Week #5: Linear Independence

- Application of Linear Independence to Systems of Linear Equations
- Strategies and Patterns in Solving Linear Systems

Midterm 1 2019 - Practice Question

#5 (c) Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are elements of a vector space $\mathbf{V}$, and we know that

- the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, and also that
- the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

Show that $\mathbf{v}_3 \in S(\mathbf{v}_1, \mathbf{v}_2)$.

Continued.

Section 6 - Linear Independence and Systems of Equations

Last week, we tied systems of linear equations back to a relationship about the span of a set of vectors.

$$\begin{aligned} x_1 + 3x_2 &= 1 \\ 2x_1 + x_2 &= 3 \\ 4x_1 - x_2 &= -3 \end{aligned} \qquad\qquad x_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

Represent the question “Does the system of equations on the left have a solution?” in new terms, using span.

Example: compare the **number of solutions** to the following systems:

$$x_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + x_2 \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\3\\4\end{pmatrix}$$

$$x_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + x_2 \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\3\\4\end{pmatrix}$$

$$x_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + x_2 \begin{pmatrix}0\\1\\1\end{pmatrix} + x_3 \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\4\end{pmatrix}$$

Example: compare the **linear independence** and **span** of the LHS vectors, in the same systems:

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Determining the number of solutions to a system of linear equations

Extract the column vectors of coefficients from the linear system.

Example:

$$\begin{aligned} x_1 + 3x_2 &= 1 \\ 2x_1 + x_2 &= 3 \\ 4x_1 - x_2 &= -3 \end{aligned} \qquad \underbrace{x_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}_{\mathbf{v}_1} + x_2 \underbrace{\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}}_{\mathbf{v}_2} = \underbrace{\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}}_{\mathbf{w}}$$

In general, we can convert any system of linear equations in $x_1, \ldots, x_p$ into a new set of equations with vectors:

Cases:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

(1) $\mathbf{w}$ is **not** in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

(2) $\mathbf{w}$ **is** in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ **and**
$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly **independent**.

(3) $\mathbf{w}$ **is** in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ **and**
$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly **dependent**.

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

Geometric Intuition in $\mathbb{R}^3$

(1) **w** is **not** in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

(2) **w is** in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$;
$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly **independent**.

(3) **w is** in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$;
$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly **dependent**.

Proofs.

(1) Prove that if **w** is **not** in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then there exists no solutions to the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$.

(2) Prove that if

- **$\mathbf{w}$ is** in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, and
- the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly **independent**, then

there is **exactly one solution** to the equation
$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$.

Continued.

(3) Prove that if

- **$\mathbf{w}$ is** in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, and
- the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly **dependent**, then

there are **infinitely many solutions** to the equation
$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$.

Continued.

New facts:

- In solving systems of equations, the RHS (as a vector) must be in the **span** of the LHS for there to even be a solution.

- If there is a solution, the linear dependence or independence of the LHS vectors determines whether the solution is unique or not.

Examples: For each of the following systems of equations, determine the number of solutions using vector properties as much as possible.

$$\begin{aligned} x + y &= 5 \\ x + 2y &= 10 \end{aligned}$$

$$\begin{aligned} 2x + 3y &= 1 \\ 4x + 6y &= 1 \end{aligned}$$

$$2x - 3y = 5$$
$$4x - 6y = 10$$

New goals:

- Can we more efficiently determine whether a vector is in the span?
- Can we more efficiently classify sets of vectors as linearly dependent or independent?

We can (soon)!
Reaching both of these goals will tie back to solving systems of linear equations, so we will start by building an efficient strategy for solving any linear system of equations!

Section 7 - Strategies and Patterns in Solving Linear Systems

Recipe for solving Systems of Linear Equations

(1) Encode the system of linear equations in a matrix (the “augmented matrix of the system”).
(2) Use row operations to get the matrix into a good form (“upper triangular”, or better “Row Reduced Echelon Form” or RREF)
(3) With the matrix in this form, read off the solution(s).

We will spend some time seeing what each of these steps means.

Definition: A **matrix** is a **rectangular array or a grid of numbers**.

$$\begin{bmatrix} 1 & 7 & 9 \\ 2 & \pi & 4 \end{bmatrix}$$

____ × ____ matrix

$$\begin{bmatrix} 1 & e \\ -3 & 1 \\ 7 & -0.5 \\ 0.1 & 2 \end{bmatrix}$$

____ × ____ matrix

(1) Encode the system of linear equations in a matrix (the "augmented matrix of the system").

Example:

$$\begin{aligned} 7x_1 + 14x_2 + 70x_3 - 6x_4 &= 11 \\ x_1 + 2x_2 + 10x_3 - x_4 &= -2 \\ 3x_1 + 7x_2 + 33x_3 - 3x_4 &= -8 \end{aligned}$$

Represent these equations using a matrix.

Patterns/Process:

Think about operations you usually use when solving equations like adding, subtracting multiples of rows. Is anything lost or gained by doing similar operations on the coefficients in matrix form?

$$\begin{array}{llll} 3x & +7y & -4z & = 2 \\ 6x & -2y & +z & = 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 7 & -4 & 2 \\ 6 & -2 & 1 & 6 \end{array}\right]$$

Matrix Row Operations

The allowable row operations are:

- Switching, swapping or permuting the rows.
- Multiplying or dividing any row by a non-zero number.
- Adding or subtracting a multiple of row i to/from row j, and putting the result in row j.

Example: If we had the matrix form below for variables x, y and z, how would we interpret it?

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Compare that with the following matrix:

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 3 \\ 2 & 1 & 1 & 4 \end{array}\right]$$

Example: Use row operations to simplify the system of equations in matrix form below. Use the variables x_1, x_2, x_3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right]$$

Continued.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right]$$

Example: Use row operations to simplify the system of equations in matrix form below. Use the variables x_1, x_2, x_3, x_4.

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right]$$

Continued.

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right]$$

Example: Use row operations to find a solution to the system of equations in matrix form below. Use the variables x_1, x_2, x_3, x_4.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right]$$

Continued.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right]$$

Continued.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right]$$

Row Reduced Echelon Form

Formal goal: use row operations to put our equations' matrix into **RREF**.

A matrix is in RREF **if**:

- In each non-zero row, there is a **leading 1**.
  - From left to right, the first non-zero value should be a 1.
  - All 0's is okay.
- If a column has a leading 1, all other values in that column are 0's.
- The leading ones form a staircase from top left to bottom right.