Week #7: Dimension of a Vector Space, Linear Transformations

Review: Generating Sets, Bases

Concept Question:

Definition: Given a vector space $\mathbf{V}$, a **generating** or **spanning set** for $\mathbf{V}$ is a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbf{V}$ for which...

Select **all** that apply:

(A) The set of vectors is linearly independent.

(B) The set of vectors is linearly dependent.

(C) The span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is equal to $\mathbf{V}$.

(D) The span of $\mathbf{V}$ is equal to $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Concept Question

Definition: Given a vector space $\mathbf{V}$, a **basis for V** is a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that...

Select **all** that apply:

(A) The set of vectors is linearly independent.

(B) The set of vectors is linearly dependent.

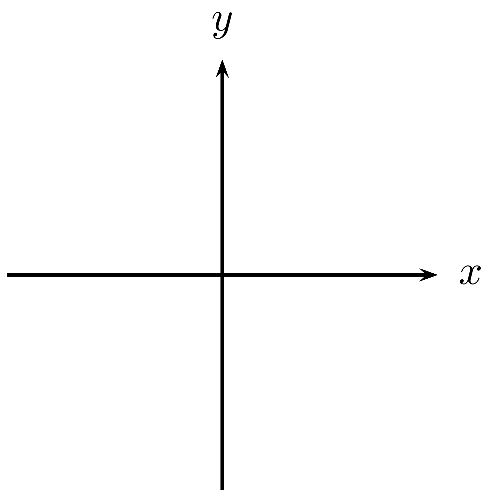
(C) The span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is equal to $\mathbf{V}$.

(D) The span of $\mathbf{V}$ is equal to $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example: Categorize each of the following vectors sets by whether they are generating sets, linearly independent, and/or a basis for $\mathbb{R}^2$.

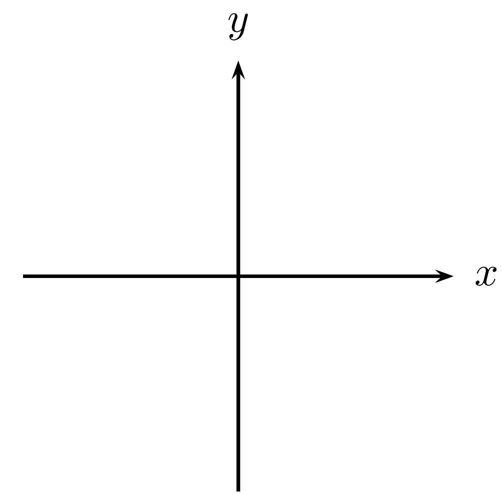
(a) The set $\{(1,0),(1,1)\}$.

Generating Set	Linearly Indep.	Basis



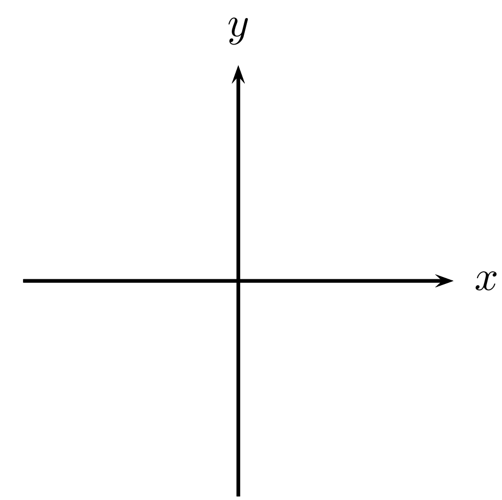
(b) The set $\{(1,0),(0,1),(1,1)\}$.

Generating Set	Linearly Indep.	Basis



(c) The set $\{(1,2),(2,4),(-1,-2)\}$.

Generating Set	Linearly Indep.	Basis



Dimension of a Vector Space

Provisional definition: Let $\mathbf{V}$ be a vector space. The *dimension* of $\mathbf{V}$, or $\dim(\mathbf{V})$, is the number of vectors in a basis for $\mathbf{V}$.

If $\mathbf{V}$ doesn't have a finite basis then $\dim(\mathbf{V})$ is infinite.

Example: what vector spaces have we seen with finite dimension?

Example: what vector spaces have we seen with infinite dimension?

Issue: we need to confirm that **any** two bases for the same vector space have the same number of elements.

E.g. that in $\mathbb{R}^3$, it isn't possible to get a basis with 2 vectors, or with 4, if we choose them just right.

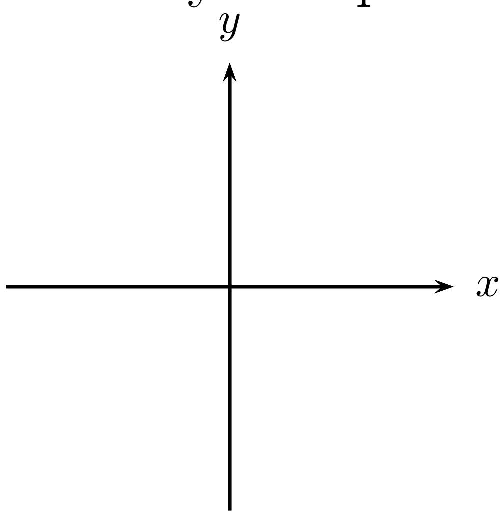
We will prove this with the help of a "lemma".
From Wikipedia: *In mathematics, a lemma* [...] *is a generally minor, proven proposition which is used as a stepping stone to a larger result. For that reason, it is also known as a "helping theorem" or an "auxiliary theorem".*

Key Lemma: Let $\mathbf{V}$ be a vector space, and suppose that $\{u_1, u_2, \ldots, u_a\}$ are a set of linearly independent vectors in $\mathbf{V}$, and that $\{v_1, v_2, \ldots, v_b\}$ are a spanning set for $\mathbf{V}$.
Then $a \leq b$.
Illustrate this with a diagram.

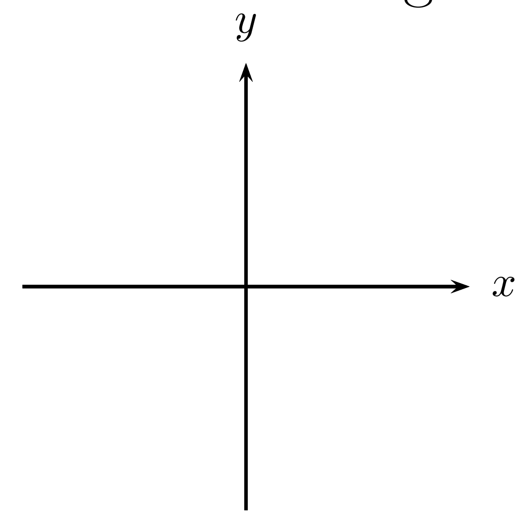
Key Lemma: Let $\mathbf{V}$ be a vector space, and suppose that $\{u_1, u_2, \ldots, u_a\}$ are a set of linearly independent vectors in $\mathbf{V}$, and that $\{v_1, v_2, \ldots, v_b\}$ are a spanning set for $\mathbf{V}$.
Then $a \leq b$.

Example: Illustrate this with examples of vector sets in $\mathbb{R}^2$.

Linearly Independent

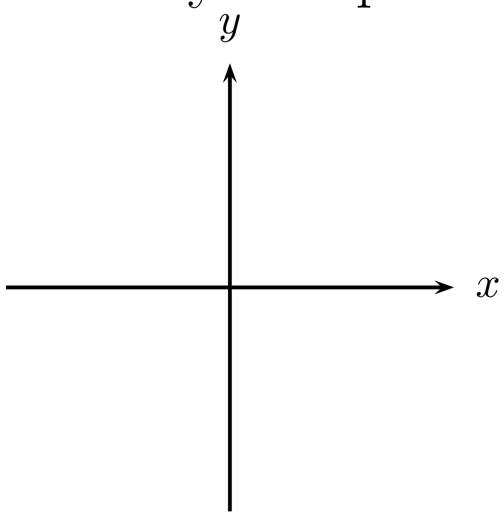


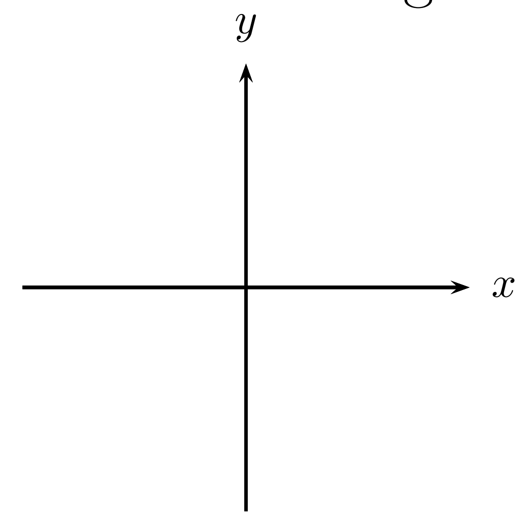
Generating Set

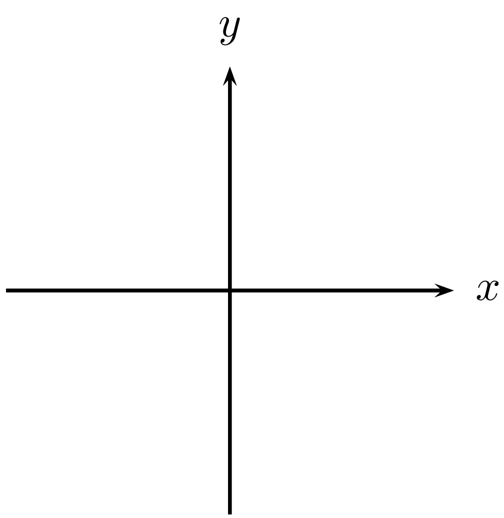


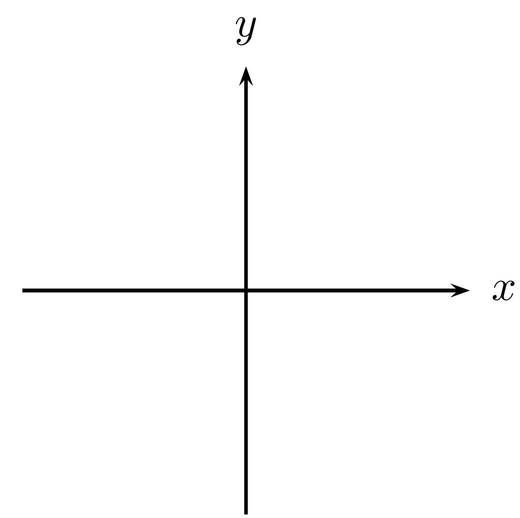
Linearly Independent

Generating Set









The Key Lemma can be proven in several ways. On request, there is a PDF with a proof based on systems of linear equations, and another proof based on induction. They are quite long though, so we won't cover them in these lectures.

However, with the use of the Key Lemma, it is easy to prove the result we need to uniquely define dimensionality.

Theorem 14: Let $\mathbf{V}$ be a vector space and $\{v_1, v_2, \ldots v_p\}$ and $\{w_1, w_2, \ldots, w_q\}$ be two bases for $\mathbf{V}$. Then $p = q$.

Paraphrase this theorem.

Theorem 14: Let **V** be a vector space and $\overbrace{\{v_1, v_2, \ldots v_p\}}^{\text{Set } A}$ and $\overbrace{\{w_1, w_2, \ldots, w_q\}}^{\text{Set B}}$ be two bases for **V**. Then the sets must be the same size, i.e. $p = q$.

Proof. Use the Key Lemma to show that $p = q$.

Example: Consider the vector subspace

$$\mathbf{V} = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\} \subset \mathbb{R}^3.$$

Show that the pair of vectors $\{v_1, v_2\} = \{(1, 0, -1), (0, 1, -1)\}$ is a basis for $\mathbf{V}$.

$$\mathbf{V} = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\} \subset \mathbb{R}^3.$$

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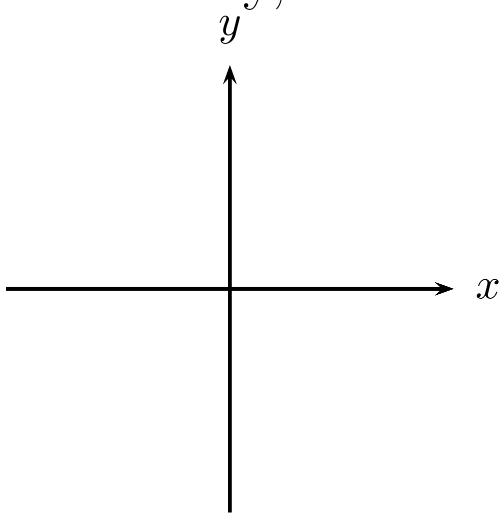
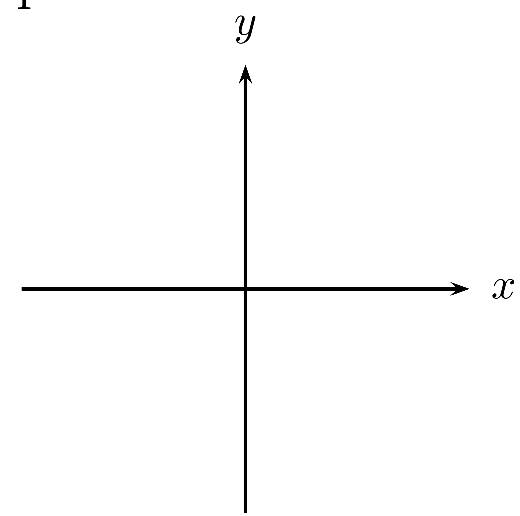
Use that result to determine the dimension of $\mathbf{V}$.

Example: Make a case for the dimensionality of $C^{\infty}(\mathbb{R})$ being infinite.

Corollaries Related to Dimension

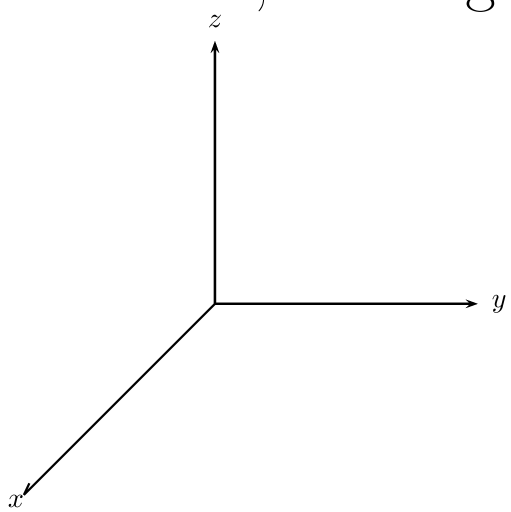
Intuitively, can we have 3 or more vectors in $\mathbb{R}^2$ that are linearly **in**dependent?

