Week #12: Eigenvalues and Eigenvectors - Continued

Section 15 - Eigenvalues and Eigenvectors - Continued

Recall: Eigenvalues and eigenvectors let us summarize the effect of a linear transformation, in an alternative to the matrix form of the transform.

Example: Consider the matrix $A = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

This matrix has the following eigenvalues and corresponding eigenvectors:

- $\lambda_1 = 1$ with $\bar{v}_1 = [1, 1]$, and
- $\lambda_2 = 0.8$ with $\bar{v}_2 = [-1, 1]$.

for any $\bar{v} \in \mathbb{R}^2$

$\bar{v} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

basis for $\mathbb{R}^2$

$A\bar{v} = A\left(a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

Summarize the effect of this matrix when transforming an input vector from $\mathbb{R}^2$.

- Any vector parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (eg. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} -10 \\ -10 \end{bmatrix}$) will be scaled by $\lambda_1 = 1$ (or will be unchanged)
- Any vector parallel to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ will be scaled by $\lambda_2 = 0.8$.

Example: Find the eigenvalues for the matrix $B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

Eigen equation: $B\bar{v} = \lambda \bar{v}$

Eigenvalue condition: $\det(B - \lambda I) = 0$

$$|B - \lambda I| = \begin{vmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{vmatrix} = (7-\lambda)(-1-\lambda) - 9 \overset{\text{Set}}{=} 0$$

expand: $-7 - 6\lambda + \lambda^2 - 9 = 0$

tidy: $\lambda^2 - 6\lambda - 16 = 0$

$(\lambda - 8)(\lambda + 2) = 0.$

so $\lambda_1 = 8$, $\lambda_2 = -2$ are the eigenvalues for B.

Problem: How many eigenvalues can there be for a 2×2 matrix?

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \underbrace{(a-\lambda)(d-\lambda) - bc}_{\text{polynomial in } \lambda\text{, quadratic or degree 2.}}$$

$\Rightarrow$ 2 roots to that polynomial

$\Rightarrow$ 2 eigenvalues (real or complex; possibly repeated)

Problem: How many eigenvalues can there be for an $n \times n$ matrix?

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & \sim & & - \\ \sim & a_{22}-\lambda & & \\ \sim & \sim & a_{33}-\lambda & \\ & & & \ddots \\ & & & a_{nn}-\lambda \end{vmatrix}$$

$\rightarrow$ polynomial in λ, degree n.

Fund'l Thm of Algebra:

n^{th} degree poly'l has n roots*, complex

$\Rightarrow$ $\boxed{n \text{ eigenvalues.}}$

Definition: the *characteristic polynomial* for a matrix is the polynomial in λ defined by

$$\det(A - \lambda I) = 0$$

solutions to

eigenvalue condition

The roots of the characteristic polynomial are equal to the eigenvalues of the matrix.

Example: what was the characteristic polynomial for $B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$?

$\det(B - \lambda I) = \dots = \lambda^2 - 6\lambda - 16$

is B's characteristic polynomial.

Example: Find the eigen*vectors* for the matrix $B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

Eigenvector cond'n

$(B - \lambda I)\bar{v} = \bar{0}$

Let $\bar{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

Recall: $\lambda_1 = 8$, $\lambda_2 = -2$

For $\lambda_1 = 8$

$$(B - \lambda I)\bar{v} = \begin{bmatrix} 7-8 & 3 \\ 3 & -1-8 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

check: not invertible. ✓

$-a + 3b = 0$

or $a = 3b$

Pick: $\boxed{b=1} \to \boxed{a=3} \to \bar{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

so $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigen vector for $\lambda_1 = 8$.

$$B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}.$$

For $\lambda_2 = -2$

$$(B - \lambda I)\bar{v} = \begin{bmatrix} 7-(-2) & 3 \\ 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

check: not invertible. ✓

$$3a + b = 0$$

$$b = -3a$$

pick $\boxed{a = 1}$ or $\to \boxed{b = -3}$ so $\bar{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

is an eigenvector for $\lambda_2 = -2$.

Note: pick $\boxed{a = -7} \to \boxed{b = 21}$ so $\bar{v}_2 = \begin{bmatrix} -7 \\ 21 \end{bmatrix}$

is also an eigenvector for $\lambda_2 = -2$

Some students can be reasonably confused by the seeming arbitrariness of the eigenvector construction: what is the rationale for selecting just one vector as the eigenvector when there are an infinite number available?

While eigenvectors are useful as building blocks for describing spaces, it can be more elegant to define the *eigenspace* for each eigenvalue instead.

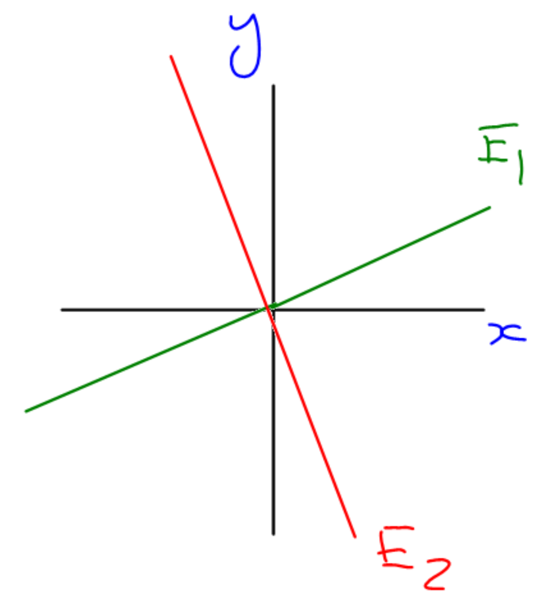
Definition: For an $n \times n$ matrix A, with an eigenvalue of λ, the vector subspace of $\mathbb{R}^n$ given by $\ker(A - \lambda I)$ is called the *eigenspace of A associated with the eigenvalue λ of A.*

all $\bar{v}$'s w/ $(A - \lambda I)\bar{v} = \bar{0}$

Example: Describe the eigenspaces for the matrix $B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

for $\lambda_1 = 8$, eigen space is $E_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = 3y \right\}$.

for $\lambda_2 = -2$ " " $E_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = -3x \right\}$



Problem: Contrast the eigenspaces and eigenvectors for the matrix $B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

eg. for $\lambda_1 = 8$

eigenspace = all the vectors w/ $x = 3y$.

eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a basis for the eigenspace

eigenspace is the span$\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$

and similarly for $\lambda_2 = -2$

Transforming Matrices and their Eigenvectors and Eigenvalues

Problem: For a matrix A with eigenvalue λ and corresponding eigenvector $\bar{v}$:
Find an eigenvalue and corresponding eigenvector for the related scaled matrix kA.

