Week #3: Vector Subspaces

- Vector subspaces, and simplified process for identifying them
- Linear combinations
- Span of a set of vectors as a subspace
- Implications of span for the solution of systems of equations

No clicker questions today (Wed)

Section 3 - Vector Subspaces

Recall: Vector Spaces are

- **A set V** combined with:
- An **addition** map/operator called "+"; and
- A **scalar multiplication** map/operator called "·".

These spaces must also satisfy a set of 8 axioms around their addition and scalar multiplication operators, e.g.

- There must be a zero element, $\mathbf{0}$, such that $0 + x = x + 0 = x$.
- Each element x must have a negative/inverse of itself, called $-x$, such that $x + (-x) = \mathbf{0}$.
- addition is commutative.
- $\ldots$

Vector Spaces	Not Vector Spaces

We now look at a somewhat simpler question: if we **already have** a vector space, when will a **subset** (with the same addition and scalar multiplication) also be a vector space?

Example: We have seen $\mathbb{R}^2$ is a vector space, with the usual addition and scalar multiplication.
Is the subset of $\mathbb{R}^2$ defined by
$V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = \mathbf{1}\}$ also a vector space?

Option 1: go through all 8 vector space axioms like last week.

Option 2: because we are just looking at a subset of a known vector space, we can take a short-cut!

Vector Subspace. Definition: Let $(\mathbf{V}, +, \cdot)$ be a real vector space. Now let $\mathbf{W}$ be a subset of $\mathbf{V}$, i.e. $\mathbf{W} \subset \mathbf{V}$, and working with the same $+$ and $\cdot$.

$(\mathbf{W}, +, \cdot)$ will also be vector space, or more commonly called a **vector subspace** of $(\mathbf{V}, +, \cdot)$ if the following axioms are satisfied:

(1) The subset $\mathbf{W}$ contains the zero element $\mathbf{0}$ of $(\mathbf{V}, +, \cdot)$.

(2) $\mathbf{W}$ is closed under addition. I.e. for any $x, y \in \mathbf{W}$, the sum $x + y \in \mathbf{W}$ too.

(3) $\mathbf{W}$ is closed under scalar multiplication. I.e. for any real α and element $x \in \mathbf{W}$, the product $\alpha x \in \mathbf{W}$ too.

Based on that shorter set of tests, is the subset of $\mathbb{R}^2$ defined by
$V_1 = \{(x, y) \in \mathbb{R}^2 : x + y = \mathbf{1}\}$ a vector subspace?

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Example: Is the subset of $\mathbb{R}^2$ defined by
$V_0 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$ a vector subspace?

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Example: Is the subset of any vector space $(\mathbf{V}, +, \cdot)$ defined simply by $W = \{\mathbf{0}\}$ a vector subspace?

Example: Is the empty set, i.e. which is a subset of any vector space $(\mathbf{V}, +, \cdot)$, a vector subspace?

Example: Is the set of all polynomials of degree 5 or less, $P_5(\mathbb{R})$, a vector subspace of the C^∞ vector space?

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Example: What about the set $\mathbf{U} = \{f \in C^\infty : f(0) = 3\}$: is it a vector subspace of C^∞?

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Is there a similar type of set that **would** be a vector subspace of C^∞?

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Example: Is the set $\mathbf{W} = \{f \in C^\infty : f(0) = f(3)\}$ a vector subspace of the C^∞ vector space? (Note the f on the RHS this time!)

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Example: Is the set $\mathbf{W} = \{f \in C^\infty : \frac{d}{dx}f(x) = f(x)\}$ a vector subspace of C^∞?

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Graph the members of $\mathbf{W}$, and interpret the membership condition as a differential equation.

Does the differential equation $x''(t) = -x(t)$ ring any bells from APSC 171?

Example: Is the set $\mathbf{W} = \{x(t) \in C^\infty : \frac{d^2}{dt^2}x(t) = -x(t)\}$ a vector subspace of C^∞?

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Return to the abstract case. We will be working with subspaces a lot, so having some quick rules we can use will be helpful.

Example: **Prove** that intersection of any two vector subspaces is also a vector subspace. (Ref: Theorem 3 on page 49 of the Lecture Notes.)

Continued: **Prove** that intersection of any two vector subspaces is also a vector subspace.

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Example: We now have shown that the **intersection** of any two vector subspaces is also a vector subspace. However, the Notes also indicate that the **union** of two vector subspaces **may or may not be a subspace**.

Illustrate this with examples of subsets from $\mathbb{R}^2$.

Section 4 - Linear Combinations

Once we have a real vector space $(\mathbf{V}, +, \cdot)$, we can safely:

- Add any elements of $\mathbf{V}$, and
- Multiply any elements of $\mathbf{V}$ by a real scalar.

