with linearity, knowing the scaling effect for each vector in a basis completely captures the transformational effect of a matrix.

n dimensional for input space

Assume that $\{\bar{v}_1, \bar{v}_2, \dots \bar{v}_n\}$ are a basis for V, and that $L : V \to V$ is such that

complicated ← scaling factor

$$L(\bar{v}_i) = \lambda_i \bar{v}_i,$$

i.e. that each basis vector $\bar{v}_i$ is simply scaled by its scaling factor λ_i.

Prove that for such an L we can find the transform of any vector $\bar{v} \in V$ from this.

b/c $\{\bar{v}_1 \dots \bar{v}_n\}$ is a basis,

then for any $\bar{v} \in V$, there exist coeff s.t.

$\bar{v} = a_1 \bar{v}_1 + \dots a_n \bar{v}_n$ decomp of $\bar{v}$ along basis

Then $L(\bar{v}) = L(a_1\bar{v}_1 + \dots + a_n \bar{v}_n)$

$= a_1 L(\bar{v}_1) + \dots + a_n L(\bar{v}_n)$ by linearity of L

$= a_1 \lambda_1 v_1 + \dots + a_n \lambda_n v_n$ by our info about L.

fully defined

Question: how can we identify the key transforms of a matrix "$\bar{v}_i$ direction gets λ_i scaling?" in general?

The Eigenvalue/Eigenvector Equation

Definition: Given a linear transform $L : V \to V$, an **eigenvector** for/of L is a non-zero vector $\bar{v} \in V$ such that $L(\bar{v})$ is a multiple of $\bar{v}$. That is:

$$L(\underline{\bar{v}}) = \underline{\lambda\bar{v}}$$

(scaled version of the same $\bar{v}$

In a matrix form,

$$A\bar{v} = \lambda\bar{v}$$

The λ value is called the **corresponding eigenvalue** for that eigenvector.

Example: for the matrix $A = \begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}$ we used earlier, identify the eigenvectors and their corresponding eigenvalues. Recall:

$$\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}3\\ 0\end{bmatrix} = 3\begin{bmatrix}1\\ 0\end{bmatrix} \rightarrow$$

so $\bar{v}_1 = \begin{bmatrix}1\\ 0\end{bmatrix}$ is an eigenvector of A, with corresponding eigenvalue of $\lambda_1 = 3$

⇕

A will scale $\begin{bmatrix}1\\ 0\end{bmatrix}$ by 3 (or multiples)

$A \cdot \bar{v}$

$$\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix} \neq \lambda\begin{bmatrix}0\\ 1\end{bmatrix}$$

so $\begin{bmatrix}0\\ 1\end{bmatrix}$ is <u>not</u> an eigenvector of A

$$\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix} = \begin{bmatrix}-2\\ 2\end{bmatrix} = 2\begin{bmatrix}-1\\ 1\end{bmatrix}$$

so $\bar{v}_2 = \begin{bmatrix}-1\\ 1\end{bmatrix}$ is an eigenvector of A, w/ corresponding eigenvalue $\lambda_2 = 2$.

Computing the Eigenvalues of a Matrix

Starting with the eigenvalue/eigenvector equation, determine the condition for a value λ to be an eigenvalue of a matrix A.

given

scalar

matrix

$$\boxed{A \underbrace{\bar{v}}_{\text{vect}} = \lambda \underbrace{\bar{v}}_{\text{vector}}}$$

what is special/defines λ?

$$I\,(A\bar{v}) = I(\lambda\bar{v})$$

$$\det(A - \lambda I) = 0$$

$$IA = A$$

$$\underbrace{A}_{n\times n}\bar{v} = \underbrace{\lambda I}_{n\times n}\bar{v}$$

$$A\bar{v} - \lambda I\bar{v} = \bar{0}$$

so $$(A - \lambda I)\bar{v} = \bar{0}$$

Assume $\bar{v}$ be a non-zero vector. $\Rightarrow$ $\bar{v}$ is in the Kernel of $(A - \lambda I)$

$\Rightarrow$ b/c $\bar{v}$ is non-zero

$\Rightarrow$ $(A - \lambda I)$ is not invertible

Eigenvalue condition: $\boxed{\det(A - \lambda I) = 0}$

Example: Find the eigenvalues of the matrix $A = \begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}$

