Week #1, Lecture 2: Fundamentals

- Notation, Sets and More

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Section 0 - Sets, Quantifiers, and Mappings

A **set** is a collection of objects. Examples might include:

Classify each of these sets as finite or infinite.

Common Mathematical Sets

Some of our favourite sets are:

- $\mathbb{R}$:
- $\mathbb{C}$:
- $\mathbb{N}$:
- $\mathbb{Z}$:
- $\mathbb{R}^2$:

- C^∞:

- P_n:

- $\emptyset$:

Thinking of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^2$, C^∞, P_n, and $\emptyset$.

Check-In: how many of these sets were known to you before today?

A. All of them

B. All of number-based ones

C. Just the reals and complex.

D. Never really catalogued these before...

Any other sets you can think of? You can type them into the Zoom chat.

Set Notation

Examples:

- Simple Lists

- Set builder notation:
  { variable : test(s) element has to be pass to be in the set. }

- (Informal) The implied . . ....

Comments: order listing in a set does **not** matter.

Challenges: define the following sets using set notation.

(a) All the integers between -2 and 5 inclusive.

(b) All the real numbers between -2 and 5 inclusive.

(c) The set of all squared natural numbers.

(d) The set of all infinitely differentiable functions which satisfy $\frac{d}{dx}f(x) = 1$.

Cartesian Product Sets

The $\times$ symbol between two sets represents the **Cartesian product** of the sets.
The Cartesian product set (or sometimes just "product set") is a new set whose elements are all the ordered pairs (a, b) with $a \in A$ and $b \in B$.
Example: If $A = \{0, 1, 2\}$ and $B = \{5, 7\}$, define the following Cartesian products:
(a) $A \times B =$

(b) $A \times A =$

(c) $B \times B \times B =$

How many elements will each product set have?

Describe how order is and is not important in the product sets.

Some of our favourite product sets:

- $\mathbb{R} \times \mathbb{R} =$

- $\mathbb{R} \times \mathbb{R} \times \mathbb{R} =$

- $\underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ times}} =$

- $\underbrace{\mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C}}_{n \text{ times}} =$

Example: Create a product set that could represent all possible 8-digit binary numbers.

Membership, Set Equality, and Subsets

Symbol denoting that a single object belongs to/is an element of a set:

Symbol denoting that a single object **is not** an element of a set:

In the following examples, determine which symbol is appropriate between the object and the set:

15.7 $\mathbb{R}$	-3+2i $\mathbb{R}$	(x^2+1) C^∞
15.7 $\mathbb{N}$	-3 +2i $\mathbb{C}$	(x^2+1) P_2
15.7 $\mathbb{Z}$	-3 + 2i $\mathbb{Z}$	(x^2+1) P_1

Equality between sets. Definition: Let A and B be two sets. A is said to be equal to B if they have exactly the same elements. In other words, for two sets, $A = B$ only if the following two conditions are met:

1. Every element in A is also an element of B, and
2. Every element in B is also an element of A.

Example: Determine if the two sets $A = \{5, 2, 3, 4, 6\}$ and $B = \{x \in \mathbb{Z} : x > 1 \text{ and } x \leq 6\}$ are equal.

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Example: Determine if two sets $A = \{1, 4, 9, \ldots\}$ and $B = \{s : s = x^2, x \in \mathbb{N}\}$ are equal.

Subsets. Definition: Let A and B be two sets. A is said to be a **subset** of B if every element of A is also an element of B; we write this as $A \subset B$.

Example: Determine the subset relationships between $A = \{1, 4, 9, \ldots\}$ and $B = \{x^2 : x \in \mathbb{N}\}$.

Example: Determine the subset relationships between the following sets:

$\mathbb{Z} \quad \mathbb{R}$

$\mathbb{Z} \quad \mathbb{N}$

$\mathbb{Z} \quad \mathbb{Z}$

$\mathbb{R} \quad \mathbb{C}$

$\mathbb{R} \quad \mathbb{R}^2$

$\mathbb{C} \quad \mathbb{R}^2$

Comment: We can state that two sets A and B are **equal** *if and only if* both $A \subset B$ and $B \subset A$.

Example: Use subsets to show that $A = \{x \in \mathbb{Z} : x \geq 0\}$ and $B = \mathbb{N}$ are equal.

Week #1, Lecture 3 : Fundamentals

- Set operators
- Functions
- Linear Systems of Equations

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Section 0 - Set Operators, Functions and Mappings

Last lecture we saw sets, and we saw ways to compare them through:

- equality: $A = B$, and
- subsets: $A \subset B$.

We will also need ways to **combine** existing sets.

Set Intersection. Definition: Let S and T be sets. We denote $S \cap T$ as the set of all elements that are in **both** S **and** in T. We call $S \cap T$ the **intersection** of the sets S and T.

Set Union. Definition: Let S and T be sets. We denote $S \cup T$ as the set of all elements that are in **either** S **or** in T. We call $S \cup T$ the **union** of the sets S and T.

Set Difference. Definition: Let S and T be sets. We denote $S \setminus T$ as the set of all elements that **are** in S **but not** in T. We call $S \setminus T$ the **set difference** of the sets S and T.

Example: Let $A = \{0, 1, 2, 3\}$, and $B = \{-1, 1, 3, 5\}$.

