Week #12: Eigenvalues and Eigenvectors - Continued

Section 15 - Eigenvalues and Eigenvectors - Continued

Recall: Eigenvalues and eigenvectors let us summarize the effect of a linear transformation, in an alternative to the matrix form of the transform.

Example: Consider the matrix $A = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$

This matrix has the following eigenvalues and corresponding eigenvectors:

- $\lambda_1 = 1$ with $\bar{v}_1 = [1, 1]$, and

- $\lambda_2 = 0.8$ with $\bar{v}_2 = [-1, 1]$.

Summarize the effect of this matrix when transforming an input vector from $\mathbb{R}^2$.

Example: Find the eigenvalues for the matrix $B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

Problem: How many eigenvalues can there be for a 2×2 matrix?

Problem: How many eigenvalues can there be for an $n \times n$ matrix?

<u>Definition</u>: the *characteristic polynomial* for a matrix is the polynomial in λ defined by

$$\det(A - \lambda I)$$

The roots of the characteristic polynomial are equal to the eigenvalues of the matrix.

Example: what was the characteristic polynomial for $B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$?

Example: Find the eigen*vectors* for the matrix $B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

$$B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}.$$

Some students can be reasonably confused by the seeming arbitrariness of the eigenvector construction: what is the rationale for selecting just one vector as the eigenvector when there are an infinite number available?

While eigenvectors are useful as building blocks for describing spaces, it can be more elegant to define the *eigenspace* for each eigenvalue instead.

<u>Definition</u>: For an $n \times n$ matrix A, with an eigenvalue of λ, the vector subspace of $\mathbb{R}^n$ given by $\ker(A - \lambda I)$ is called the *eigenspace of A associated with the eigenvalue λ of A.*

Example: Describe the eigenspaces for the matrix $B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

Problem: Contrast the eigenspaces and eigenvectors for the matrix $B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

Transforming Matrices and their Eigenvectors and Eigenvalues

Problem: For a matrix A with eigenvalue λ and corresponding eigenvector $\bar{v}$:

Find an eigenvalue and corresponding eigenvector for the related scaled matrix kA.

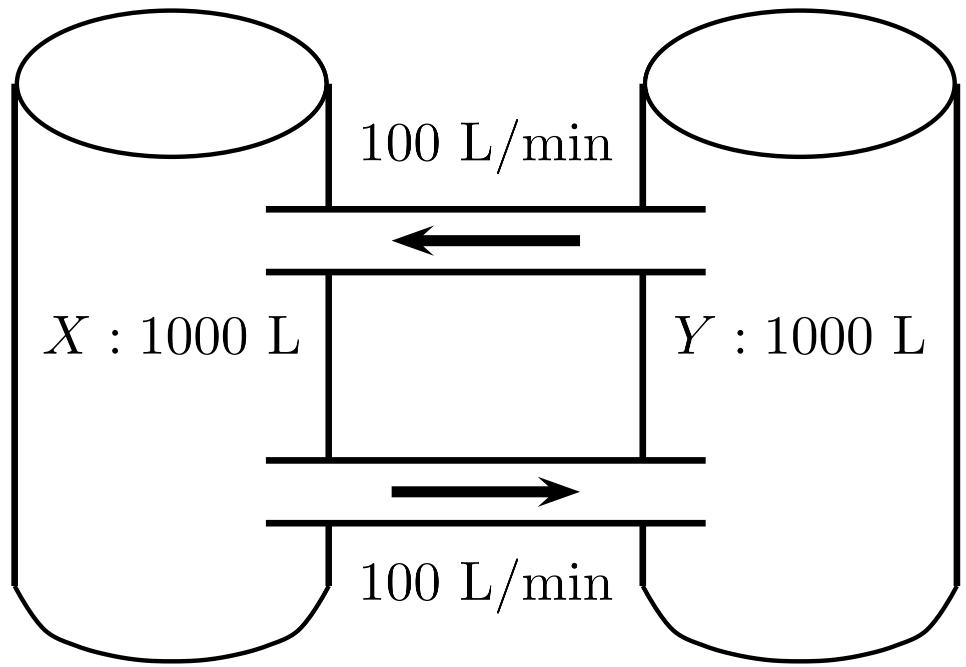
Problem: For a matrix A with eigenvalue λ and corresponding eigenvector $\bar{v}$:
Find an eigenvalue and corresponding eigenvector for the inverse matrix A^{-1}.