Know $\quad A\bar{v} = \lambda \bar{v} \quad$ Eigen equation

so Study

$$(kA)\bar{v} \overset{\text{regroup}}{=} k(A\bar{v})$$

E-eq

$$(kA)\bar{v} = k(\lambda \bar{v})$$

$$\underbrace{(kA)}_{\text{new matrix}}\bar{v} = \underbrace{(k\lambda)}_{\text{new scaling}}\bar{v}$$

so $\bar{v}$ is also an eigenvector for (kA)
w/corresponding eigenvalue $(k\lambda)$

[Think scaling
- A scales $\bar{v}$ by λ
- kA " $\bar{v}$ by $(k\lambda)$]

Problem: For a matrix A with eigenvalue λ and corresponding eigenvector $\bar{v}$:
Find an eigenvalue and corresponding eigenvector for the inverse matrix A^{-1}.

Know $\quad A\bar{v} = \lambda\bar{v}$

mult on left by A^{-1}

$$\underbrace{A^{-1}(A}_{I}\bar{v}) = A^{-1}(\cancel{\lambda}\bar{v})$$

$$\frac{1}{\lambda}\bar{v} = \frac{\cancel{\lambda}}{\lambda}A^{-1}\bar{v}$$

$$\underbrace{A^{-1}}_{\text{new matrix}}\bar{v} = \underbrace{\frac{1}{\lambda}}_{\text{new scaling.}}\bar{v}$$

so $\bar{v}$ is also an eignvector of A^{-1}, w/ corresponding eigevalue $\frac{1}{\lambda}$

Think scaling

- A scales $\bar{v}$ by λ
- A^{-1} scales $\bar{v}$ by $1/\lambda$

Problem: For a matrix A with eigenvalue λ and corresponding eigenvector $\bar{v}$:
Find an eigenvalue and corresponding eigenvector for the related matrix $A + cI$.

Know: $A\bar{v} = \lambda \bar{v}$

Study $(A+cI)\bar{v} \underset{\text{expand}}{=} A\bar{v} + cI\bar{v}$

$= \lambda\bar{v} + c\bar{v}$

$\underbrace{(A+cI)}_{\text{new matrix}}\bar{v} \overset{\text{group}}{=} \underbrace{(\lambda+c)}_{\text{new scaling}}\bar{v}$

so $\bar{v}$ is also an eigenvector of $(A+cI)$

w/ corresponding eigenvalue of $(\lambda+c)$

Problem: For a matrix A with eigenvalue λ and corresponding eigenvector $\bar{v}$:
Find an eigenvalue and corresponding eigenvector for the power A^{10}. matrix multiplication

Know: $A\bar{v} = \lambda \bar{v}$

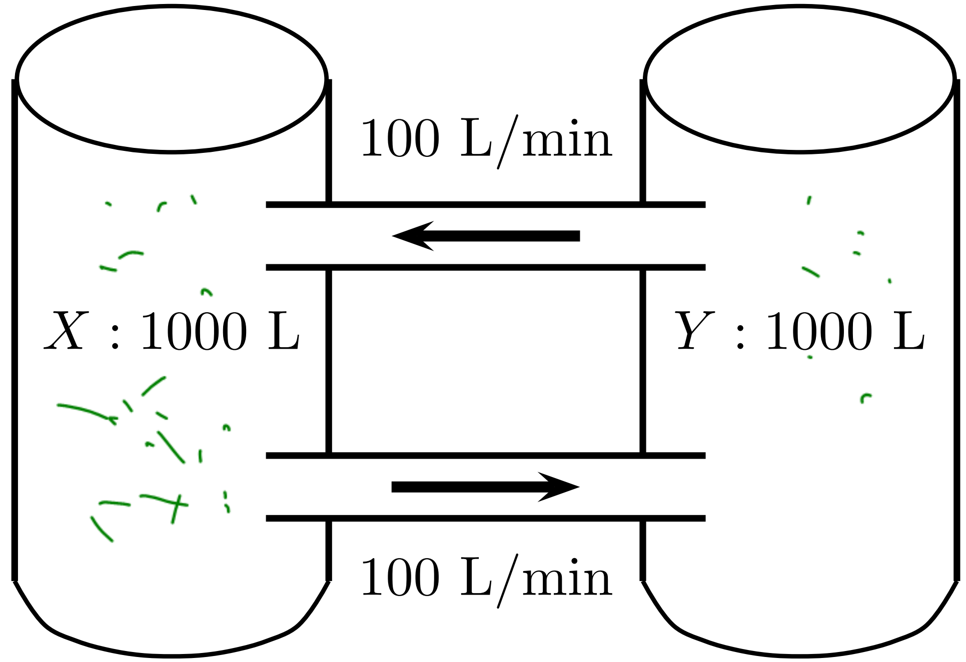
Study
$$A^{10}\bar{v} = \underbrace{A \cdot A \cdot \ldots A}_{10 \text{ times}}[A\bar{v}]$$
$$= \underbrace{A \cdot A \cdot \ldots \cdot A}_{9 \text{ times}}[A(\lambda \bar{v})]$$

$$A^{10}\bar{v} = \lambda^{10}\bar{v}$$

so $\bar{v}$ is also an eigenvector of A^{10} (or A^n)
w/ corresponding eigenvalue of λ^{10} (w λ^n)

Eigenvector Application: The Tank Problem

Problem: we have two tanks, X and Y, which each hold 1,000 L of water.