Find λ by building A−λI, find determinant, set = 0.

$$(A-\lambda I) = \begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 3 \\ -2 & -2-\lambda \end{bmatrix}$$

Now find $\det(A-\lambda I)$

$$= (5-\lambda)(-2-\lambda) - (-6) = -10 - 3\lambda + \lambda^2 + 6$$

tidy $= \lambda^2 - 3\lambda - 4$ set $= 0$ solve quadratic

$$(\lambda - 4)(\lambda + 1) = 0$$

so $\lambda_1 = 4$, $\lambda_2 = -1$ are the eigenvalues for $A = \begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}$

A scales
- a dir'n by 4,
- another dir by −1

Example: Find the eigenvalues of the matrix $B = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix}$

Eigenvalue eq'n is

$$\det(B-\lambda I) \underline{\underline{=0}}$$

$$\det(B-\lambda I): \quad \underbrace{\begin{matrix} 3-\lambda & 1 & -1 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 4-\lambda \end{matrix}}_{B-\lambda I} \left| \begin{matrix} 3-\lambda & 1 \\ 0 & 2-\lambda \\ 0 & 0 \end{matrix} \right.$$

$$= \left[(3-\lambda)(2-\lambda)(4-\lambda) + 0 + 0\right] - \left[0 + 0 + 0\right]$$

$$= (3-\lambda)(2-\lambda)(4-\lambda) \overset{\text{set}}{=} 0$$

$\lambda = 3, 2, 4$ are the three eigenvalues for B.

B scales one dir'n by 3, a second dir'n by 2, and 3rd dir'n by 4

Computing the Corresponding Eigenvectors of a Matrix

Starting with the eigenvalue/eigenvector equation, **and a specific λ eigenvalue**, determine the condition for a vector $\bar{v}$ to be an eigenvector corresponding to λ for a matrix A.

matrix

scalar

$$A\bar{v} = \lambda\bar{v}$$

or

$$\underbrace{A}_{\text{matrix}}\bar{v} = \underbrace{\lambda I}_{\text{matrix}}\bar{v}$$

or

$$A\bar{v} - \lambda I\bar{v} = \bar{0}$$

or

$$(\underbrace{A - \lambda I}_{\text{know}})\underbrace{\bar{v}}_{\text{want}} = \bar{0}$$

$\Rightarrow$ need to solve a system of linear equations.

(for a non-zero $\bar{v}$)

Eigenvector condition: $(A - \lambda I)\bar{v} = \bar{0}$

Example: Find the eigenvectors of the matrix $A = \begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}$, given the eigenvalues are $\boxed{\underline{\lambda = 4}}, \underline{-1}$.

Take each eigenvalue one at a time

$\bar{v}_1$ = unknown, $= \begin{bmatrix} a \\ b \end{bmatrix}$

$\underline{\lambda_1 = 4}$

$(A - \lambda I)\bar{v}_1 = \bar{0}$

$$\begin{bmatrix} 5-4 & 3 \\ -2 & -2-4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

×3

or $\begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow a + 3b = 0$

or $a = -3b$

pick $\boxed{b = 1} \rightarrow \boxed{a = -3} \rightarrow \boxed{\bar{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}}$

check: $\underline{\text{not}}$ invertible ✓

Note: there are an infinite # of solutions

$$A = \begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}$$

$$\lambda = 4, -1$$

$$\boxed{\lambda_2 = -1} \qquad (A - \lambda I)\bar{v}_2 = \bar{0}$$

$$\begin{bmatrix} 5-(-1) & 3 \\ -2 & -2-(-1) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

mult

or $$\begin{bmatrix} 6 & 3 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{aligned} -2a - b &= 0 \\ -2a &= b \end{aligned}$$

check: not invertible ✓

pick $\boxed{a = 1} \rightarrow \boxed{b = -2} \rightarrow \boxed{\bar{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$

Summarize the effect of A as a linear transform.

- along $\bar{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, A scales vectors by 4
- " $\bar{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, A scales vectors by (-1) / flips.