Problem: For a matrix A with eigenvalue λ and corresponding eigenvector $\bar{v}$:
Find an eigenvalue and corresponding eigenvector for the related matrix $A + cI$.

Problem: For a matrix A with eigenvalue λ and corresponding eigenvector $\bar{v}$:
Find an eigenvalue and corresponding eigenvector for the power A^{10}.

Eigenvector Application: The Tank Problem

Problem: we have two tanks, X and Y, which each hold 1,000 L of water.



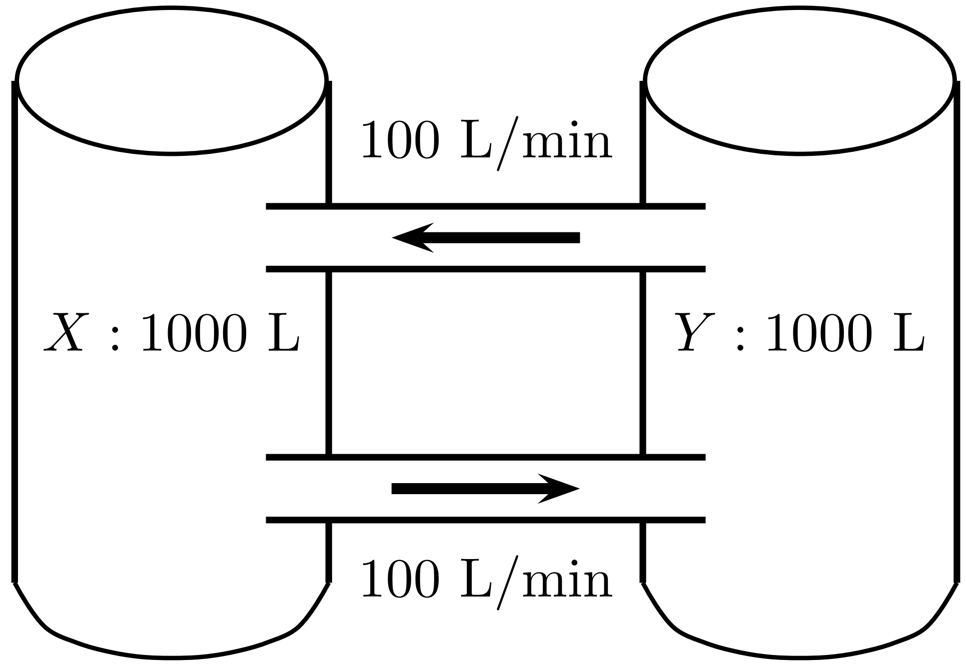
- Each minute 100 L of water is pumped from X to Y, and also from Y to X.

- At time $t = 0$, we put 80 kg of chemical Z in tank X, and

- Also at time $t = 0$, we put 20 kg of chemical Z in tank Y.

Question: How much of chemical Z is in each tank after n minutes?

Note: we are working at discrete times only, so we will use subscripts like x_n instead of function notation, $x(t)$.