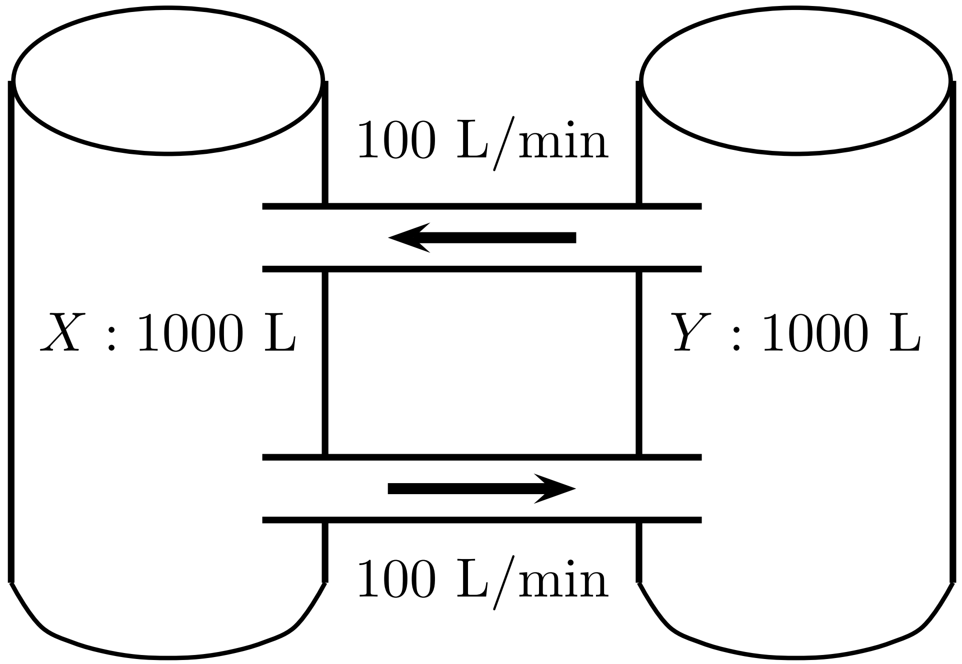
- Each minute 100 L of water is pumped from X to Y, and also from Y to X.
- At time $t = 0$, we put 80 kg of chemical Z in tank X, and
- Also at time $t = 0$, we put 20 kg of chemical Z in tank Y.

Predict

Question: How much of chemical Z is in each tank after n minutes? as mixing occurs.

$n = 0, 1, 2, 3 \ldots$

Note: we are working at discrete times only, so we will use subscripts like x_n instead of function notation, $\cancel{x(t)}$.



We define:

- x_n = mass of chemical Z in tank X at time n minutes. (Kg)

- y_n = mass of chemical Z in tank Y at time n minutes. (Kg)

Thus we can define the **state** of the system as a vector: $[x_n, y_n]$.

We know the starting state, $[x_0, y_0] = [80, 20]$.

kg kg

Problem: given the state $[x_n, y_n]$, find a way to compute the next state, $[x_{n+1}, y_{n+1}]$.

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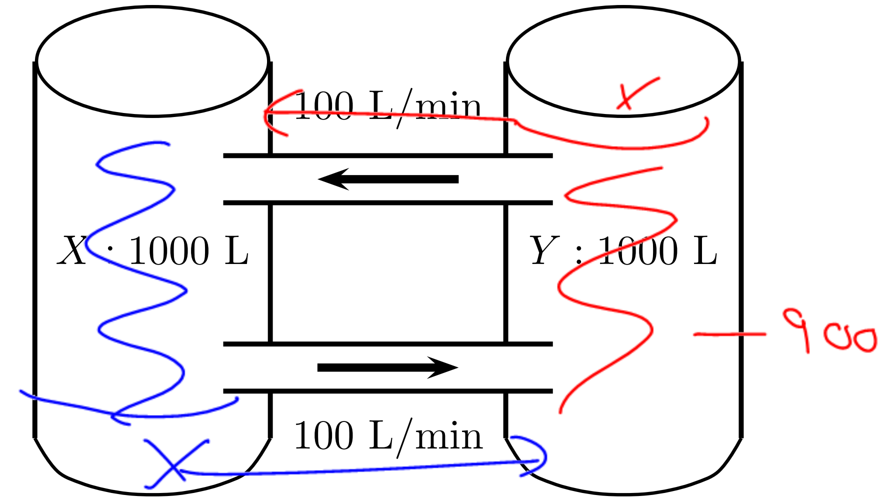


conc in tank X $= \frac{x_n}{1000}\left(\frac{kg}{L}\right)$

conc in tank Y $= \frac{y_n}{1000}\left(\frac{kg}{L}\right)$

after 1 min,

amt of chem in Tank X

mass

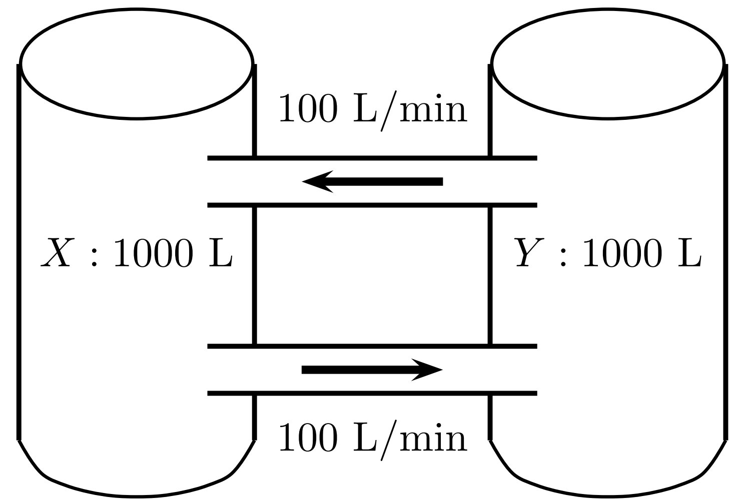
$$x_{n+1} = \underset{\text{tank } x_n \text{ water}}{(900\,\cancel{L})}\left(\frac{x_n}{1000}\frac{kg}{\cancel{L}}\right) + \underset{\text{carried from } Y}{(100\,\cancel{L})}\left(\frac{y_n}{1000\cancel{L}}kg\right) = 0.9x_n + 0.1y_n$$

and $$y_{n+1} = \underset{\text{carried from } x}{(100L)}\left(\frac{x_n}{1000}kg\right) + (900L)\left(\frac{y_n}{1000L}kg\right) = 0.1x_n + 0.9y_n$$

or $$\boxed{\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}}$$

a linear transformation

Use the linear system found to predict the mass of chemical Z in each tank for $n = 1, 2$ and 3. Recall that $[x_0, y_0] = [80, 20]$. start



$$\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \underset{y_0}{\overset{x_0}{\begin{bmatrix} 80 \\ 20 \end{bmatrix}}} = \begin{bmatrix} 74 \\ 26 \end{bmatrix} \text{kg}$$

and $$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \underset{y_1}{\overset{x_1}{\begin{bmatrix} 74 \\ 26 \end{bmatrix}}} = \begin{bmatrix} 69.2 \\ 30.8 \end{bmatrix} \text{kg}$$

and $$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 69.2 \\ 30.8 \end{bmatrix} = \begin{bmatrix} 65.36 \\ 34.64 \end{bmatrix} \text{kg.}$$

n minutes later

Find an expression for the state $[x_n, y_n]$, given the masses at time zero, $[x_0, y_0]$.

