Week #2: Vector Spaces

- The addition and scalar multiplication operators
- Axioms of a vector space
- Identifying vector spaces

Don't forget to log in to `https://qlicker.queensu.ca` at the start of the lecture.
Qlicker enrolment code is **QM3CXK**

Section 2 - Vector Spaces

A vector spaces: a set combined with two specific functions.

Definition: A **real vector space V** is a set V along with **two functions** or **operations**:

1. An addition map/operator called "+".

$$+ :(V \times V) \to V$$
$$(v, w) \to v + w$$

2. A scalar multiplication map/operator called "·".

$$\cdot :(\mathbb{R} \times V) \to V$$
$$(a, w) \to a \cdot w$$

if those functions satisfy specific properties.

Note: The 'real' in 'real vector space' comes from the scalar multiplication part. If we let people multiply by e.g. complex numbers, $\mathbb{C} \times V$, then we would be building a '*complex* vector space'.

We will stick mostly to *real* vector spaces thankfully!

Example: the way we usually use it, the set of real numbers $\mathbb{R}$ fits the defintion of a 'real vector space'.

$$
\begin{aligned}
+ :&(\mathbb{R} \times \mathbb{R}) \to \mathbb{R} \\
&(a, b) \to a + b
\end{aligned}
$$

$$
\begin{aligned}
\cdot :&(\mathbb{R} \times \mathbb{R}) \to \mathbb{R} \\
&(a, b) \to ab
\end{aligned}
$$

Example: Also, the set of functions C^∞, infinitely differentiable functions, is a real vector space.

$$\begin{aligned} + :&(C^\infty \times C^\infty) \to C^\infty \\ &(f(x), g(x)) \to f(x) + g(x) \end{aligned}$$

$$\begin{aligned} \cdot :&(\mathbb{R} \times C^\infty) \to C^\infty \\ &(a, f(x)) \to af(x) \end{aligned}$$

Now we hold up for a second though: we haven't actually **defined** 'addition' and 'scalar multiplication by a real'. In two previous examples we just did what we usually do, but without knowing how that might work in other contexts.

With those two operations in place, **V** is a real vector space if the following axioms hold.

0. Both the $+$ and $\cdot$ are well-defined functions.

1. The operation $+$ is *associative* and *commutative*.

2. There exists in V an element called the **zero vector**, $\mathbf{0}$, such that for any $v \in V$ we have
$$v + \mathbf{0} = v, \text{ and } \mathbf{0} + v = v$$

3. Each $v \in V$ has at least one negative/opposite/inverse called $-v$, such that
$$v + (-v) = \mathbf{0}, \text{ and } (-v) + v = \mathbf{0}.$$

4. The scalar multiplication is *associative*.

5. The scalar multiplication and addition combinations are distributive:
$$a(v + w) = av + aw, \text{ and } (a + b)v = av + bv$$

6. For any $v \in V$, we must have $1 \cdot v = v$.

Check-in: how familiar are you with the terms *associative*, *commutative* and *distributive*?

Reminder:

- An operation is *associative* if you can change groupings of the inputs in a triple operation with no change in the final value. E.g. $(3+2)+5 = 3+(2+5)$, or $(3\cdot2)\cdot5 = 3\cdot(2\cdot5)$

- An operation is *commutative* if you can swap the order of the inputs with no change in the final value. E.g. $2+7 = 7+2$, or $2\cdot7 = 7\cdot2$.

- Two operations are *distributive* if you can expand them without changing the value. *Edited* E.g. $3\cdot(2+7) = 3\cdot7 + 3\cdot2$, or $(2+7)\cdot3 = 2\cdot3 + 7\cdot3$.

Notation: when we deal with vector spaces, we will write either:

- $\mathbf{V}$, or

- $(\mathbf{V}, +, \cdot)$

Wait! If these $+$ and $\cdot$ are just functions, why don't we call them $f(u, v)$ or $g(a, v)$ like we did earlier?

Example: Our most-used vector space: $\mathbb{R}^n$.

In the notation $(\mathbf{V}, +, \cdot)$, what are each of the vector space ingredients?

Example: The natural numbers $(\mathbb{N}, +, \cdot)$ **are not** a vector space if we use regular addition and real scalar multiplication.

