Week #4: Linear Independence

- Linear Combinations - Continued
- Definitions of Linear Independence and Linear Dependence

Section 4 - Linear Combinations and Span - Continued

Recall the relationship between

- a vector set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and
- the **span** of that vector set,
  $S = \{\mathbf{w} \in \mathbf{V} : \mathbf{w} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_p \mathbf{v}_p\}$ for some $\alpha_i \in \mathbb{R}$.

Illustration:

Example: Consider the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ below.
Is the vector $\mathbf{w} \in \mathbb{R}^4$ in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ -2 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ -3 \\ 0 \end{pmatrix}$$

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Example: Consider the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ below.
Is the vector $\mathbf{w} \in \mathbb{R}^4$ in the Span$(\mathbf{v}_1, \mathbf{v}_2)$?

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 0 \\ 12 \\ -4 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 0 \\ 12 \\ -4 \end{pmatrix}$$

Example: Is the function (vector) $\mathbf{w} \in C^{\infty}(\mathbb{R})$ in the span of the two functions (vectors) $\mathbf{v}_1$ and $\mathbf{v}_2$ shown below?

$$\mathbf{v}_1(x) = 3 + x^2$$
$$\mathbf{v}_2(x) = -2x + x^2 + x^3$$
$$\mathbf{w}(x) = 3 + 2x - x^3$$

$$\mathbf{v}_1(x) = 3 + x^2$$
$$\mathbf{v}_2(x) = -2x + x^2 + x^3$$
$$\mathbf{w}(x) = 3 + 2x - x^3$$

Note: Last week, we saw that

"Does this system of linear equations have a solution?"

could be turned into

"Is the RHS (as a vector) in the span of the LHS (as vectors)?"

$$\begin{aligned} x_1 + 3x_2 &= 1 \\ 2x_1 + x_2 &= 3 \\ 4x_1 - x_2 &= -3 \end{aligned} \qquad x_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

Now we are seeing that, to answer the question "is the vector $\mathbf{w}$ in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$?", we set up and try to solve a linear system of equations!

Section 5 - Linear Independence and Linear Dependence

Recall: if we pick a set of two vectors in $\mathbb{R}^2$, when will the **span** of those vectors **not** be all of $\mathbb{R}^2$?

Span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$: $S \subset \mathbf{V}$ consisting of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

Example: if we pick a set of **three** vectors in $\mathbb{R}^3$, when will the span of those vectors **not** be all of $\mathbb{R}^3$?

Example: Compare the span of the following sets of functions from C^∞:

$\{1, 1+x, 1+x^2\}$ $\{1, 1-x^2, 3+2x^2\}$

Linear Independence and Linear Dependence: generalizing the 2D ‘vectors are parallel’ idea.

Definition: The finite subset $p = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of a vector space $\mathbf{V}$ is **linearly independent** if, for any
$\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R}$, the relation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_p\mathbf{v}_p = \mathbf{0}$$

implies that all the $\alpha_1, \alpha_2, \ldots, \alpha_p$ must be zero.

Example: show that $\{[1, 0], [0, 2]\}$ is a linearly independent set.

Alternative definition: a set of vectors is linearly **dependent** if there exists a non-zero set of coefficients α_i such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_p\mathbf{v}_p = \mathbf{0}$$

Example: show that $\{[2,-5],[-4,10]\}$ is a linearly dependent set.

Example: show that $\{[2,0,1],[3,1,1],[4,2,1]\}$ is a linearly dependent set.

Are any of these vectors parallel?

What part of $\mathbb{R}^3$ would be spanned by these three vectors?

Common interpretation:
If p is a linearly **dependent** subset of $\mathbf{V}$, then at least one of the elements of p can be written as a linear combination of the other elements of p.

Or: p is linearly dependent if at least one vector $\mathbf{v}_p$ is in the span of the others, $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$.

From Lecture Notes, Section 5 Problem 2. All the $\mathbf{v}_i \in \mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Example: Is the set $\{\mathbf{v}_1\}$ linearly independent, or linearly dependent?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Example: Is the set $\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5\}$ linearly independent, or linearly dependent?

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix}2\\1\\0\\0\end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \qquad \mathbf{v}_4 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \qquad \mathbf{v}_5 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

Example: Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ linearly independent, or linearly dependent?

From Lecture Notes, Section 5, similar to Problem 3. All the $f_i \in C^\infty(\mathbb{R})$.

$$f_1(t) = 1 \qquad f_2(t) = 2 + t \qquad f_3(t) = 1 + t + t^2$$

$$f_4(t) = t^2 + t^3 \qquad f_5(t) = t + t^3$$

Example: Is the set $\{f_1\}$ linearly independent, or linearly dependent?

$$f_1(t) = 1 \qquad f_2(t) = 2 + t \qquad f_3(t) = 1 + t + t^2$$

$$f_4(t) = t^2 + t^3 \qquad f_5(t) = t + t^3$$

Example: Is the set $\{f_1, f_4, f_5\}$ linearly independent, or linearly dependent?

$$f_1(t) = 1 \qquad f_2(t) = 2 + t \qquad f_3(t) = 1 + t + t^2$$
$$f_4(t) = t^2 + t^3 \qquad f_5(t) = t + t^3$$

Example: Is the set $\{f_1, f_2, f_3, f_4, f_5\}$ linearly independent, or linearly dependent?