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \overbrace{A \cdots A \, A \cdot \underbrace{A \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}}^{n \text{ times}}$$

$$\underbrace{A \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Note any complications with computing this value for large n.

finding powers of A is tedious,
requires complicated/long matrix mult at each step

initial amts

predicted chemical amts

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (\underbrace{A\,A\,\ldots\,A}_{n\text{ times}}) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

The key insight to simplifying this is that computing $A^n\,\bar{v}$ is simple **if $\bar{v}$ is an eigenvector of A.**

Example: Compare $A^n\,\bar{v}$ for the case when $\bar{v}$ is and is not an eigenvector.

$\bar{v}$ not an eigen vector

$A^n\,\bar{v}$

$= \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \cdots \begin{bmatrix} & \\ & \end{bmatrix}$

$\vdots$

ick!

If $\bar{v}$ is an eigenvector

then $A\,\bar{v} = \lambda\,\bar{v}$

$\Rightarrow \quad A^n\,\bar{v} = \lambda^n\,\bar{v}$

and we're done!

So much easier if we use the right vectors.

To understand what occurs when we apply A repeatedly, it will help if we know the eigenvalues and eigenvectors of A. Find them for $A = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}$.

Eigenvalue condition: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 9/10 - \lambda & 1/10 \\ \frac{1}{10} & \frac{9}{10} - \lambda \end{vmatrix} = \left(\frac{9}{10} - \lambda\right)\left(\frac{9}{10} - \lambda\right) - \frac{1}{10}\,\frac{1}{10} \overset{\text{Set}}{=} 0$$

expand $\quad \frac{81}{100} - \frac{18}{10}\lambda + \lambda^2 - \frac{1}{100} = 0$

tidy $\quad \left(\lambda^2 - \frac{18}{10}\lambda + \frac{8\cancel{0}}{10\cancel{0}} = 0\right) \times 10$

$$10\lambda^2 - 18\lambda + 8 = 0 \quad \xrightarrow{\div 2} \quad 5\lambda^2 - 9\lambda + 4 = 0$$

Quad formula $\quad \lambda = \frac{9 \pm \sqrt{81 - 4(5)(4)}}{2(5)} = \frac{9}{10} \pm \frac{\sqrt{1}}{10} = \frac{9}{10} \pm \frac{1}{10} = \boxed{1, \frac{8}{10}}$

are the eigenvalues of A.

$\boxed{\text{For } \lambda_1 = 1}$

Eigenvector condition

$$(A - \lambda I)\bar{v} = \bar{0}$$

let $\bar{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $A = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}$

$$\begin{bmatrix} 0.9-1 & 0.1 \\ 0.1 & 0.9-1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

check: not invertible. ✓

$$0.1a - 0.1b = 0$$

$$a = b$$

pick: $a = 1$, $b = 1$

$$\boxed{\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$\boxed{\text{For } \lambda_2 = 0.8}$

$$\begin{bmatrix} 0.9-0.8 & 0.1 \\ 0.1 & 0.9-0.8 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

check. ✓

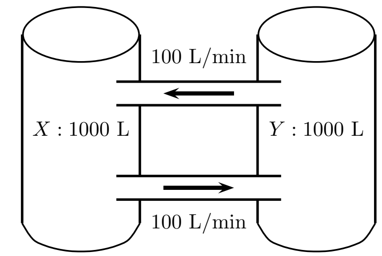
$$0.1a + 0.1b = 0$$

$$a = -b$$

pick $b = 1 \rightarrow a = -1$

so $\boxed{\bar{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

$$A = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}$$



$$\begin{bmatrix}x_n\\y_n\end{bmatrix} = A^n\begin{bmatrix}x_0\\y_0\end{bmatrix}, \text{ with } \lambda_1 = 1, \bar{v}_1 = \begin{bmatrix}1\\1\end{bmatrix} \text{ and } \lambda_2 = 0.8, \bar{v}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$$

Returning to the tank context, use the eigenvectors and eigenvalues to re-write the predicted chemical levels, starting with $[x_0, y_0] = [80, 20]$.

future
↓ time

initial state

$$\begin{bmatrix}x_n\\y_n\end{bmatrix} = \underbrace{A^n}_{\text{hard!}}\begin{bmatrix}80\\20\end{bmatrix}$$

but $\begin{bmatrix}80\\20\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}-1\\1\end{bmatrix}$ basis of eigenvectors

solve: find $\alpha = 50$, $\beta = -30$

or

$$= A^n\left(50\begin{bmatrix}1\\1\end{bmatrix} + (-30)\begin{bmatrix}-1\\1\end{bmatrix}\right) = 50\left(\underbrace{A^n\begin{bmatrix}1\\1\end{bmatrix}}_{1^n\begin{bmatrix}1\\1\end{bmatrix}}\right) + (-30)\left(\underbrace{A^n\begin{bmatrix}-1\\1\end{bmatrix}}_{=\,(0.8)^n\begin{bmatrix}-1\\1\end{bmatrix}}\right)$$

so

$$\begin{bmatrix}x_n\\y_n\end{bmatrix} = 50\begin{bmatrix}1\\1\end{bmatrix} - 30(0.8)^n\begin{bmatrix}-1\\1\end{bmatrix}$$

Simple! b/c of our use of eigenv's

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ with } \lambda_1 = 1, \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \lambda_2 = 0.8, \bar{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Predict after 10 minutes? Easy!

$$\begin{bmatrix} x_{10} \\ y_{10} \end{bmatrix} = 50 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-30)(0.8)^{10} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 53.2 \\ 46.8 \end{bmatrix} \text{kg}$$