If $v_1, v_2, \ldots v_p$ are vectors/elements in $\mathbf{V}$, what kinds of new vectors can we create? Suppose $\alpha_i \in \mathbb{R}$.

Linear Combination. Definition: let $v_1, \ldots, v_p$ be a finite collection of elements of the vector space $\mathbf{V}$ (with $p \geq 1$). The expression

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p,$$

with $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$, is called a **linear combination** of the vectors $v_1, \ldots, v_p$.

Example: write a linear combination of $5 \in \mathbb{R}$ and $7 \in \mathbb{R}$.

Example: write a linear combination of $[1, 0, 0], [0, 0, 3]$, $[1, -1, 2]$, all $\in \mathbb{R}^3$.

Example: write a linear combination of $\cos(t), \sin(t)$, both $\in C^\infty$.

Example: If we take the vectors $v_1 = [1, 2], v_2 = [0, 1] \in \mathbb{R}^2$, what parts of $\mathbb{R}^2$ can we reach with linear combinations of v_1 and v_2?

Example: If we take the vectors $w_1 = [1, 2], w_2 = [-2, -4] \in \mathbb{R}^2$, what parts of $\mathbb{R}^2$ can we reach with linear combinations of w_1 and w_2?

Example: Follow the process below for $\mathbb{R}^3$.

- We start with a known vector space $\mathbb{R}^3$.
- We pick a finite number of elements, $v_1, \ldots, v_p \in \mathbb{R}^3$ as 'seeds' or 'building blocks'.
- We imagine the subset of $\mathbb{R}^3$ we can cover or reach with *linear combinations* of $v_1, \ldots, v_p$.

Follow the process below for C^∞.

- We start with a vector space C^∞.

- We pick a finite number of elements, $v_1, \ldots, v_p \in C^\infty$ as 'seeds' or 'building blocks'.

- We imagine the subset of C^∞ we can cover or reach with *linear combinations* of $v_1, \ldots, v_p$.

Linear Combinations as Vector Subspaces. Proposition: Let $(\mathbf{V}, +, \cdot)$ be a real vector space, and $v_1, \ldots, v_p$ be a finite number of elements of the vector space $\mathbf{V}$ (with $p \geq 1$). The subset $S \subset \mathbf{V}$ consisting of all linear combinations of $v_1, \ldots, v_p$, is a vector subspace of the original $\mathbf{V}$.
The set S is called the **span** (or 'linear span') of the vectors $v_1, \ldots, v_p$.

Example: defining vector subspaces of $\mathbb{R}^n$.

Known vector space R^n

Rule for selecting subset of R^n

Pick some elements of R^n

Verify 3 rules for vector subspace

Span will automatically be a vector subspace

Example: defining vector subspaces of C^∞

Known vector space C^∞

Rule for selecting subset of C^∞

Pick some elements of C^∞

Verify 3 rules for vector subspace

Span will automatically be a vector subspace

Example: Prove, using the 3 requirements for a vector subspace, that the span of any finite set $v_1, \ldots, v_p \in \mathbf{V}$ will always be a vector subspace.

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(2) $\mathbf{W}$ is closed under addition. I.e. for any $x, y \in \mathbf{W}$, the sum $x + y \in \mathbf{W}$ too.
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Vector spaces give us a new way to think about solutions to systems of linear equations.
Example: Consider the system of linear equations below.

$$x_1 + 3x_2 = 1$$
$$2x_1 + x_2 = 3$$
$$4x_1 - x_2 = -3$$

Extract the x_i's as *coefficients* of $\mathbb{R}^3$ vectors:

What does the LHS of the equation now look like?

$$x_1 + 3x_2 = 1$$
$$2x_1 + x_2 = 3$$
$$4x_1 - x_2 = -3$$

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

Represent the question "Does this system of equations have a solution?" in new terms, using span.

Note: this hasn't yet helped us find an answer faster, but it *is* a new way to look at an old problem!

Example: Let us consider a system of equations related to $\mathbb{R}^2$.

$$\begin{aligned} x_1 + 2x_2 &= a \\ 4x_1 - 3x_2 &= b \end{aligned}$$

Use a span argument to prove that, no matter what values are picked for a and b, there **will** be a solution to this system.

Example continued.

Example: Let us consider another system of equations related to $\mathbb{R}^2$.