Intuition

If a set of vectors is
linearly **independent**:

- ___________ vectors can be removed without changing the span.

- There is/isn't any redundancy in the vectors, for making new vectors through sums, scaling.

If a set of vectors is
linearly **dependent**:

- _______________ vectors can be removed without changing the span.

- There is/isn't any redundancy in the vectors, for making new vectors through sums, scaling.

Proof that linearly dependent $\implies$ redundancy. Show that if a set of vectors is linearly **dependent**, then one of the vectors **can** be written as a linear combination of the others.

Proof that redundancy $\implies$ linearly dependent. Show that if one element in set of vectors **can** be written as a linear combination of the others, then the set is linearly **dependent**.

Example: single-vector edge cases.

Prove that the set made up of just the zero vector, $\{\mathbf{0}\}$, is linearly **dependent**.

Prove that a set made up of a single **non**-zero vector,
$s = \{\mathbf{v}_1 \in \mathbf{V} : \mathbf{v}_1 \neq \mathbf{0}\}$, is linearly **independent**.

Example: the zero vector with others. Consider any set $s = \{\mathbf{0}, \mathbf{v}_1, \ldots, \mathbf{v}_p\}$, $\mathbf{v}_i \in \mathbf{V}$ and $\mathbf{v}_i \neq \mathbf{0}$.
Prove that this s is linearly **dependent**.

Arguments with linear independence: Suppose that 4 vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ are linearly **in**dependent.

What can you say about the linear independence of the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Prove your answer.

Suppose that you find the relationship that $\mathbf{v}_5$ is in the span of $\{\mathbf{v}_1, v_2, v_3, v_4\}$. What can you say about the linear independence of the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$? Prove your answer.

Further examples: In $\mathbf{W}_2 = \{(x, y) : x > 0, y > 0\}$, are $\mathbf{v}_1 = (2, 3), \mathbf{v}_2 = (4, 9)$ linearly independent or linearly dependent?

$$+ :(W_2 \times W_2) \to W_2$$
$$(x, y) + (a, b) \to (xa, yb)$$

$$\cdot :(\mathbb{R} \times W_2) \to W_2$$
$$\alpha \cdot (x, y) \to (x^\alpha, y^\alpha)$$

Continued

$$\begin{aligned} + :&(W_2 \times W_2) \to W_2 \\ &(x, y) + (a, b) \to (xa, yb) \end{aligned}$$

$$\begin{aligned} \cdot :&(\mathbb{R} \times W_2) \to W_2 \\ &\alpha \cdot (x, y) \to (x^\alpha, y^\alpha) \end{aligned}$$

Example: In $C^\infty(\mathbb{R})$: are $f_1 = \sin(x)$ and $f_2 = \cos(x)$ linearly independent or linearly dependent?

$$\{\sin(x), \cos(x)\}$$

$$\{\sin(x), \cos(x)\}$$

Example: In $C^\infty(\mathbb{R})$: are $f_1 = e^x$ and $f_2 = e^{3(x-1)}$ linearly independent or linearly dependent?

$$\{e^{3x}, e^{3(x-1)}\}$$

Example: In $C^\infty(\mathbb{R})$: is the set $\{x^2, 4|x|^2\}$ linearly independent or linearly dependent?

$$\{e^{3x}, e^{3(x-1)}\}$$

Defining subspaces with span and with conditions.

Consider the two sets:

- $\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$
- $\mathbf{S} = \text{Span}([1, 0, 2], [-2, 1, 0])$

Describe both sets.

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$$
$$\mathbf{S} = \text{Span}(\mathbf{v}_1 = [1, 0, 2], \mathbf{v}_2 = [-2, 1, 0])$$

Verify that both $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbf{W}$.

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$$
$$\mathbf{S} = \text{ Span}(\mathbf{v}_1 = [1, 0, 2], \mathbf{v}_2 = [-2, 1, 0])$$

Knowing that both $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbf{W}$, what do you think the relationship is between the sets $\mathbf{S}$ and $\mathbf{W}$?

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$$
$$\mathbf{S} = \text{ Span}(\mathbf{v}_1 = [1, 0, 2], \mathbf{v}_2 = [-2, 1, 0])$$

Show that $\mathbf{S} \subset \mathbf{W}$.

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$$
$$\mathbf{S} = \text{Span}(\mathbf{v}_1 = [1, 0, 2], \mathbf{v}_2 = [-2, 1, 0])$$

Show that $\mathbf{W} \subset \mathbf{S}$.

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : 2x + 4y - z = 0\}$$
$$\mathbf{S} = \text{Span}(\mathbf{v}_1 = [1, 0, 2], \mathbf{v}_2 = [-2, 1, 0])$$

What have you proven about the relationship between **S** and **W**?