We define:

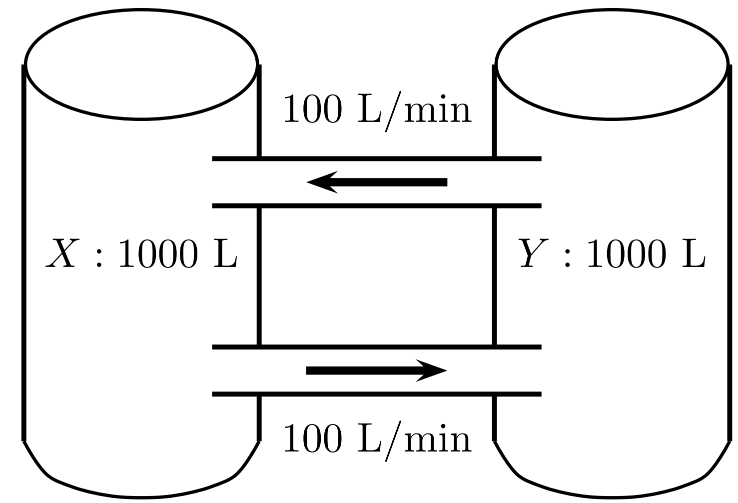
- x_n = mass of chemical Z in tank X at time n minutes.

- y_n = mass of chemical Z in tank Y at time n minutes.

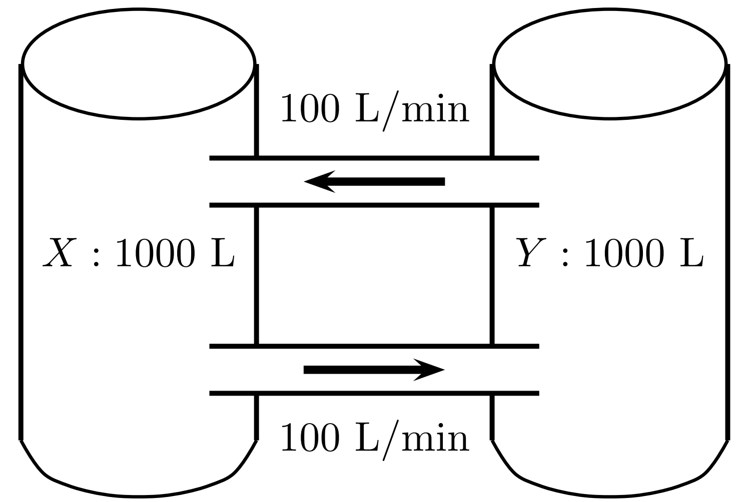
Thus we can define the **state** of the system as a vector: $[x_n, y_n]$.

We know the starting state, $[x_0, y_0] = [80, 20]$.

Problem: given the state $[x_n, y_n]$, find a way to compute the next state, $[x_{n+1}, y_{n+1}]$.


100 L/min
X : 1000 L
Y : 1000 L
100 L/min

Use the linear system found to predict the mass of chemical Z in each tank for $n = 1, 2$ and 3. Recall that $[x_0, y_0] = [80, 20]$.



Find an expression for the state $[x_n, y_n]$, given the masses at time zero, $[x_0, y_0]$.

Note any complications with computing this value for large n.

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (\underbrace{A\,A\,\ldots\,A}_{n \text{ times}}) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

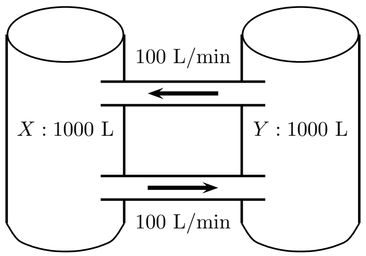
The key insight to simplifying this is that computing $A^n\,\bar{v}$ is simple **if $\bar{v}$ is an eigenvector of A.**

Example: Compare $A^n\,\bar{v}$ for the case when $\bar{v}$ is and is not an eigenvector.