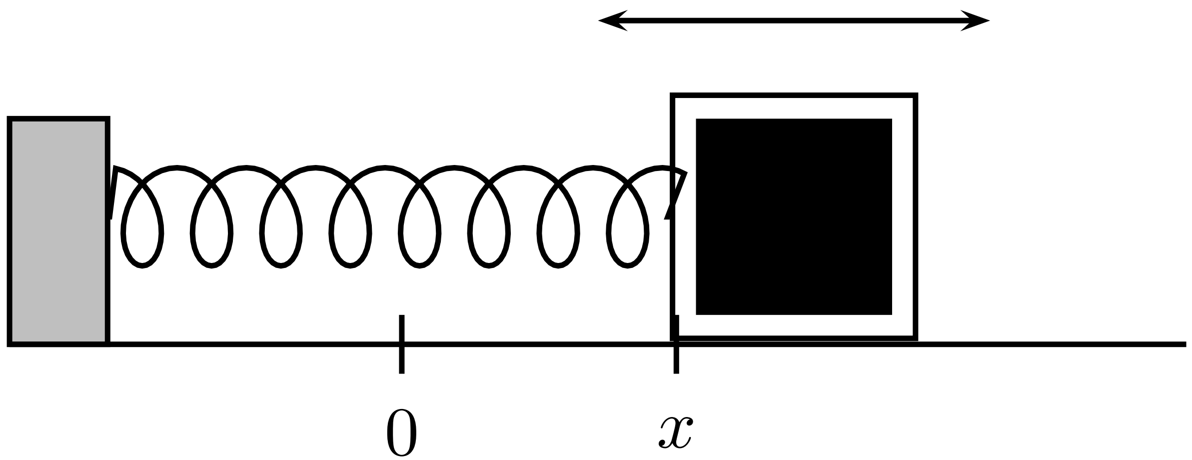
$$\begin{aligned} x_1 - 2x_2 &= a \\ -3x_1 + 6x_2 &= b \end{aligned}$$

Use a span argument to prove that, for some a and b values for this system, there will **not** be a solution.

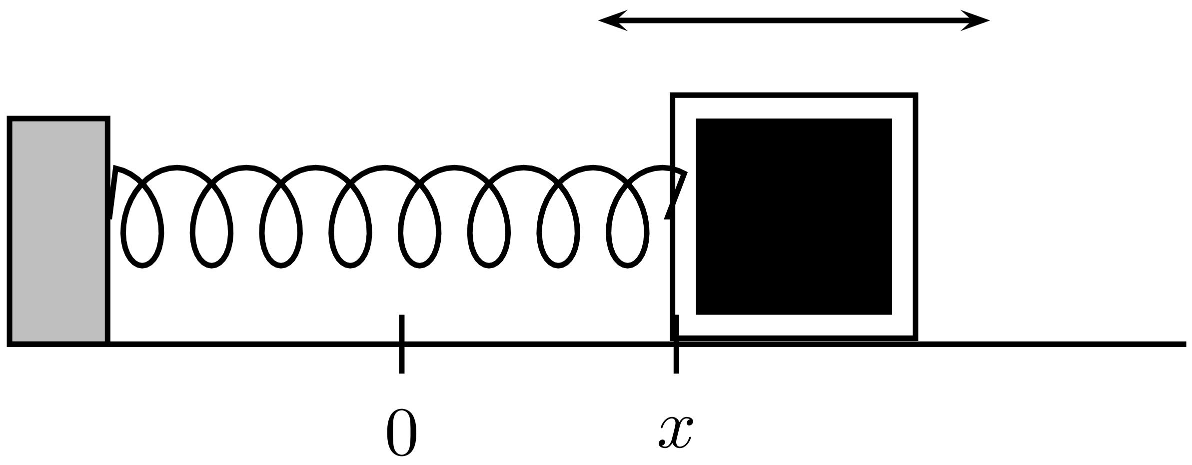
Example continued.

Vector spaces can also help us to better understand the solutions to the spring/mass differential equation from APSC 171 and APSC 112.

Example: Consider the spring system shown below.



What is the differential equation that the motion of the mass must follow?



$$mx''(t) + kx(t) = 0$$

What does a **solution**, $x(t)$, to the differential equation represent?

Prove that the set of all solutions to $mx''(t) + kx(t) = 0$ is a vector subspace of C^∞:
$\mathbf{W} = \{x(t) \in C^\infty \;:\; x(t) \text{ satisfies } mx''(t) + kx(t) = 0\}$

$$mx''(t) + kx(t) = 0$$

Continued.

$$mx''(t) + kx(t) = 0$$

$$\text{or } x''(t) = -\frac{k}{m}x(t)$$

Now, consider two **known** solutions to the differential equation (found in APSC 171):

- $x_1(t) =$

- $x_2(t) =$

Describe how the following two sets are related:

- set of all possible solutions to differential equation

$$mx'' + kx = 0, \text{ and}$$

- the span of the two simple solutions

$$\left\{ \sin\left(\sqrt{\frac{k}{m}}\, t\right), \cos\left(\sqrt{\frac{k}{m}}\, t\right) \right\}.$$

If time available, further examples or proofs of subspace and span relationships.