

Week #6: Solving to RREF, Interpreting Solutions from RREF, Generating Sets and Basis Sets

Qlicker

Systems of Equations and Augmented Matrices

$$3x + 7y - 4z = 2$$

$$6x - 2y + z = 6$$

$$\left[\begin{array}{ccc|c} 3 & 7 & -4 & 2 \\ 6 & -2 & 1 & 6 \end{array} \right]$$

Ease of Identifying Solutions In Different Forms

These 3 systems are equivalent, and have the same solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 1 & -3 & 2 & -4 \\ -3 & 3 & -2 & -2 \end{array} \right]$$

Names for Matrix Forms

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 1 & -3 & 2 & -4 \\ -3 & 3 & -2 & -2 \end{array} \right]$$

Gaussian Elimination

Gaussian elimination is the process of applying *matrix row operations* until a matrix is in *row-reduced echelon form (RREF)*.

Matrix Row Operations

The allowable row operations are:

- Switching, swapping or permuting the rows.
- Multiplying or dividing any row by a non-zero number.
- Adding or subtracting a multiple of row i to/from row j , and putting the result in row j .

Row Reduced Echelon Form

A matrix is in RREF **if**:

- In each row, there should be a **leading 1**.
 - From left to right, the first non-zero value should be a 1.
 - All 0's is okay.
- If a column has a leading 1, all other values in that column should be 0's.
- The leading rows form a staircase from top left to bottom right.

Example: Solve the following system in x_1, x_2, x_3 using Gaussian elimination.

$$\left[\begin{array}{ccc|c} 2 & 6 & 1 & 8 \\ 3 & 9 & 2 & 14 \end{array} \right]$$

(Qlicker part-way through)

Example: Solve the following system in x_1, x_2, x_3 after first reducing the system to RREF.

$$\left[\begin{array}{ccc|c} 2 & -1 & 7 & -15 \\ 5 & -4 & 13 & -48 \\ 1 & -1 & 2 & -11 \end{array} \right]$$

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$$\left[\begin{array}{ccc|c} 2 & -1 & 7 & -15 \\ 5 & -4 & 13 & -48 \\ 1 & -1 & 2 & -11 \end{array} \right]$$

Example: Solve the following system in x_1, x_2, x_3, x_4 using Gaussian elimination.

$$\left[\begin{array}{cccc|c} 7 & 14 & 70 & -6 & -11 \\ 1 & 2 & 10 & -1 & -2 \\ 3 & 7 & 33 & -3 & -8 \end{array} \right] \quad \text{Qlicker}$$

$$\left[\begin{array}{cccc|c} 7 & 14 & 70 & -6 & -11 \\ 1 & 2 & 10 & -1 & -2 \\ 3 & 7 & 33 & -3 & -8 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 7 & 14 & 70 & -6 & -11 \\ 1 & 2 & 10 & -1 & -2 \\ 3 & 7 & 33 & -3 & -8 \end{array} \right]$$

Interpreting RREF forms

Example: Determine the number of solutions and the general solution for the following system in RREF.

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right]$$

Example: Determine the number of solutions and the general solution for the following system in RREF.

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{array} \right]$$

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$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

General Procedure for writing down all the solutions (from RREF)

Special Case: a row like $\left[\begin{array}{cccc|c} \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \dots & 0 & 1 \end{array} \right]:$

(Remember back to the span of the columns...)

More general case, **with** a solution:

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & 0 & 3 & 0 & | & 2 \\ 0 & 0 & 1 & 0 & -5 & 0 & | & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & | & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & | & 6 \end{array} \right]$$

- Write the variable names on the top of the matrix.
- Columns **without** leading 1's: **independent** variables, can take on **any value**.
 - For each of these, pick a new free variable, usually t or s .
- Columns **with** leading 1's: **dependent** variables, will get $x_i = \dots$ equations.

$$\left[\begin{array}{cccccc|c} 1 & 4 & 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & -5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{array} \right]$$

Example: Find the vector form of the solution to the system below.

$$\left[\begin{array}{cccccc|c} 1 & 0 & -2 & 0 & 5 & 0 & -1 \\ 0 & 1 & 4 & 0 & -6 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & -2 & 0 & 5 & 0 & -1 \\ 0 & 1 & 4 & 0 & -6 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

Section 8 - Generating Sets, Bases and Coordinates

After looking at systems of equations for awhile, we now go (mostly) back to looking at vector sets and spaces.