Example: Find the eigenvectors of the matrix $B = \begin{bmatrix}3 & 1 & -1\\ 0 & 2 & 5\\ 0 & 0 & 4\end{bmatrix}$, given the eigenvalues are $\lambda = 2, 3, 4$.

eigenvector condition

$(B - \lambda I)\bar{v} = \bar{0}$, $\bar{v}$ is non-zero, let $\bar{v} = \begin{bmatrix}a\\ b\\ c\end{bmatrix}$

$\boxed{\lambda_1 = 2}$

$$(B - \lambda I)\bar{v} = \vec{0} \Rightarrow \begin{bmatrix}3-2 & 1 & -1\\ 0 & 2-2 & 5\\ 0 & 0 & 4-2\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

tidy

$$\begin{bmatrix}1 & 1 & -1\\ 0 & 0 & 5\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

check: not invertible

$a + b - c = 0$

$2c = 0$ $\boxed{c = 0}$

$a + b = 0$

$a = -b$

pick $\boxed{b = 1} \rightarrow \boxed{a = -1}$

$$\boxed{v_1 = \begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}}$$

$$B = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

with $\lambda = 2, 3, 4$

$\boxed{\lambda_2 = 3}$

$(B - \lambda I)\bar{v} = \bar{0}$

$$\begin{bmatrix} 3-3 & 1 & -1 \\ 0 & 2-3 & 5 \\ 0 & 0 & 4-3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

check: not invertible ✓

$\boxed{c = 0}$

$-b + \cancel{5c}^{0} = 0 \rightarrow \boxed{b = 0} \rightarrow \boxed{\bar{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

$0a = 0$

a can be anything!

Pick $\boxed{a = 1}$

$$B = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

with $\lambda = 2, 3, 4$

$\underline{\lambda_3 = 4}$ $\quad (B - \lambda I)\bar{v} = \bar{0}$

$$\begin{bmatrix} 3-4 & 1 & -1 \\ 0 & 2-4 & 5 \\ 0 & 0 & 4-4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

check: $\underline{\text{not}}$ invertible. ✓

$-a + b - c = 0$ ①

$-2b + 5c = 0$

$2b = 5c$

$b = 5/2\, c$

Pick $\boxed{c = 2} \rightarrow \boxed{b = 5}$

① $-a = -b + c$

$$\boxed{a = b - c = 5 - 2 = 3}$$

$\rightarrow \bar{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$

Summarize the effect of B as a linear transform.

- scales $\bar{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ by $\lambda_1 = 2$
- scales $\bar{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ by $\lambda_2 = 3$
- scales $\bar{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ by $\lambda_3 = 4$

Summary: Computing Eigenvalues and Eigenvectors

Eigen equation: $A\bar{v} = \lambda\bar{v}$

for special/characteristic directions $\bar{v}$,
A simply scales vectors parallel to $\bar{v}$
by λ.

1) Find the eigenvalues for A using the eigenvalue condition:

$$\det(A - \lambda I) = 0$$

$$\Updownarrow$$

$A - \lambda I$ is not invertible

2) Find each corresponding eigenvector for A using the eigenvector condition, once for each eigenvalue λ:

$$\underbrace{(A - \lambda I)}_{\text{known matrix}}\underset{\nwarrow}{\bar{v}} = \bar{0}$$

solve for $\bar{v}$.

one representative.

Note: eigenvectors are a basis for a subspace, so are not unique. If you find $\bar{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for a particular λ,

then $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$, $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$, etc. will also be eigenvectors.

Prove that any multiple of an eigenvector for a given λ will also be an eigenvector.

