Week #8: Null Space, Image, Matrix Forms

Review:

Definition: for a linear mapping $L : \mathbf{V} \to \mathbf{W}$, the set of all **input** vectors that are mapped to $\mathbf{0_W}$ is called the:

(A) Function of L.

(B) Image of L.

(C) Kernel of L.

(D) Zeros of L.

The kernel of a linear mapping $L : \mathbf{V} \to \mathbf{W}$ is:

(A) some subset of the input space $\mathbf{V}$.

(B) some subset of the output space $\mathbf{W}$.

(C) a vector subspace of the input space $\mathbf{V}$.

(D) a vector subspace of the output space $\mathbf{W}$.

Kernel and Image of Linear Transformations

Where we are heading: for a linear mapping $L : \mathbf{V} \to \mathbf{W}$, we will be able to divide up the dimensions of the **input** space, $\mathbf{V}$.

Let $\dim(\mathbf{V}) = n$ be the starting dimension.

- The **kernel** of L, with $L(\mathbf{v}) = \mathbf{0}_{\mathbf{W}}$, will have some dimension k, and
- The **image** of L, all the output $\mathbf{w}$'s we can get from some $L(\mathbf{v}) = \mathbf{w}$, will have dimension $n - k$.

Definition: Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation. We define the **image** of L, $\text{Im}(L)$, by

$$\text{Im}(L) = \{\mathbf{w} \in \mathbf{W} : \text{There is some } \mathbf{v} \in \mathbf{V} \text{ such that } L(\mathbf{v}) = \mathbf{w}\}$$

Note: the whole output space $\mathbf{W}$ is called the **co-domain** or **target set**.

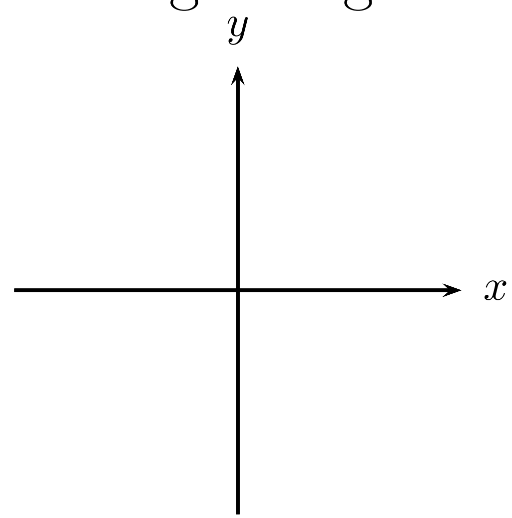
Note that for a linear transformation $L : \mathbf{V} \to \mathbf{W}$,

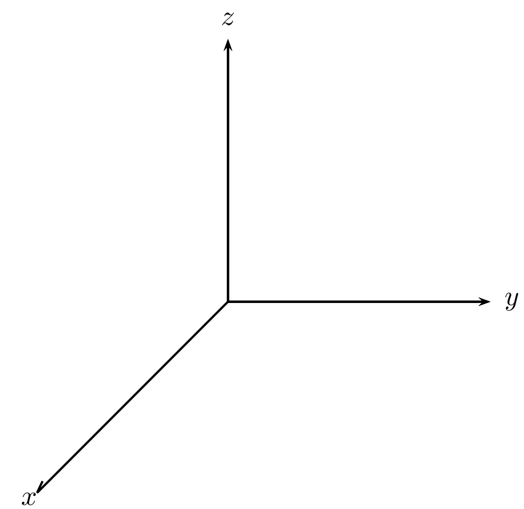
- the kernel $\mathrm{Ker}(L)$ is a subset of the **input** space $\mathbf{V}$, while

- the image $\mathrm{Im}(L)$ is a subset of the **output** space $\mathbf{W}$.