To understand what occurs when we apply A repeatedly, it will help if we know the eigenvalues and eigenvectors of A. Find them for $A = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}$.

$$A = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}$$



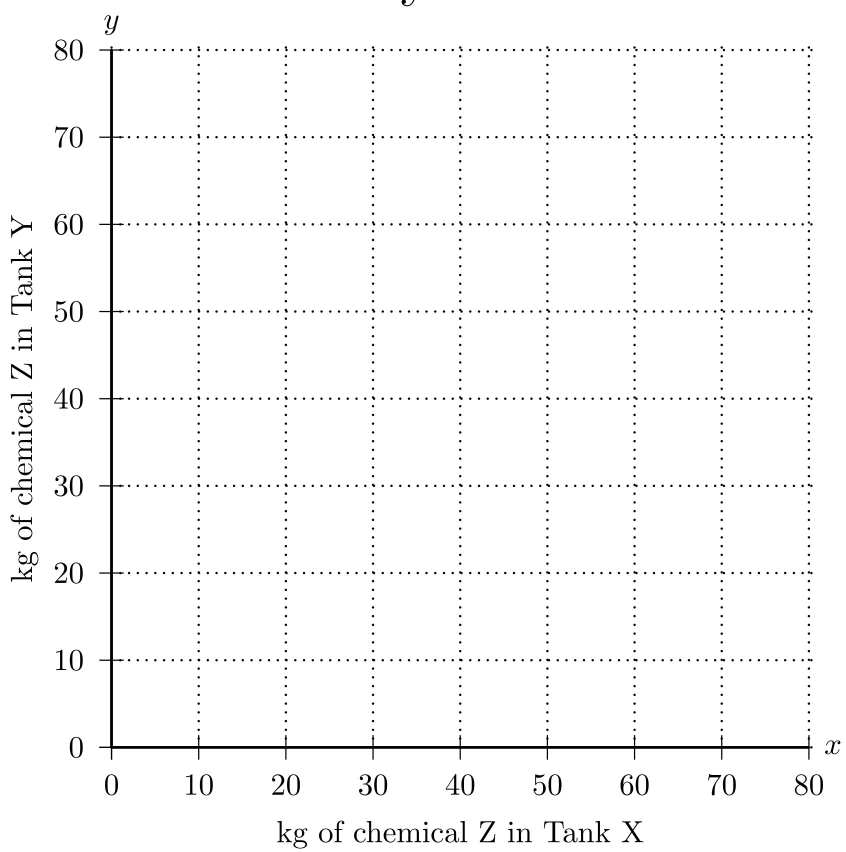
$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ with } \lambda_1 = 1, \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \lambda_2 = 0.8, \bar{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

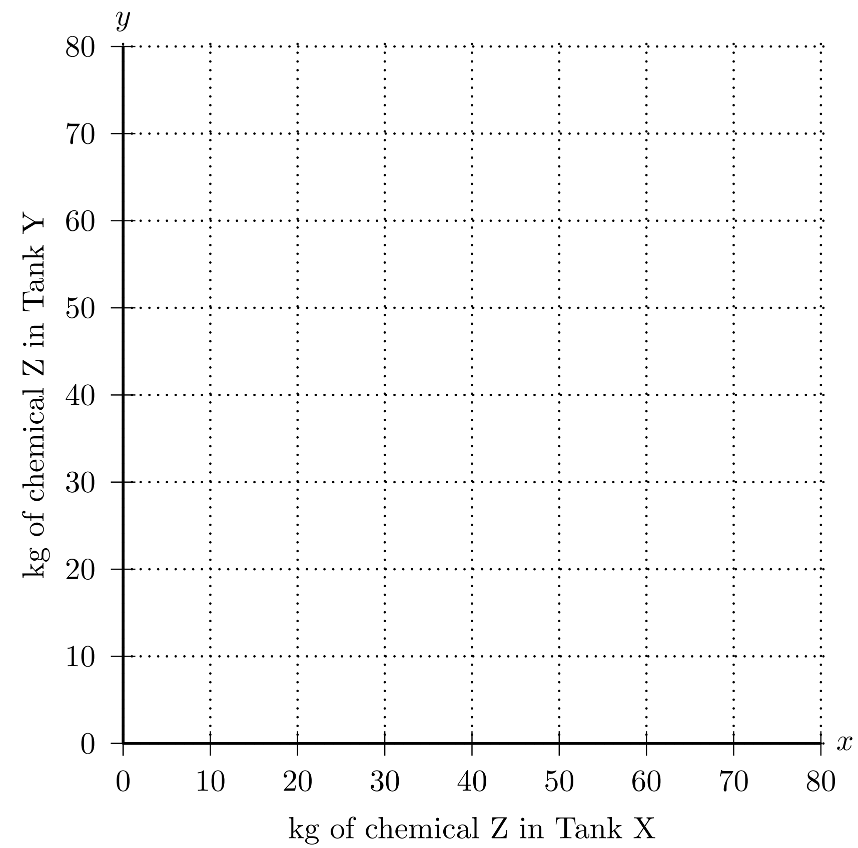
Returning to the tank context, use the eigenvectors and eigenvalues to re-write the predicted chemical levels, starting with $[x_0, y_0] = [80, 20]$.