Predict long term, $\lim\limits_{n \to \infty}$?

$$\lim_{n \to \infty} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \lim_{n \to \infty} \left(50 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-30)\underbrace{(0.8)^n}_{\to 0} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = 50 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{kg}$$

$$= \begin{bmatrix} 50 \\ 50 \end{bmatrix} \text{kg}$$

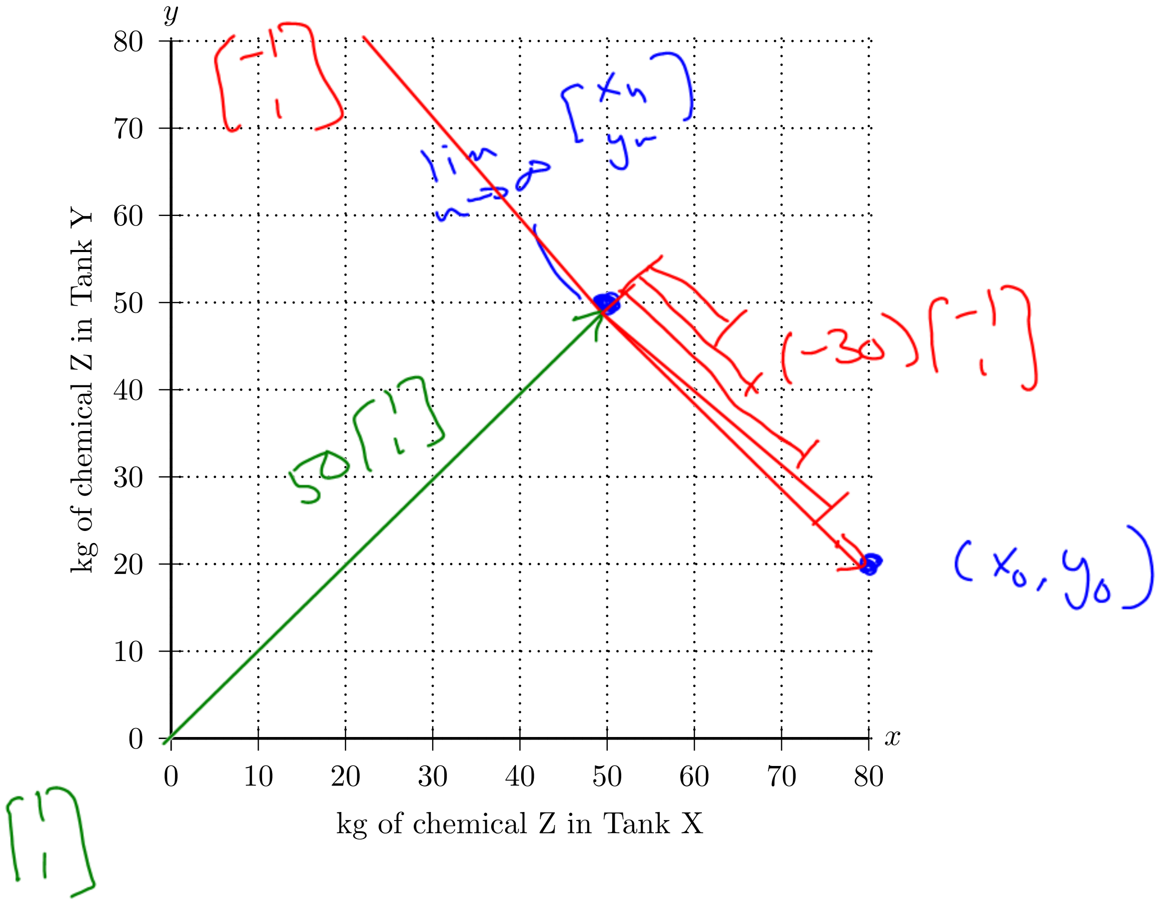
long run, both tanks contain 50 kg of the chemical.

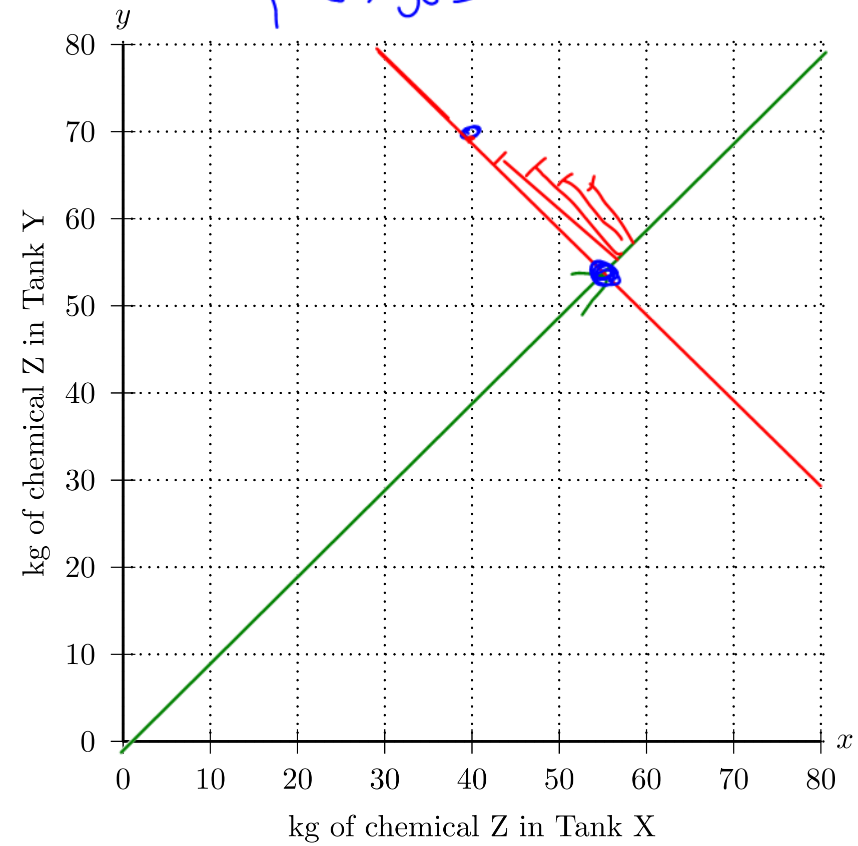
$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ with } A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.8 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