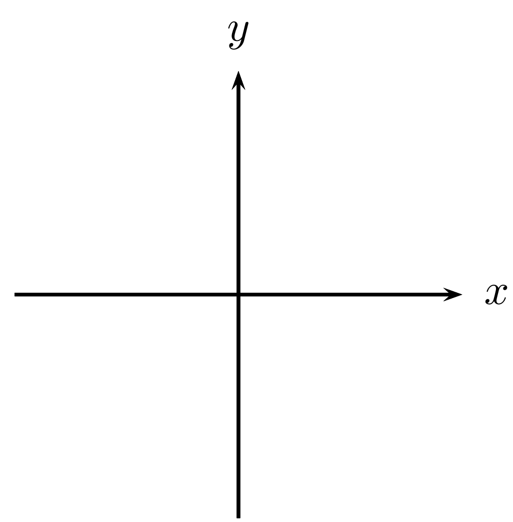
- the image **may or may not** be all of $\mathbf{W}$.

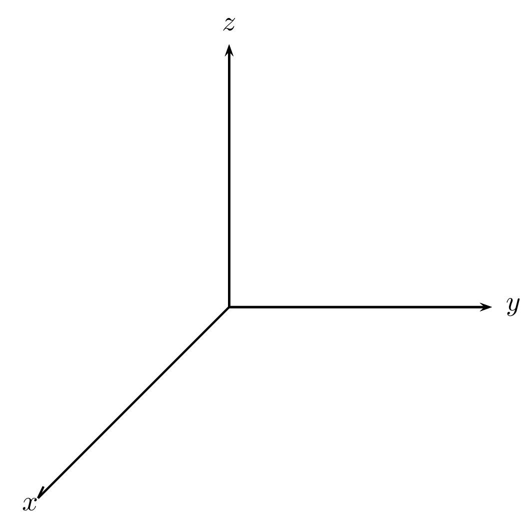
Problem: Sketch possible the following configurations for linear mappings from $\mathbb{R}^2 \to \mathbb{R}^3$:





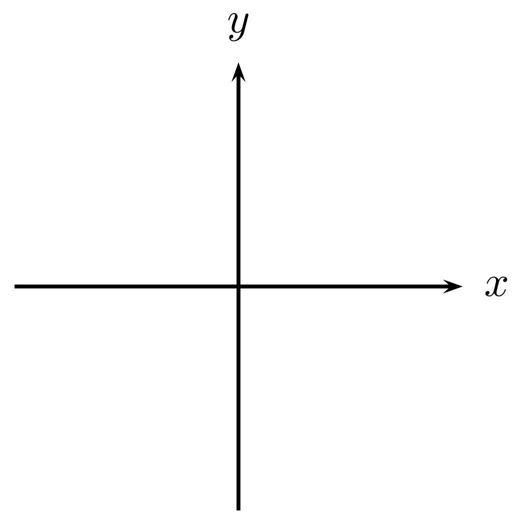
$f(x, y) = [x, y, x + y]$

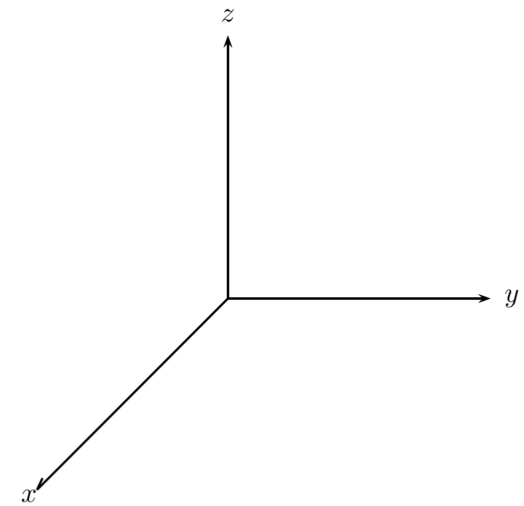




$g(x, y) = [0, x - y, y - x]$

$\mathbb{R}^2 \to \mathbb{R}^3$ examples.





$$h(x, y) = [0, 0, 0]$$

Theorem 19 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation, and

$$\mathrm{Im}(L) = \{\mathbf{w} \in \mathbf{W} : \text{There is some } \mathbf{v} \in \mathbf{V} \text{ such that } L(\mathbf{v}) = \mathbf{w}\}$$

Then $\mathrm{Im}(L)$ is a **subspace** of $\mathbf{W}$ (not just a subset).

Proof

Theorem 19 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
Then $\mathrm{Im}(L)$ is a subspace of $\mathbf{W}$.

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Here is a theorem that will be helpful when we study dimensions related to transforms. It ties into our idea that Ker(L) measures how much L "collapses" the input space $\mathbf{V}$, under its linear transformation.

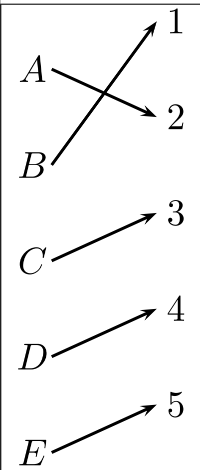
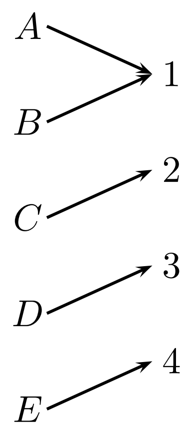
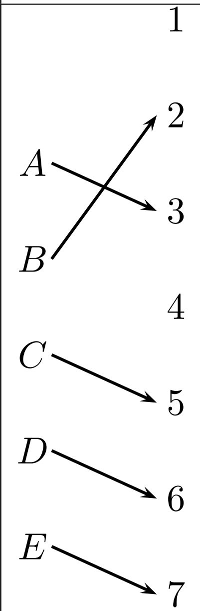
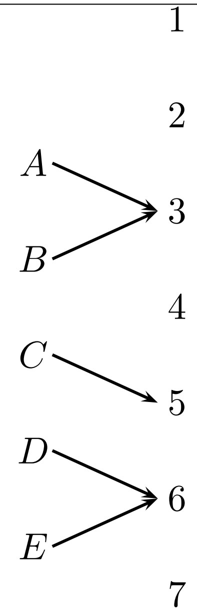
Theorem 18 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
L is **injective** or one-to-one if and only if $\text{Ker}(L) = \mathbf{0}_{\mathbf{V}}$.

Concept Question: *injective* functions are which of these?

(a) Let $f : S \to T$. $\forall x, y \in S$: $x \neq y \Rightarrow f(x) \neq f(y)$.

(b) Let $f : S \to T$. $\forall t \in T, \exists x \in S : f(x) = t$.

Diagrams

	Injective	Not Injective
Surjective		
Not surjective		

Theorem 18 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
L is **injective** if and only if $\mathrm{Ker}(L) = \mathbf{0}_{\mathbf{V}}$.

Proof: $\mathrm{Ker}(L) = \{\mathbf{0}_{\mathbf{V}}\} \implies L$ is injective.

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Theorem 18 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
L is **injective** if and only if $\text{Ker}(L) = \mathbf{0_V}$.

Proof: L is injective $\implies \text{Ker}(L) = \{\mathbf{0_V}\}$.

Theorem 18 Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation.
L is **injective** if and only if $\text{Ker}(L) = \mathbf{0}_{\mathbf{V}}$.

Example: Determining if a point is in $\text{Im}(L)$.
Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear mapping defined by

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

Show that the point $(-6, 2, 10)$ **is** in the $\text{Im}(L)$.

Continued

$L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear mapping defined by

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

Example: Show that the point $(1, 2, 3)$ is **not** in the $\mathrm{Im}(L)$.

Continued

Tie these ideas about linear transformations back to our earlier ideas about span and linear systems of equations.

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

$$\begin{aligned} 2x - 3y &= a \\ x + y &= b \\ 4x + 5y &= c \end{aligned}$$

Bases in transformations and Im(L)

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

Earlier we used a bit of trial and error and equation solving to see if a vector was or wasn't in the image of L. Can we use some insight instead to develop a **basis for the image**?

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

We will study this basis-of-image question in more detail next week!

Function Composition

Let $\mathbf{V}$ be a vector space with elements $u, v \in \mathbf{V}$. We then add transforms $T_1 : \mathbf{V} \to \mathbf{V}$ and $T_2 : \mathbf{V} \to \mathbf{V}$ which are linear transforms such that:

$T_1(u) = 3u - 6v$ and $T_1(v) = -u + 3v$, and

$T_2(u) = -4u + 7v$ and $T_2(v) = 2u + 2v$.