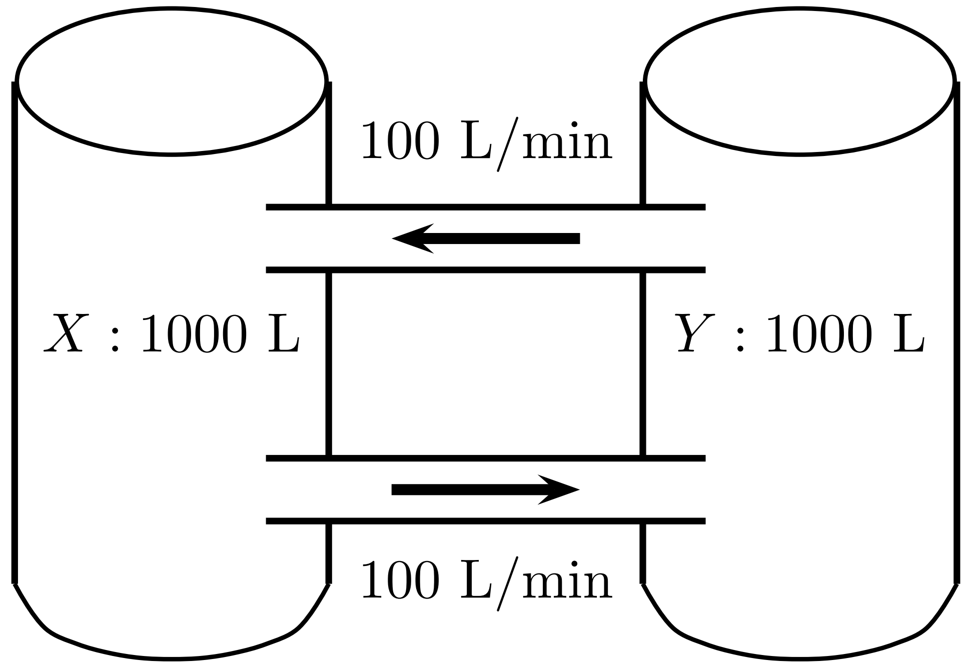
$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ with } \lambda_1 = 1, \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \lambda_2 = 0.8, \bar{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ with } A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.8 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Geometric Interpretation: Consider $[x_0, y_0] = [80, 20]$ and other starting quantities, and sketch how they would evolve over time.







Chemical Interpretation: Explain in chemistry terms why we see the behaviour we observed for $[x_n, y_n]$, and specifically the long-term amounts as $n \to \infty$.

Summary of insights from the tank problem:

- If we are repeatedly applying a linear transform A to a vector $\bar{v}$, the result of n applications will be the matrix power $A^n\,\bar{v}$.

- If the vector $\bar{v}$ **is** an eigenvector, $A^n\,\bar{v}$ simply becomes $\lambda^n\,\bar{v}$, where λ is the corresponding eigenvalue for $\bar{v}$.

- If the vector $\bar{v}$ **is not** an eigenvector, then we can (under certain conditions) write $\bar{v}$ as a linear combination of A's eigenvectors, and then use linear combinations to get a simple formula for $A^n\,\bar{v}$.

(This is a major feat, because without eigenvectors, the product $A^n\,\bar{v}$ is impossible to express in a simple closed form.)