Prove that, using the vector space axiom/property that:

"Each $v \in V$ has a negative/opposite/inverse called $-v$, such that

$$v + (-v) = \mathbf{0}, \text{ and } (-v) + v = \mathbf{0}."$$

Example: Use similar logic to show that the set of non-negative reals, $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, defined with the usual real number addition and scalar multiplication **is not** a vector space.

Example: The integers $(\mathbb{Z}, +, \cdot)$ **are also not** a vector space if we use regular addition and real scalar multiplication.

Show why not, this time looking at the scalar multiplication, $\cdot$, defined as a function:

$$\begin{aligned} \cdot :& (\mathbb{R} \times \mathbb{Z}) \to \mathbb{Z} \\ & (a, n) \to a \cdot n \end{aligned}$$

Example: The set of infinitely differentiable functions $(C^\infty(\mathbb{R}), +, \cdot)$ **is** a vector space, if we define:

- Addition as:

- Real scalar multiplication as:

What is the **zero element** in $(C^\infty, +, \cdot)$?

How can we build the opposite/negative of any element in $(C^\infty, +, \cdot)$?

Summary: Vectors Spaces are sets on which we can define an appropriate + and · operation.

Not Vector Spaces	Vector Spaces

Example: Consider the subset $V_0 \subset \mathbb{R}^2$ which satisfies $V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$, with the usual $\mathbb{R}^2$ addition and scalar multiplication.

What is the shape of this set, geometrically, in $\mathbb{R}^2$?

$V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$.

Is $(V_0, +, \cdot)$ a vector space? To show this we would need to show that **all** the axioms for a vector space are fulfilled.

(We will just show some of these...)

0. Both the $+$ and $\cdot$ are well-defined functions.

1. The operation $+$ is *associative* and *commutative*.

$$V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}.$$

2. There exists in V an element called the **zero vector** $\mathbf{0}$, such that for any $v \in V$ we have

$$v + \mathbf{0} = v, \text{ and } \mathbf{0} + v = v$$

3. Each $v \in V$ has a negative/opposite/inverse called $-v$, such that

$$v + (-v) = \mathbf{0}, \text{ and } (-v) + v = \mathbf{0}.$$

$$V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}.$$

4. The scalar multiplication is *associative*.

5. The scalar multiplication and addition combinations can be expanded:

$$a(v + w) = av + aw, \text{ and } (a + b)v = av + bv$$

Example: Consider the similar subset $V_1 \subset \mathbb{R}^2$ which satisfies $V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = \mathbf{1}\}$, with the usual $\mathbb{R}^2$ addition and scalar multiplication. What is the shape of this set, geometrically, in $\mathbb{R}^2$?

$V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = \mathbf{1}\}$.

Does this space satisfy the criterion that:

"0. Both the $+$ and $\cdot$ are well-defined functions."

$V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = \mathbf{1}\}$.

Does V_1 violate any other vector space axioms?

Intuitions: what seems to help make a vector space different from non-vector-spaces?

Example: Let W_2 be a subset of $\mathbb{R}^2$, with
$W_2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$.
Is W_2 a real vector space, if we use the usual addition and scalar multiplication for $\mathbb{R}^2$ vectors? Why or why not?

Now let's try expanding our ideas of what "+" and "$\cdot$" can mean. Imagine the same **set** W_2, but making a different vector space by defining **new addition and scalar multiplication** rules.
First, addition:

$$+ :(W_2 \times W_2) \to W_2$$
$$(x, y) + (a, b) \to (xa, yb)$$

Example: Try 'adding' the following in $(W_2, +, \cdot)$:

(a) (1,2)+(3,4)

(d) (2,2)+(4,5)

(b) (4,5)+(2,2)

(e) (0,5)+(7,2)

Will this "addition" be a well-defined function on W_2?

$W_2 = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

Next, scalar multiplication:

$\cdot :(\mathbb{R} \times W_2) \to W_2$

$\alpha \cdot (x,y) \to (x^\alpha, y^\alpha)$

Example: Try "scalar multiplying" the following in $(W_2, +, \cdot)$:

(a) $2 \cdot (1,2)$

(d) $0 \cdot (2,3)$

(b) $3 \cdot (0.5, 4)$

(e) $-\pi \cdot (2,3)$

Will this "scalar multiplication" be a well-defined function?