If $A\bar{v} = \lambda\bar{v} \Rightarrow \bar{v}$ is an eigenvector of A

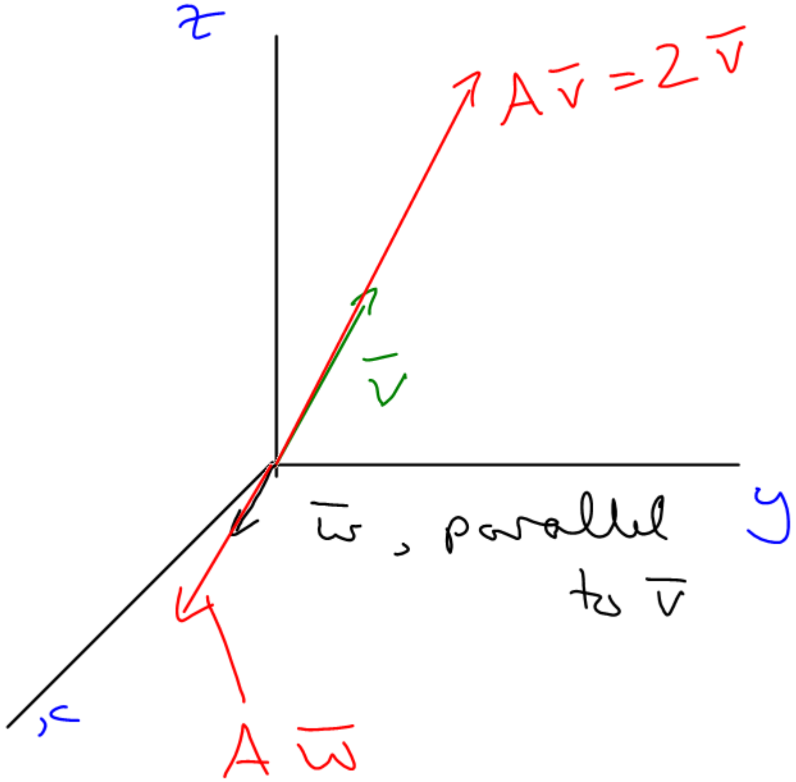
then $k(A\bar{v}) = k(\lambda\bar{v})$ for some $k \in \mathbb{R}$

$$A(k\bar{v}) = \lambda(k\bar{v})$$

$$\Downarrow$$

$k\bar{v}$ is also an eigenvector.

Interpret this result in the context of the scaling interpretation of eigenvalues and eigenvectors.



e.g. $\bar{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an eigenvector for A, w/ $\lambda = 2$

Guide/Commentary on the 3Blue1Brown video "Eigenvectors and Eigenvalues"

YouTube link - https://youtu.be/PFDu9oVAE-g

Timestamp 1:30

Recall that "i hat" or $\hat{i}$ is another name for the standard first basis vector, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Similarly "j hat" or $\hat{j}$ is another name for the standard second basis vector, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Timestamp 2:00

Note that the matrix used, $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, is the same 2×2 matrix we used as an example page 13, 14 and 17.

- The eigenvalue $\lambda_1 = 3$ corresponded to the eigenvector $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- The eigenvalue $\lambda_2 = 2$ corresponded to the eigenvector $\bar{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Timestamp 4:10

In the 3D rotation transformation, we see a variation on the alternatives for describing a matrix.

- We can report all 9 elements of the 3×3 matrix, through the complicated formula shown with sines and cosines, **or**
- We can know the axis of rotation, and the angle that we rotate by.

Both fully describe the same linear transformation, but the second version is much easier to understand.

Note: for the rotation, there is only one **real** eigenvector that is scaled by 1. As you might suspect with rotations-as-multipliers (instead of stretches), the rotation is most easily represented using **complex** numbers. You'll see that in your 2nd year Differential Equation class.

Timestamp 7:30

The idea of imagining the $(A-\lambda I)$ transform as λ changes is cool visually, but honestly I don't think about the idea of a sliding scale for λ when I'm actually doing problems.

Timestamp 9:10

How to move through all the alternative forms of the $A\bar{v} = \lambda\bar{v}$ is worth asking about in office hours if they don't make sense to you!
Those alternate forms are exactly what we used to construct our eigenvalue condition and eigenvector condition.

Timestamp 10:50

Note that in all of our examples this week, we *did* have 2 eigenvectors for our 2×2 matrices, and 3 eigenvectors for our 3×3 matrices. As indicated in the video though, for some matrix you may have fewer or even no eigenvectors.

We will explore that possibility and what it means next week. (Though just to talk about the rotation matrix again briefly, if $A\bar{v} = \lambda\bar{v}$ allows *complex-valued* scalings, then we *can* represent rotations as (complex-valued) eigenvalues and eigenvectors. A brief note on that appears as a 1-second flash at 11:34...

Timestamp 12:30

The case where you have *more than one dimension* associated with an eigenvalue is also something new (not covered in this week's notes). Again, next week we will look into how and when this can occur.

Timestamp 13:15

This last part of the video is new material which previews the ideas we will cover next week, including computing powers of matrices and diagonalization.