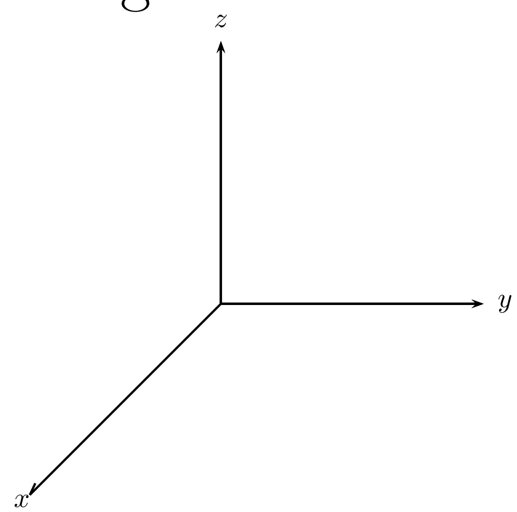


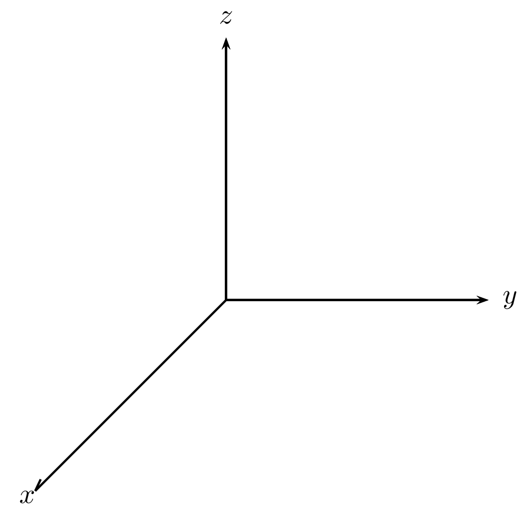


Prove in general than if $\dim(V) = N$ then any set of $N+1$ or more vectors in V must be linearly **de**pendent.

And looking at the reverse of this, if we have a linearly **in**dependent set, or are trying to build one, how big can that set get? E.g. in $\mathbb{R}^3$.







Prove in general than if $\dim(V) = N$, and we have a set $\{v_1, \ldots, v_p\}$ of linearly **in**dependent vectors, then $p \leq N$.

These results give us **alternative definitions** of the dimension of a vector space $\mathbf{V}$:

- $\dim(\mathbf{V})$ is the ________________ number of vectors in any generating set for $\mathbf{V}$.

- $\dim(\mathbf{V})$ is the ________________ number of vectors in any linearly independent set for $\mathbf{V}$.

- $\dim(\mathbf{V})$ is the ________________ number N such that all sets of $N + 1$ vectors must be linearly **de**pendent.

Linear Transformations

Back from Week 1: **Mappings and Functions.**

Definition: Let S and T be two sets. A **mapping** or **function** from S to T is a rule which assigns to each element of S **one and only one** element of T.

Notation:

$$f : S \to T$$
$$x \to f(x)$$

S is called the ______________________________

T is called the ______________________________

New special kind of functions: *linear* functions.

Definition: Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces.

A function $L : \mathbf{V} \to \mathbf{W}$ is called a **linear function** if it passes two tests.

1. Addition Test: For all $v_1, v_2 \in \mathbf{V}$,

$$L(v_1 + v_2) = L(v_1) + L(v_2)$$

2. Scalar Multiplication Test: For all $\alpha \in \mathbb{R}$, and $v \in \mathbf{V}$,

$$L(\alpha v) = \alpha L(v)$$

Equivalent terms: linear **function**, **transformation**, **map** and **mapping**.

Example: the **identity map** $L : \mathbf{V} \to \mathbf{V}$ is defined by $L(v) = v$. Show that this is a linear map.

$$L(v) = v$$

Example: Show that the function $L : \mathbb{R} \to \mathbb{R}$ given by $L(x) = \sqrt{|x|}$ is **not** a linear transformation.

Example: Determine whether the function $L : \mathbb{R} \to \mathbb{R}$ given by $L(x) = 3x + 1$ is a linear transformation or not.

Quick fact: helpful for non-linearity check:

In any linear function $L : \mathbf{V} \to \mathbf{W}$, $L(\mathbf{0_V}) = \mathbf{0_W}$.

Why is this the case?

Example: Use this to show that $L(x, y, z) = (x + 1, y + z)$ is not a linear transformation.

Example: show this test isn't always sufficient by giving examples of non-linear transformations that still satisfy $L(\mathbf{0_V}) = \mathbf{0_W}$.