Where might you find an eigenvector after APSC 174?

Geology, Civil, and Mech Materials - Stress Tensor

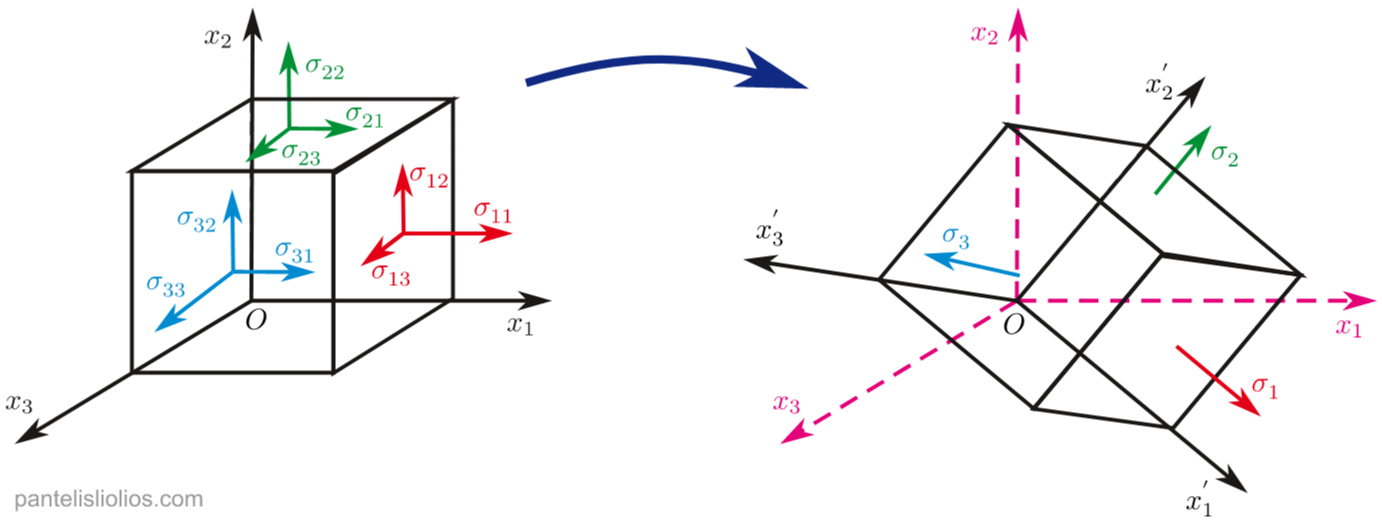


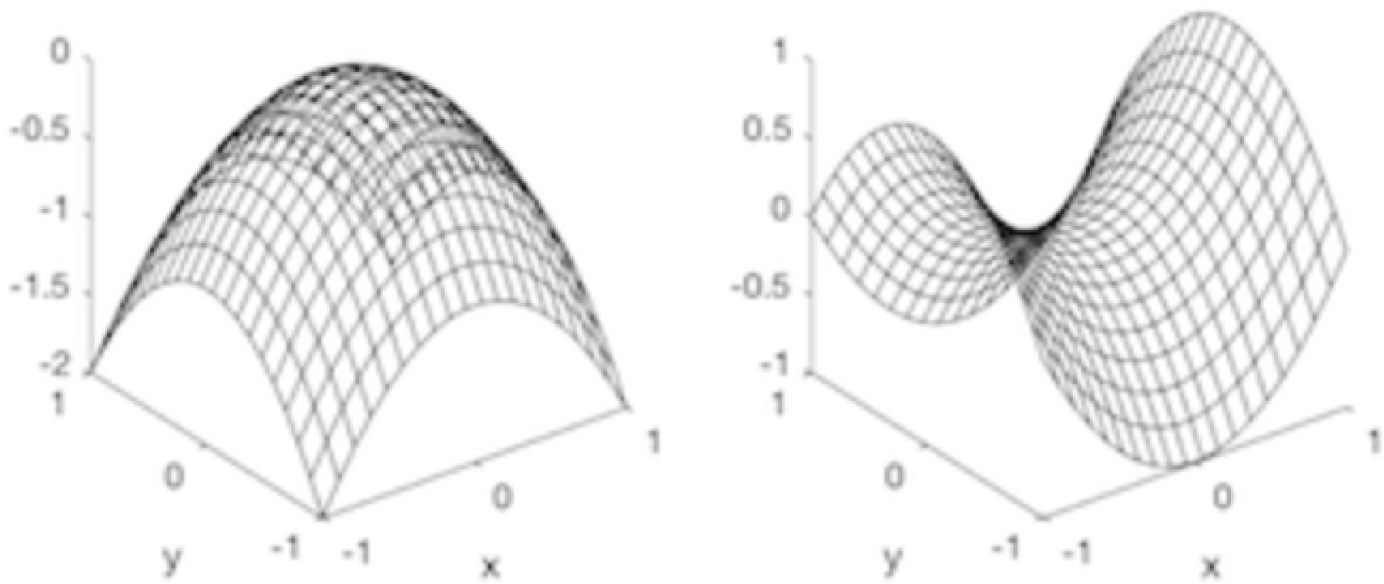
Figure 1: Principal stresses and their direction with respect to the initial coordinate system.