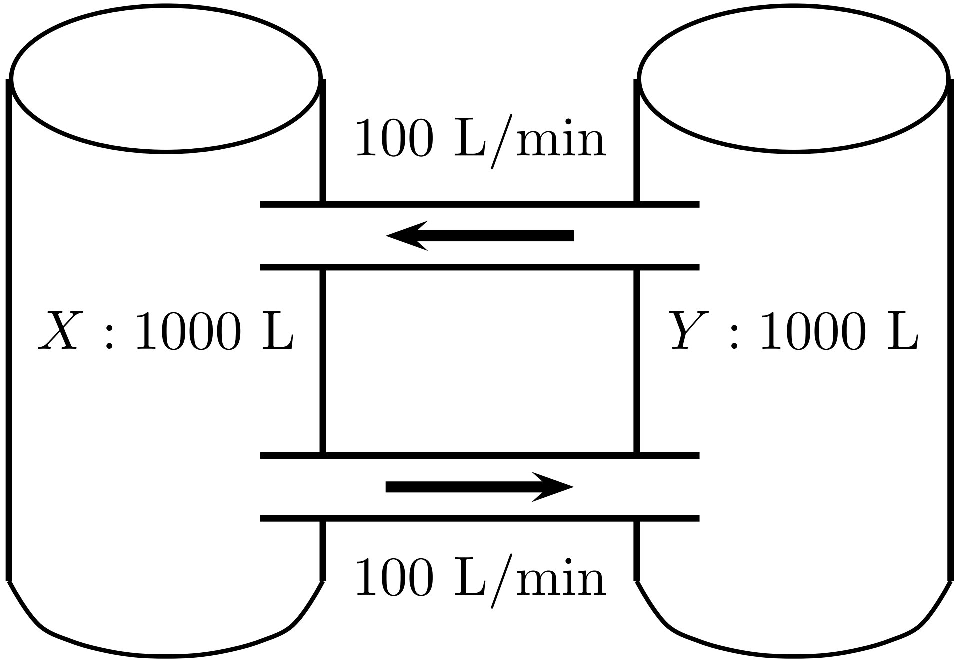
shrinks

Geometric Interpretation: Consider $[x_0, y_0] = [80, 20]$ and other starting quantities, and sketch how they would evolve over time.

$[x_0, y_0] = [40, 70]$







Chemical Interpretation: Explain in chemistry terms why we see the behaviour we observed for $[x_n, y_n]$, and specifically the long-term amounts as $n \to \infty$.

as $n \to \infty$, $A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \to c \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ c \end{bmatrix}$ = equal amounts in both tanks b/c all the water is well mixed

Summary of insights from the tank problem:

- If we are repeatedly applying a linear transform A to a vector $\bar{v}$, the result of n applications will be the matrix power $A^n \bar{v}$.

  usually complicated

- If the vector $\bar{v}$ **is** an eigenvector, $A^n \bar{v}$ simply becomes $\lambda^n \bar{v}$, where λ is the corresponding eigenvalue for $\bar{v}$.

  Easy!

  like $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

- If the vector $\bar{v}$ **is not** an eigenvector, then we can (under certain conditions) write $\bar{v}$ as a linear combination of A's eigenvectors, and then use linear combinations to get a simple formula for $A^n \bar{v}$.

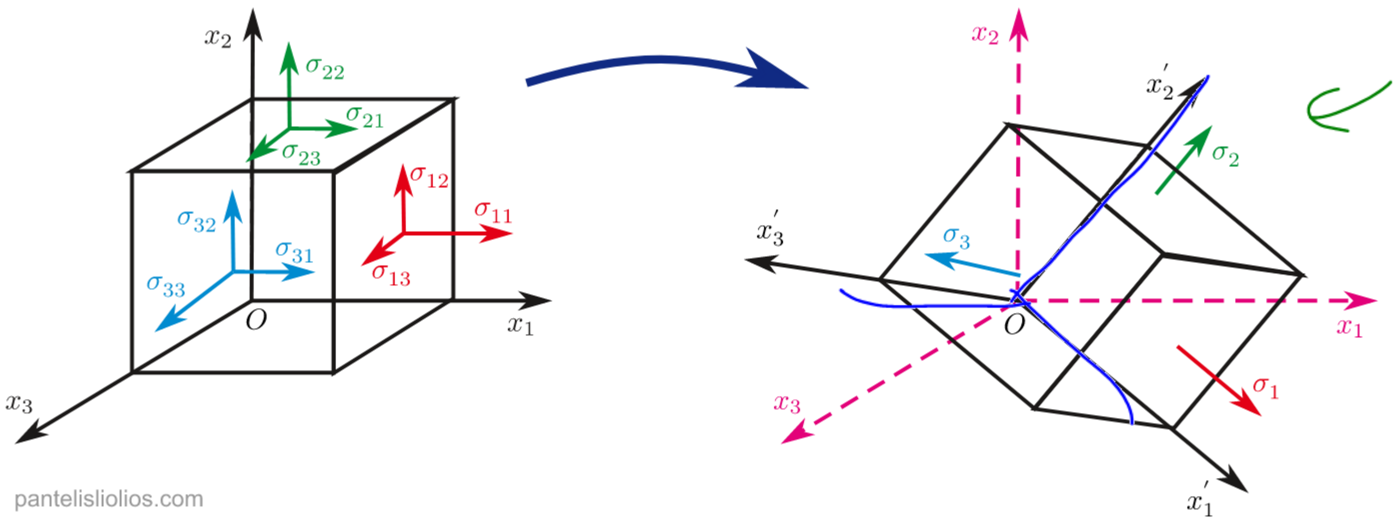
  given $\bar{v} = \alpha_1 \bar{v}_1 + \dots + \alpha_p \bar{v}_p$ ($\bar{v}_1 \dots \bar{v}_p$ are eigenvectors of A)

  then $A^n \bar{v} = \underbrace{\alpha_1 (\lambda_1^n) \bar{v}_1 + \dots + \alpha_p (\lambda_p^n) \bar{v}_p}_{\text{lin comb of vectors; easy!}}$

(This is a major feat, because without eigenvectors, the product $A^n \bar{v}$ is impossible to express in a simple closed form.)