Summary of the $L(\mathbf{0_V}) = \mathbf{0_W}$ test:

- If $L(\mathbf{0_V}) = \mathbf{0_W}$ then ________________________

- If $L(\mathbf{0_V}) \neq \mathbf{0_W}$ then ________________________

Example: Determine whether the definite integral over a fixed interval $[a, b]$ is a linear transformation.

$L :$ ______ $\to$ ______

$L(f) =$

The Kernel or Null Space of Linear Transformations

Example: consider the transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ with $L(x, y, z) = (x - y, y - z)$.
What is the input set or *domain* of L, and what is its dimension?

What is the output set or *co-domain* of L, and what is its dimension?

$$L(x, y, z) = (x - y, y + z)$$

Compute $L(1, 1, -1) =$

Compute $L(-4, -4, 4) =$

What do you think all the points in the input space $(x, y, z) \in \mathbb{R}^3$ satisfying $L(x, y, z) = \mathbf{0}$ will have in common?

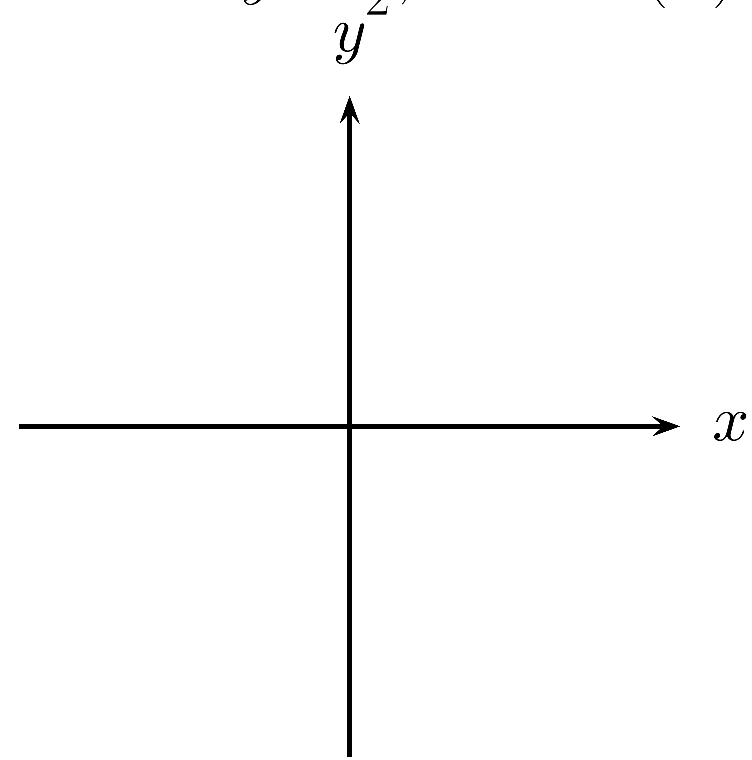
What is another name for that set of points for which $L(x, y, z) = \mathbf{0}$?

Definition: for a linear mapping $L : \mathbf{V} \to \mathbf{W}$, the set of all **input** vectors that are mapped to $\mathbf{0_W}$ is called the **kernel** of L, or the **null space** of L. Formally:

$$\mathrm{Ker}(L) = \{\mathbf{v} \in V : L(\mathbf{v}) = \mathbf{0_W}\}$$

Example: for $L : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R}), L(f) = \frac{d}{dx}f(x)$, find the kernel of L.

Example: for $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(x, y) =$ projection of (x, y) onto the line $y = \frac{x}{2}$, find $\text{Ker}(L)$.



Kernel as a Vector Subspace

From these examples, we get an idea that the kernel is not just some subset of the input space $\mathbf{V}$, but rather a highly structured subset...

Theorem 17. Let $\mathbf{V}$ and $\mathbf{W}$ be real vector spaces, and let $L : \mathbf{V} \to \mathbf{W}$ be a **linear** mapping. Then the kernel of L, $\mathrm{Ker}(L)$, is a **vector subspace** of $\mathbf{V}$.