Definition: Given a vector space \mathbf{V} , a **generating** or **spanning set** for \mathbf{V} is a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbf{V} whose span is all of \mathbf{V} .

Equivalent: Every vector in \mathbf{V} is a linear combination of the set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

Example: In the plane \mathbb{R}^2 , what options are there for possible spanning sets?

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Example: In $\mathbf{W} = \{f(x) \in C^\infty(\mathbb{R}) : f''(x) = 0\}$, what options are there for possible generating sets?

Bases vs Spanning Sets

Definition: Given a vector space \mathbf{V} , a **basis for \mathbf{V}** is a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ such that:

- $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a generating or spanning set for \mathbf{V} , **and**
- $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent.

Example: Contrast the sets of vectors

$\{(1, 0), (0, 1)\}$ and

$\{(1, 0), (0, 1), (1, 1)\}$ in \mathbb{R}^2 .

Note: in linear algebra, the plural of **basis** is **bases**.

Coordinates with Respect to a Basis

A basis is useful because it provides us a unique way to represent any vector in \mathbf{V} .

I.e. For a given basis $B = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, a vector \mathbf{w} has a p -tuple $(\alpha_1, \dots, \alpha_p)_B$ called **the coordinates of \mathbf{w} with respect to the basis B** , if

$$\mathbf{w} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_p \mathbf{v}_p.$$

Question: the *uniqueness* of the representation relies on the term *basis*, rather than spanning set. Explain why using linear algebra terms.

Example: Show diagrammatically how the idea of a basis and coordinates in that basis might be used in a physics problem.

The set

$$B = \left\{ \begin{bmatrix} -5 \\ -6 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \end{bmatrix} \right\}$$

is a basis for \mathbb{R}^2 . Find the coordinates of the vector $\vec{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ relative to the basis B .

$$[\vec{x}]_B = \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix}$$

The set $B = \{4 + 2x^2, 12 - 4x + 6x^2, 37 - 8x + 18x^2\}$ is a basis for P_2 . Find the coordinates of $p(x) = 44 - 16x + 22x^2$ relative to this basis:

$$[p(x)]_B = \begin{bmatrix} \boxed{} \\ \boxed{} \\ \boxed{} \end{bmatrix}$$

Midterm 2 - 2016 - Problem 2

Consider the system of linear equations

$$3x_1 + x_2 - 2x_3 + x_4 = 3$$

$$2x_1 - 3x_2 + 4x_3 - 2x_4 = 2$$

$$-x_1 + 2x_2 + 3x_3 + 3x_4 = 8$$

$$4x_1 + x_2 + 15x_3 + 6x_4 = 31$$

where we wish to solve for the quadruple (x_1, x_2, x_3, x_4) of real numbers that satisfy this system.

(a) Write the augmented matrix for this system.

(b) Transform the augmented matrix into reduced row-echelon form (RREF) using a sequence of elementary row operations (and clearly indicate the elementary row operations you use at each step).

$$\left[\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 3 \\ 2 & -3 & 4 & -2 & 2 \\ -1 & 2 & 3 & 3 & 8 \\ 4 & 1 & 15 & 6 & 31 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 3 \\ 2 & -3 & 4 & -2 & 2 \\ -1 & 2 & 3 & 3 & 8 \\ 4 & 1 & 15 & 6 & 31 \end{array} \right]$$

(c) Using (b), find the set of all solutions to the original system.

$$\left[\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 3 \\ 2 & -3 & 4 & -2 & 2 \\ -1 & 2 & 3 & 3 & 8 \\ 4 & 1 & 15 & 6 & 31 \end{array} \right]$$