Compute:

(a) $A \cup B$

(b) $A \cap B$

(c) $A \setminus B$

Example: Let $A = \{c : c = x^3, x \in \mathbb{Z}\}$, and $B = \mathbb{N}$.
Compute:

(a) $A \cup B$

(b) $A \cap B$

(c) $A \setminus B$

Question: Does the order of the sets listed matter for intersection, union and set difference?

Quantifiers

- $\forall$ - "For all ...", or "For any..."

  Example: Rewrite the formula $A \subset B$ using $\forall$:

- $\exists$ - "There exists ..."

  Example: Rewrite the statement $A \setminus B \neq \emptyset$ using $\exists$:

Mappings and Functions. Definition: Let S and T be two sets. A **mapping** or **function** from S to T is a rule which assigns to each element of S **one and only one** element of T.

Notation:

$$f : S \to T$$
$$x \to f(x)$$

S is called the ________________________________

T is called the ________________________________

Example: Write out some functions we know from calculus in this new style.

(a) $y = \sin(x)$

(b) $y = e^{3x}$

(c) $y = \frac{1}{x}$

Note any issues with these functions.

Name other mappings from $\mathbb{R} \to \mathbb{R}$ that you have seen that would **not** be functions under this definition.

Example: Consider the three mappings

$$\begin{aligned} f :& \mathbb{N} \to \mathbb{N} \\ & x \to x^2 \end{aligned}$$

$$\begin{aligned} g :& \mathbb{Z} \to \mathbb{Z} \\ & x \to x + 1 \end{aligned}$$

$$\begin{aligned} h :& \mathbb{N} \to \mathbb{Z} \\ & x \to x + 1 \end{aligned}$$

Note they differ in how they cover the target set, and whether have unique or non-unique outputs. It is often useful to classify functions by these properties.

Injective. Definition: Let $f : S \to T$. f is said to be **injective** if $\forall x, y \in S$: $x \neq y \Rightarrow f(x) \neq f(y)$.
Paraphrase:

Surjective. Definition: Let $f : S \to T$. f is said to be **surjective** if $\forall t \in T$, $\exists x \in S : f(x) = t$.
Paraphrase:

Bijective. Definition: Let $f : S \to T$. f is said to be **bijective** if $\forall t \in T$, there exists one and only one element $x \in S : t = f(x)$.
Paraphrase:

Diagrams

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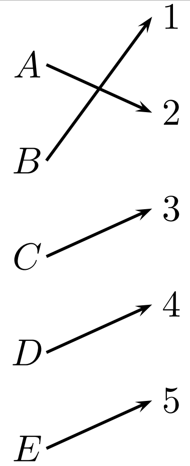
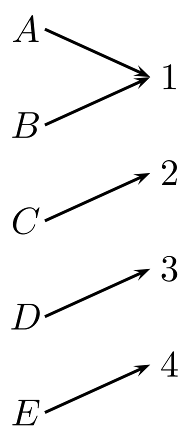
<tr><td>Surjective</td><td>
</td><td>
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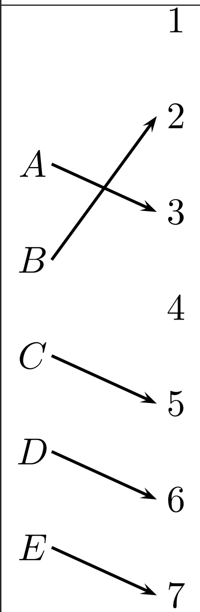
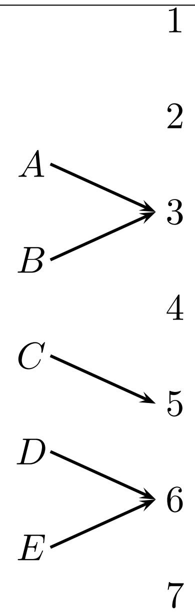
<tr><td>Not surjective</td><td>
</td><td>
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Example: Identify functions from calculus, with the sets $f : \mathbb{R} \to \mathbb{R}$ that would be:

- Injective, but not surjective.

- Surjective, but not injective.

- Bijective.

Could the classification of those functions be changed if we redefined either the domain set or the target set for the functions?

Composition

Suppose that $f : A \to B$, and $g : B \to C$.

The we can combine or **compose** these maps to get a map directly from $A \to C$:

Notation is $g \circ f$ reads as either:

- g follows f, or
- g composed with f.

Section 1 - Systems of Linear Equations

Number of solutions to equations

When we solve equation(s), we get a **set** of solutions.

Examples:

(a) Find the set of real solutions to $x^2 = 9$.

(b) Find the set of (possibly complex) solutions to $y^4 = 1$.

(c) Find the set of **only real** solutions to $u^2 = -1$.

Those equations are simple but still relatively challenging to solve because they are **non-linear** equations.
When we look at systems of **purely linear** equations though, we find some simple patterns in the possible solutions.
Example A: Find the set of solutions to the set of linear equations

$$3x + y = 4$$
$$2x + y = 1$$

Example B: Find the set of solutions to:

$$\begin{aligned} 3x + y &= 1 \\ 6x + 2y &= 4 \end{aligned}$$

Example C: Find the set of solutions to:

$$\begin{aligned} 3x + y &= 2 \\ 6x + 2y &= 4 \end{aligned}$$

Categorize the types of solutions we found:

- System A:
- System B:
- System C:

It will take some work over the next few weeks, but we will prove that these are in fact the **only** possible outcomes when we solve systems of linear equations.

Next week: Section 2, Vector Spaces, or "Sets with Math".