Recall: What are the requirements for a subset $S \subset \mathbf{V}$ to be a vector subspace?

1)

2)

3)

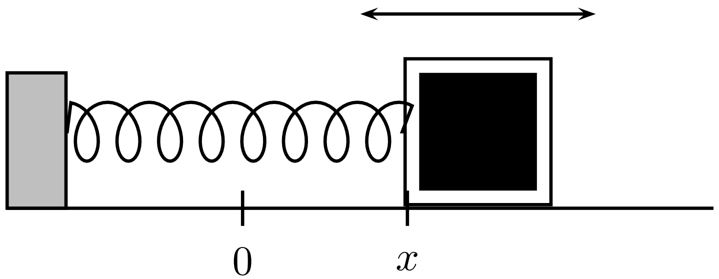
Theorem 17. Let $\mathbf{V}$ and $\mathbf{W}$ be real vector spaces, and let $L : \mathbf{V} \to \mathbf{W}$ be a **linear** mapping. Then the kernel of L, $\mathrm{Ker}(L)$, is a vector subspace of $\mathbf{V}$.

Proof

Theorem 17. Let $\mathbf{V}$ and $\mathbf{W}$ be real vector spaces, and let $L : \mathbf{V} \to \mathbf{W}$ be a **linear** mapping. Then the kernel of L, $\text{Ker}(L)$, is a vector subspace of $\mathbf{V}$.

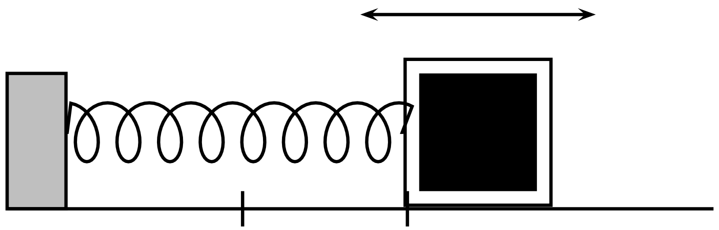
Theorem 17. Let $\mathbf{V}$ and $\mathbf{W}$ be real vector spaces, and let $L : \mathbf{V} \to \mathbf{W}$ be a **linear** mapping. Then the kernel of L, $\text{Ker}(L)$, is a vector subspace of $\mathbf{V}$.

The Kernel: Application to Differential Equations



Example: For the spring system shown above, what is the differential equation that governs the position of the mass over time, $x(t)$?

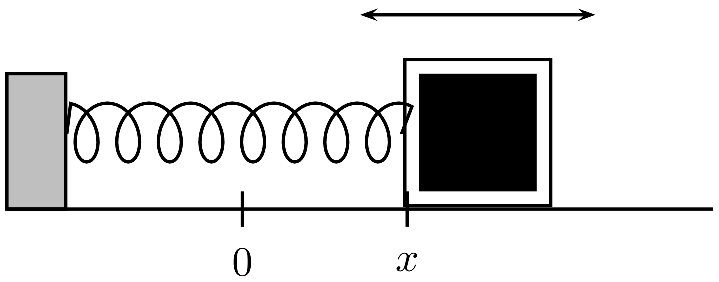
For the specific case where the mass is 1 kg, and the spring constant is 1 N/m, what is the differential equation?



By inspecting $x''(t) = -x(t)$, we found that possible solutions were
$x_1(t) =$____________ and
$x_2(t) =$____________.

We showed earlier that the set of solutions to
$x''(t) = -x(t)$
were a vector subspace of $C^\infty(\mathbb{R})$, by showing the set satisfied the 3 subspace axioms.

We could also then conclude that functions of the form
$x(t) =$________________________
were solutions too. Provide a rationale for that statement.



Prove again that the set of solutions to $x''(t) = -x(t)$ is a vector subspace, but now using a linear transformation argument.