Find the images of u and v under the composition:

$T_2(T_1(u)) =$

$T_2(T_1(v)) =$

Matrices Defined by Linear Transformations

Any linear transform from $\mathbb{R}^n \to \mathbb{R}^m$ can be represented by an $m \times n$ matrix.

Bonus: the matrix form of a linear transform, after reducing to RREF, will let us

- find the dimension of Ker(L), and

- find a basis for Ker(L);

- find the dimension of Im(L), and

- find a basis for Im(L);

Sizes of matrices

Example: identify the size of each of the matrices below. Make sure the dimensions are presented in the standard order.

$$\begin{bmatrix} 2 & 5 & 8 \\ 0 & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we fix m and $n \geq 1$, we define the set of all $m \times n$ matrices as $M_{m,n}(\mathbb{R})$.

Fun fact: this set of matrices is a vector space!

- We define $M_1 + M_2$ addition as element-by-element addition, and
- We define scalar multiplication αM as multiplying every element of the matrix M by α.

Note: we only have a vector space $M_{m,n}(\mathbb{R})$ if we keep the dimensions of the matrices constant.

Example: try adding two 2×3 matrices, then separately a 2×2 and a 3×1 matrix.

Key Property of Linear Transformations

Suppose that $L : \mathbf{V} \to \mathbf{W}$ is a linear transformation, and that we have a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in \mathbf{V}$ for which we know $L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_p)$.
Then, for any new vector $\mathbf{v}$ we can easily deduce $L(\mathbf{v})$, so long as $\mathbf{v}$ is in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

Example: $L : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation, and we learn:

- $L([1, 0]) = [1, 2, 3]$, and
- $L([0, 1]) = [-1, 1, 2]$.

Find $L([3, 2])$.

$L([1,0]) = [1,2,3]$, and $L([0,1]) = [-1,1,2]$.

Find in general, $L([x,y])$ for any x, y.

$L([1,0]) = [1,2,3]$, and $L([0,1]) = [-1,1,2]$.

This can give us the idea to encode a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ using an $m \times n$ matrix.

Find the **standard matrix** associated with L.

Definition For a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, we associate to L the $m \times n$ **standard matrix** M for which where:

- 1st column of M is $L(1, 0, \ldots, 0)$,
- 2nd column of M is $L(0, 1, \ldots, 0)$,
- $\ldots$
- nth column of M is $L(0, 0, \ldots, 1)$,

Example: Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by

$$L(x, y, z, w) = (5x - 2y,\;\; 3z - w,\;\; y - 7z + 8w).$$

Find the standard matrix for L.

$$L(x, y, z, w) = (5x - 2y,\ \ 3z - w,\ \ y - 7z + 8w)$$

Example: Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$L(x, y, z) = (-2x - 4y + 3z,\ 11x + 15y + 8z).$$

Find the standard matrix for L.

Concept Check: What is the size of the standard matrix for a linear transformation $L : \mathbb{R}^7 \to \mathbb{R}^5$?

M is ______ $\times$ ______

Concept Check: If the standard matrix for L is 6 rows $\times$ 8 columns,

then L: ______ $\rightarrow$ ______

Moving between standard matrix and the matching linear transform

Example: $A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix}$

If A is the standard matrix for a transform L_A, what are the dimensions of the input and output spaces of L?

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix}$$

Interpret A as the transform of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix}$$

Compute $L_A(x, y, z)$.

Note any patterns in the resulting vector.

Since we often start with a matrix representation for a linear transform, we would like a simpler process for defining the output of $L_A(\mathbf{v})$.

We define the (**non-scalar**) matrix product $A\mathbf{v}$, with $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $\mathbf{w}_i$ as the ith column of A as:

$$A\mathbf{v} = \mathbf{w}_1 x_1 + \mathbf{w}_2 x_2 + \dots \mathbf{w}_n x_n$$

What kind of product (from calculus class) does this look like?

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix}$$

Evaluate $L_A(2, 4, -1)$.

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 0 & 8 \\ 4 & 1 & 6 \\ 0 & 7 & 0 \end{bmatrix}$$

Evaluate $L_A(0, -1, 1)$.

Next week: matrix multiplication, and the dimensions/bases for the kernel and image!