$W_2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

$+ :(x, y) + (a, b) \to (xa, yb)$

$\cdot :\alpha \cdot (x, y) \to (x^\alpha, y^\alpha)$

Example: What is the $\mathbf{0}$ element in W_2? Recall that this requires $\mathbf{0} + (x, y) = (x, y)$.

$W_2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

$+ :(x, y) + (a, b) \to (xa, yb)$

$\cdot :\alpha \cdot (x, y) \to (x^\alpha, y^\alpha)$

Example: For any $(x, y) \in W_2$, how do you build the negative/opposite value such that $-(x, y) + (x, y) = \mathbf{0}$?

Discussion: If most of our examples are just about $\mathbb{R}^n$ regular vectors anyway, what is the point of this abstract definition of a vector space?

We can use intuition about the **geometry** of $\mathbb{R}^2$ and $\mathbb{R}^3$ vectors to inform us about the **algebra** of other harder-to-visualize vector spaces.

Concept	Picture	Algebra
Vector addition $\vec{u} + \vec{v}$		

Concept	Picture	Algebra
All vectors are linear combinations of a core set of "building block" vectors.		

Other similar features we will see in Section 3.

Deductions from the Vector Space Axioms

We call a set $(V, +, \cdot)$ if and only if it satisfies the vector space axioms.

However, the axioms lead to a set of secondary properties that are also then guaranteed to be true for any vector space. We will describe and prove several of these, as a nice collection of proof examples.

Theorem 1. In any vector space V, there is only one zero vector $\vec{0}$.
(Recall: the axiom was just that there exists *a* zero vector, but maybe there is more than one equivalent vector.)
<u>Proof:</u>

Theorem 2. In any vector space V, each element v has only one (i.e. unique) additive inverse $-v$.

Proof:

Theorem 3 - Cancellation property. Given a vector space V, and three vectors $u, v, w \in V$ that satisfy $u + v = w + v$, then we can conclude $u = w$.

Proof:

Theorem 4. Let V be a vector space. Then

(a) For any vector $v \in V$, $0 \cdot v = \vec{0}$

(b) For any scalar $a \in \mathbb{R}$, $a \cdot \vec{0} = \vec{0}$

<u>Proofs:</u>

Theorem 4 cont. Let V be a vector space. Then

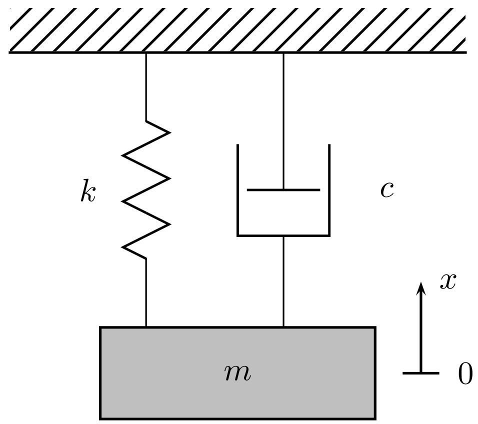
(c) For any vector $v \in V$, $-v = -1 \cdot v$

i.e. the scalar product $-1 \cdot v$ *is* the additive inverse of v.

<u>Proofs:</u>

Comment: All these theorems above can feel, well, *obvious* if we just think of $\mathbb{R}$ and $\mathbb{R}^n$: we just assume these based on our experience.
However, as we identify or encounter new vector spaces, it can be profoundly helpful to have this rock-solid bed of properties like these that we can immediately rely on, even in unfamiliar spaces.

Example: from the spring/mass system in APSC 171.



$$ma = \sum F$$

or

$$mx'' = -kx - cx'$$

In APSC 171, we found two simple solutions, e.g. $x_1(t) = e^{(a+bi)t}$ and $x_2(t) = e^{(a-bi)t}$.

We then made the statement "any linear combination of those two solutions is also a solution", e.g. $x(t) = z_1 e^{(a+bi)t} + z_2 e^{(a+bi)t}$

Relation to vector spaces: