Week #9: Kernel and Image Bases, Matrix Multiplication

For a linear transformation $L : \mathbf{V} \to \mathbf{W}$, the image $\mathrm{Im}(L)$ is:

(A) some subset of the input space $\mathbf{V}$.

(B) some subset of the output space $\mathbf{W}$.

(C) a vector subspace of the input space $\mathbf{V}$.

(D) a vector subspace of the output space $\mathbf{W}$.

For a linear transformation $L : \mathbf{V} \to \mathbf{W}$, the kernel $\mathrm{Ker}(L)$ is:

(A) some subset of the input space $\mathbf{V}$.

(B) some subset of the output space $\mathbf{W}$.

(C) a vector subspace of the input space $\mathbf{V}$.

(D) a vector subspace of the output space $\mathbf{W}$.

For a linear transformation $L : \mathbf{V} \to \mathbf{W}$, the image $\mathrm{Im}(L)$:

(A) will be all of $\mathbf{W}$.

(B) will be all or some of $\mathbf{W}$.

(C) will be just $\mathbf{0}_{\mathbf{W}}$ if $\mathrm{Ker}(L)$ is non-empty.

Definition: For a linear mapping $L : \mathbf{V} \to \mathbf{W}$, the set of all **input** vectors that are mapped to $\mathbf{0_W}$ is called the **kernel** of L, or the **null space** of L.
Formally:

Definition: Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation. We define the **image** of L, $\operatorname{Im}(L)$ by:

Recall: We have seen different forms of a linear transformation.

$$L(x, y) = (2x - 3y, x + y, 4x + 5y)$$

$$L(x, y) = \begin{bmatrix} \\ \\ \\ \end{bmatrix} x + \begin{bmatrix} \\ \\ \\ \end{bmatrix} y$$

$$A_L = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$\text{and } L(x, y) = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

But what if we don't know the transform of the simple basis vectors $(1, 0, 0 \ldots), (0, 1, 0, \ldots)$?
Example: Consider $T : \mathbb{R}^2 \to \mathbb{R}^3$ as a linear transformation such that

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 12 \\ -6 \\ 3 \end{bmatrix}$$

Find the standard matrix of T.

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 12 \\ -6 \\ 3 \end{bmatrix}$$

$$T\left(\begin{bmatrix}3\\2\end{bmatrix}\right)=\begin{bmatrix}9\\1\\5\end{bmatrix} \text{ and } T\left(\begin{bmatrix}4\\-1\end{bmatrix}\right)=\begin{bmatrix}12\\-6\\3\end{bmatrix}$$

Back to kernel and image!

Both the kernel and the image are subspaces:

-

-

Since they are subspaces, each of them must have **bases** and **dimension**. Sketch:

Note: every *linear transformation* L has a kernel and an image.

We will also used the same nomenclature for every *matrix* A, where we infer that A is the standard matrix for a linear transform L.

E.g. "The kernel of the matrix A is...", or "The image of the matrix A is..."

Theorem 23: Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and A its standard matrix. Then the **columns of** A are a **generating set** for $\mathrm{Im}(L)$.

Illustration:

$$L(x, y) = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} x + \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} y$$

$$A_L = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

We see that the columns of A are a **generating set** for $\mathrm{Im}(L)$. Is that enough for that set to be a **basis** for $\mathrm{Im}(L)$?

Example: compare the images of the two matrices below, and find a basis each for their images.

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 2 & 1 \\ 4 & 8 & 5 \end{bmatrix}$$

Now compare the RREF for each matrix and look for patterns.

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 4 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 2 & 1 \\ 4 & 8 & 5 \end{bmatrix}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{RREF}(B) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Algorithm for finding a basis of $\mathrm{Im}(L)$

Let A be the standard matrix for L.

- Put A into RREF.
- Look at the columns of A with leading ones. The corresponding columns in **the original** A are a basis for $\mathrm{Im}(L)$.

Example: $L : \mathbb{R}^4 \to \mathbb{R}^3$, with its standard matrix

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix}$$

Find a basis for $\mathrm{Im}(L)$.

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix}$$

What is the dimension of $\mathrm{Im}(L)$?

Why does this work?

Row operations **preserve linear relationships between the columns**.

When we get to RREF, columns that are linearly independent there (leading ones) where also linearly independent in the original A.

Illustration: Let

$$A = \begin{bmatrix} 2 & 1 & -4 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$
$$\quad c_1 \; c_2 \; c_3$$

Confirm that $c_3 = -3c_1 + 2c_2$.

Below are some steps in the row reduction of this matrix:

$$\begin{bmatrix} 2 & 1 & -4 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Show that the same column relationship $c_3 = -3c_1 + 2c_2$ exists at each stage.

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Follow-up: why are the columns with leading 1's in RREF guaranteed to be a linearly independent set of vectors?

Algorithm for finding a basis of Ker(L)

Let A be the standard matrix for L.

- Put A into RREF.
- Add an extra column of 0's to get an augmented matrix.
- Write down the general solution to the corresponding system in vector form.
- The vectors associated with the free variables are a basis for the kernel.

Example: $L : \mathbb{R}^4 \to \mathbb{R}^3$, with its standard matrix

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix}$$

Find a basis for Ker(L). (Start on next page)

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for $\mathrm{Ker}(L)$.

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Confirm the kernel membership of a few points.

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

How many dimensions are there in $\mathrm{Ker}(L)$?

Dimensional Analysis

Example:

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \to \text{RREF} \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$L :$ _____ $\to$ _____

$\text{Dim}(\text{Im}(L)) =$ ____

$$A = \begin{bmatrix} 2 & -6 & -1 & 2 \\ -1 & 3 & -1 & -7 \\ 1 & -3 & 1 & 7 \end{bmatrix} \to \text{RREF} \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Dim}(\text{Ker}(L)) =$ ____

General Case:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ a_{31} & a_{32} & \dots \\ \dots & & \end{bmatrix} \to \text{RREF} \begin{bmatrix} 1 & c & \dots & \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 \\ \dots & & & \end{bmatrix}$$

$L :$ _____ $\to$ _____

$\text{Dim}(\text{Im}(L)) =$ ____

$\text{Dim}(\text{Ker}(L)) =$ ____

Rank - Nullity Theorem

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$n = \text{Dim}(\text{Im}(L)) + \text{Dim}(\text{Ker}(L))$$

$$n = \text{Rank}(A) + \text{Nullity}(A)$$

Consequences of Rank and Nullity for Function Properties

What properties of a function are equivalent to the function being **injective** or **one-to-one**?

What properties of a function are equivalent to the function being **surjective** or **onto**?

Section 12 - Applications to Solutions of Linear Systems

We have seen augmented matrices before when we introduced RREF as a way to solve systems of equations.

Example: Consider the system of equations

$$\begin{aligned} x + \quad y + 2z &= -3 \\ -2y + z &= 2 \\ 2x \quad\quad + 3z &= -4 \end{aligned}$$

Big question: How many solutions does this system have?

Interim: Write this system using its matrix form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -3 \\ 0 & -2 & -1 & 2 \\ 2 & 0 & 3 & -4 \end{array}\right]$$

Interpret this system in terms of the span of a set of vectors.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -3 \\ 0 & -2 & -1 & 2 \\ 2 & 0 & 3 & -4 \end{array}\right]$$

Interpret this system in terms of the $\mathrm{Im}(L)$ for some transform L.

$$\begin{bmatrix} 1 & 1 & 2 & | & -3 \\ 0 & -2 & -1 & | & 2 \\ 2 & 0 & 3 & | & -4 \end{bmatrix} \to \text{RREF} \begin{bmatrix} 1 & 0 & \frac{3}{2} & | & -2 \\ 0 & 1 & \frac{1}{2} & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

What are the dimension of $\text{Ker}(L)$ and $\text{Im}(L)$?

Does the original system of equations have no solution, a single solution, or an infinite number of solutions?

More generally: let A be thought of as the standard matrix for a transformation, and the system of equations being written as $[A|b]$ for some vector b. Work through the cases for A and b.

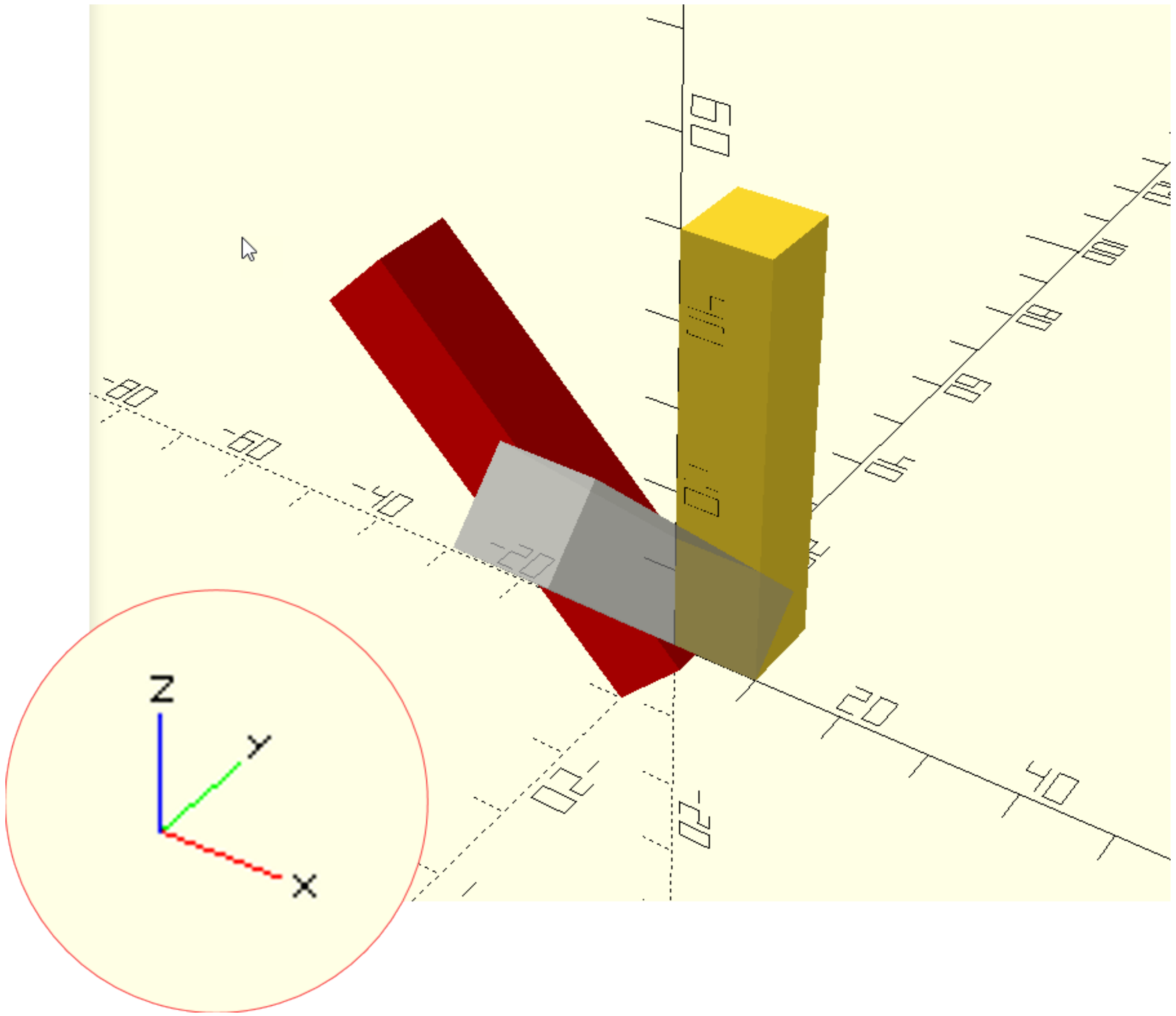
Case 1: $b \notin \mathrm{Im}(A)$.

Case 2a: $b \in \mathrm{Im}(A)$, and $\mathrm{Ker}(A) = \{\mathbf{0}\}$.

Case 2b: $b \in \mathrm{Im}(A)$, and $\mathrm{Ker}(A) \neq \{\mathbf{0}\}$.

Section 13 - Matrix Multiplication

A very handy property of linear transforms is that the **composition of two linear transforms is also linear**. For example, in graphics, we frequently want to compose rotations of an object.



Start: Yellow block.

Grey block: Rotated by 45 degrees around the x axis.

Red block: Then rotated by 90 degrees around the z axis.

Theorem: Suppose that $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^p$ are both linear transformations. Then $L = L_2 \circ L_1$ is also a linear transformation.
Reminder: interpret $L_2 \circ L_1$ in words.

Sketch the transform spaces.

If $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^p$ are linear then so is their composition, $L_2 \circ \L_1$.

Proof

If $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^p$ are linear then so is their composition, $L_2 \circ \text{Ł}_1$.

If $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^p$ are linear then so is their composition, $L_2 \circ$ Ł$_1$.

Matrix Representation of Linear Transformations

Suppose that L_A and L_B have standard matrices A and B. Since $L_B \circ L_A$ is also a linear transformation, it will have a standard matrix, say C.
Question: how is C related to the original A and B?

Example: Let L_A and L_B be defined by the standard matrices below

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 4 & 1 & 3 \\ -3 & 0 & 1 & 1 \end{bmatrix}$$

The C matrix must be _____ $\times$ _____.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 4 & 1 & 3 \\ -3 & 0 & 1 & 1 \end{bmatrix}$$

Find the standard matrix C for $L_B \circ L_A$, using the definition of this matrix as the transform of the canonical basis vectors $(1, 0)$ and $(0, 1)$.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 4 & 1 & 3 \\ -3 & 0 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 4 & 1 & 3 \\ -3 & 0 & 1 & 1 \end{bmatrix}$$

From this example, we can infer the general pattern in the matrices for composing linear transformations.

Proposition Suppose that $L_A : \mathbb{R}^n \to \mathbb{R}^m$ and $L_B : \mathbb{R}^m \to \mathbb{R}^p$ are both linear transformations with standard matrices A and B respectively. Then the standard matrix C for the composition $L = L_B \circ L_A$ is

$$C = \left[Bc_1 \mid Bc_2 \mid Bc_3 \mid \ldots \mid Bc_n \right]$$

where the c_i are the columns of A, and Bc_i is the output of $L_B(c_i)$.

This tool of combining matrices turns out to be surprisingly useful, so we define it in general for any pair of (appropriately dimensioned) matrices.

Definition If A is an $m \times n$ matrix, and B is an $p \times m$ matrix, the we define the product BA to be the $p \times n$ matrix

$$BA = \left[Bc_1 \mid Bc_2 \mid Bc_3 \mid \dots \mid Bc_n \right]$$

where the c_i are the columns of A.

Note: we can also compute the matrix product BA using the "finger-following method" related to the dot product, as we will demonstrate in an example.

Example: Find the matrix product of the matrices below, by the finger-following method across (rows in B) and (columns in A).

$$\begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ -3 & 4 \end{bmatrix}$$

Check: $\begin{bmatrix} 3 & 13 \\ -7 & 5 \end{bmatrix}$

Example: Find the product of the two matrices below.

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & 2 \\ -1 & 2 & -1 & 4 \end{bmatrix}$$

Check: $\begin{bmatrix} 3 & 8 & 1 & 8 \\ -2 & -3 & -1 & -2 \\ -5 & -4 & -3 & 0 \end{bmatrix}$

Next week: some unexpected properties of matrix multiplication!