Where might you find an eigenvector after APSC 174?

Geology, Civil, and Mech Materials - Stress Tensor



new axes

eigenvectors

Figure 1: Principal stresses and their direction with respect to the initial coordinate system.

stress

shears

3 simple stresses

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

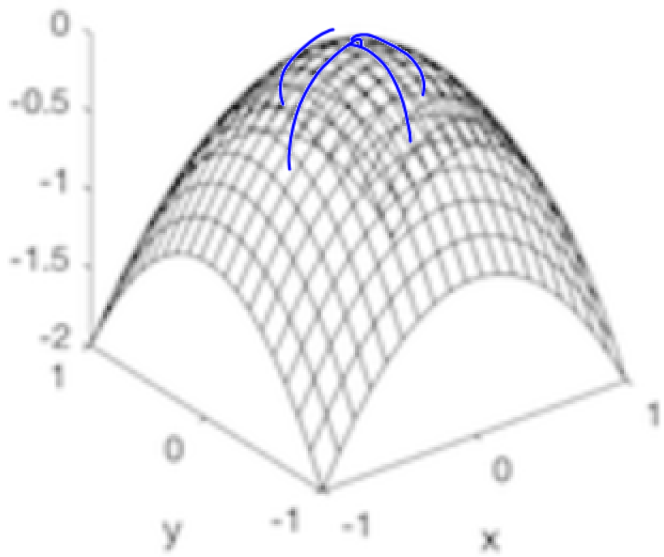
principal stresse

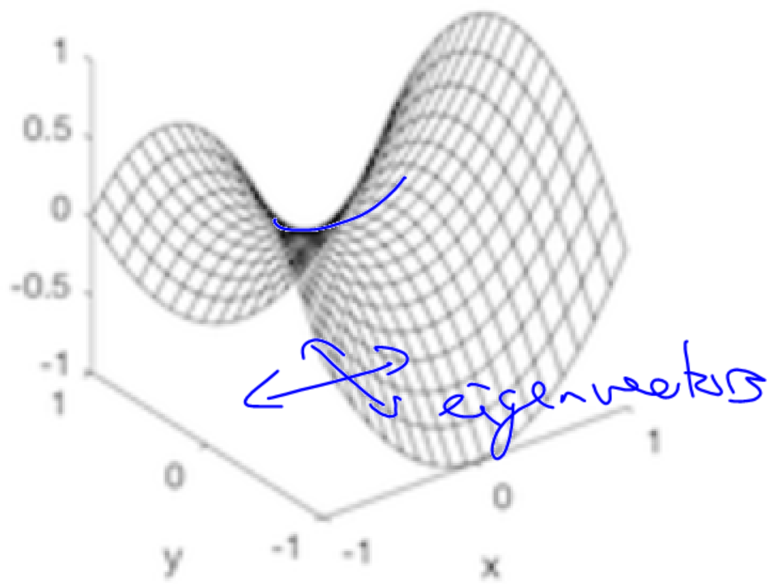
$\sigma_1, \sigma_2, \sigma_3$ are eigenvalues

Sources:

- https://en.wikipedia.org/wiki/Cauchy_stress_tensor
- https://www.pantelisliolios.com/principal-stresses-and-invariants

Vector Calculus - The Hessian Second Derivative





$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

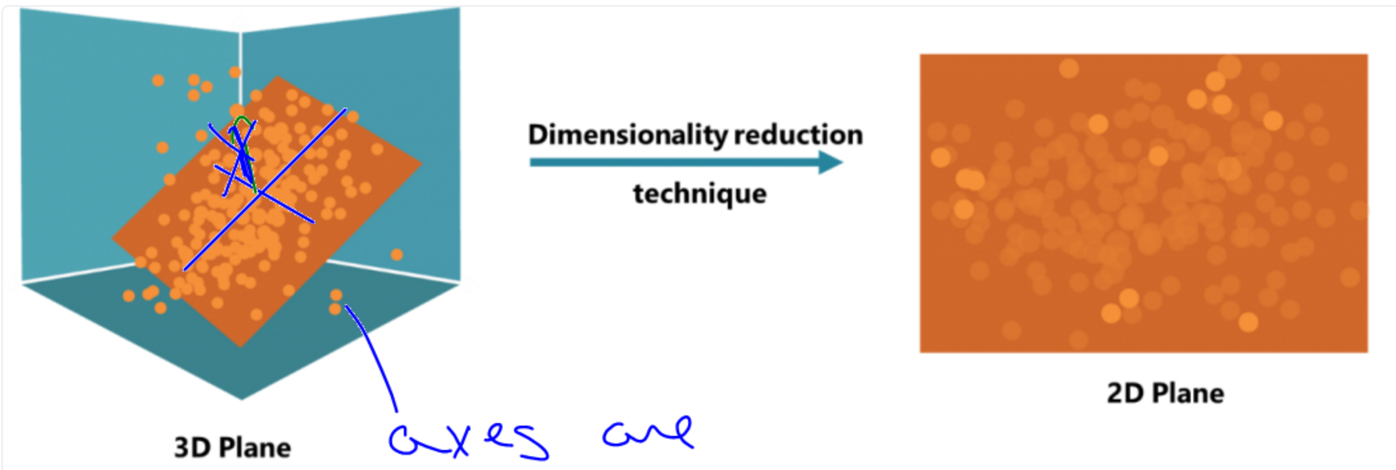
$$\begin{bmatrix} f_1'' & 0 \\ 0 & f_2'' \end{bmatrix}$$

eigenvalues

Sources:

- http://15462.courses.cs.cmu.edu/fall2019/lecture/math2/slide_050

Principal Component Analysis and Data Reduction



axes are eigenvectors of covariance matrix

Covariance

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

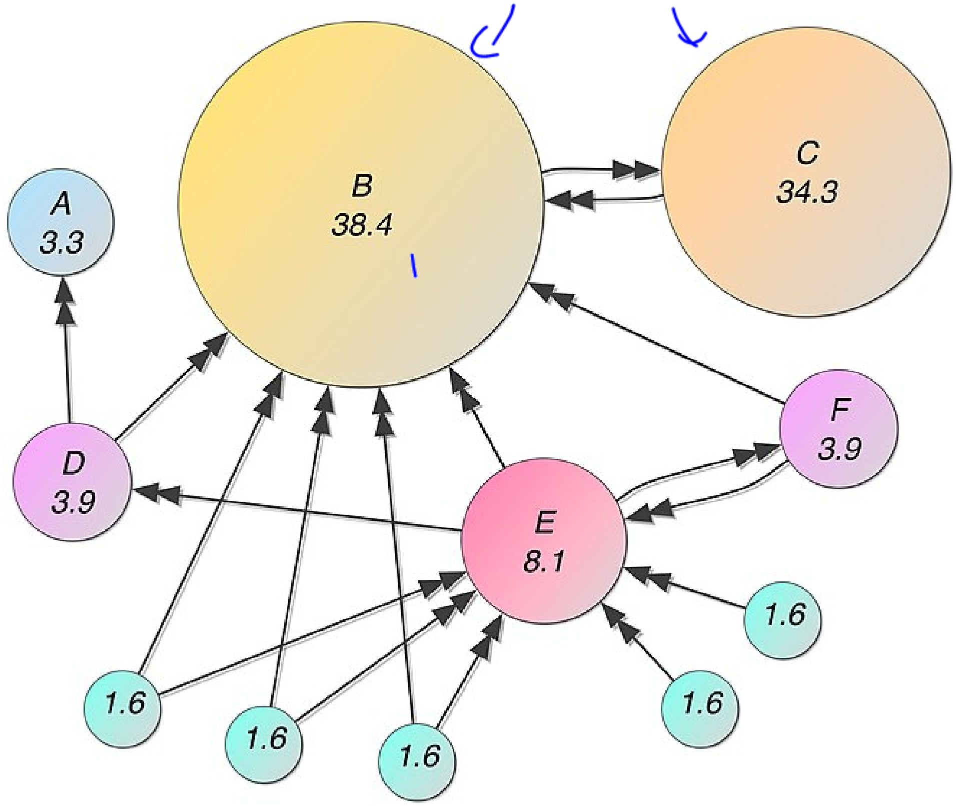
eigenvalues

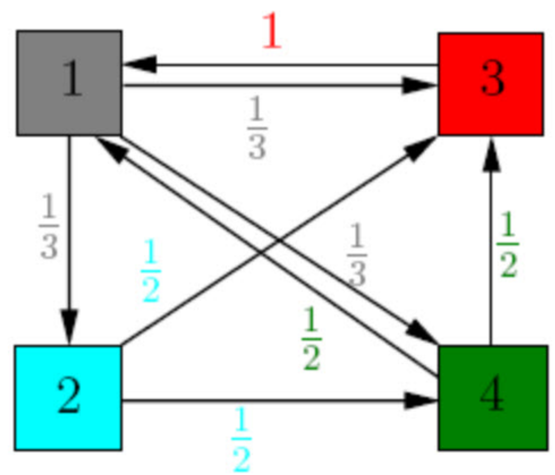
Sources:

- https://365datascience.com/tutorials/python-tutorials/principal-components-analysis/

Google PageRank

web sites





$$A = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Site importance x_i:

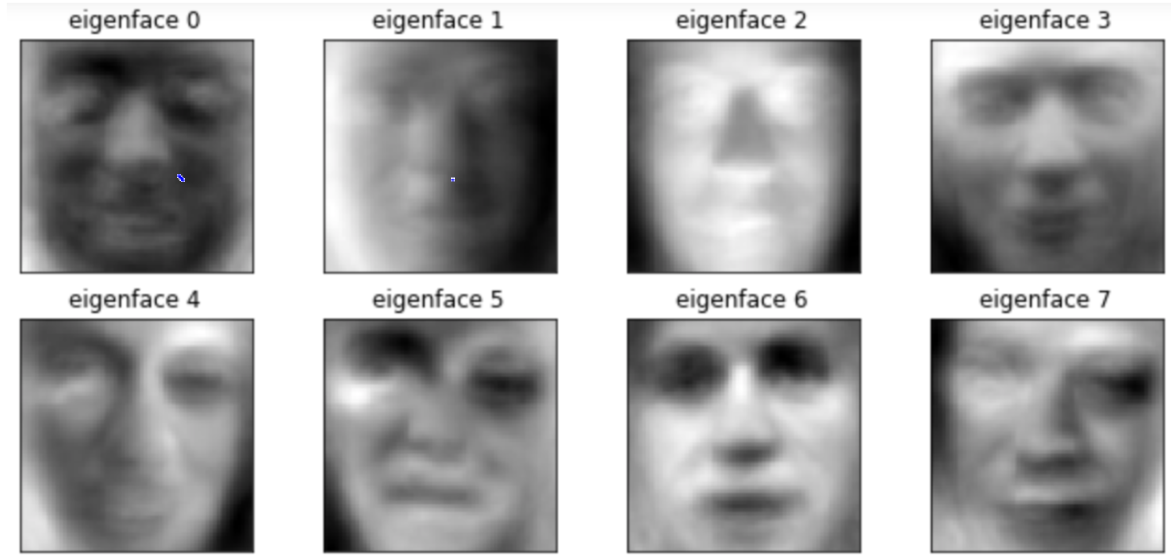
$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

eigen equation

Sources:

- http://pi.math.cornell.edu/~mec/Winter2009/RalucaRemus/Lecture3/lecture3.html
- https://en.wikipedia.org/wiki/PageRank

Image Processing: Eigenfaces



... 1000's

$$\text{Sheryl Crow} \approx 0.225\, e_0 + 0.092\, e_1 + \ldots - 0.143\, e_{1000} \ldots$$

$$\text{Your face's image} \approx 0.418\, e_0 + 0.078\, e_1 + \ldots + 0.972\, e_{1000} \ldots$$

Sources:

- https://365datascience.com/tutorials/python-tutorials/principal-components-analysis/

Physics of Waves and Quantum: Eigenfunctions

complex

scalar

$$\frac{d^2}{dx^2}X(x) = -\frac{\omega^2}{c^2}X(x)$$

eigen equation.

$$\frac{d^2}{dt^2}T(t) = -\omega^2 T(t)$$

Vibrating Strings

$$i\hbar\frac{\partial}{\partial t}\Psi(r,t) = H\Psi(r,t)$$

Schrödinger Equation

Sources:

- https://en.wikipedia.org/wiki/Eigenfunction

And we're done: Good luck on all your final exams!