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \qquad \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Sources:

- https://en.wikipedia.org/wiki/Cauchy_stress_tensor
- https://www.pantelisliolios.com/principal-stresses-and-invariants

Vector Calculus - The Hessian Second Derivative



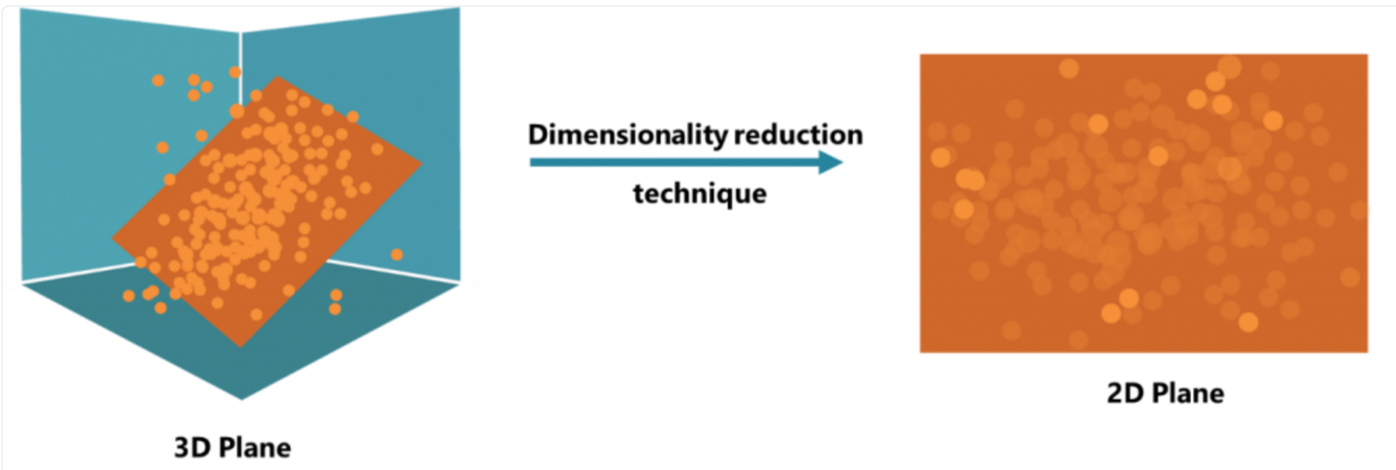
$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$\begin{bmatrix} f_1'' & 0 \\ 0 & f_2'' \end{bmatrix}$$

Sources:

- http://15462.courses.cs.cmu.edu/fall2019/lecture/math2/slide_050

Principal Component Analysis and Data Reduction



Covariance

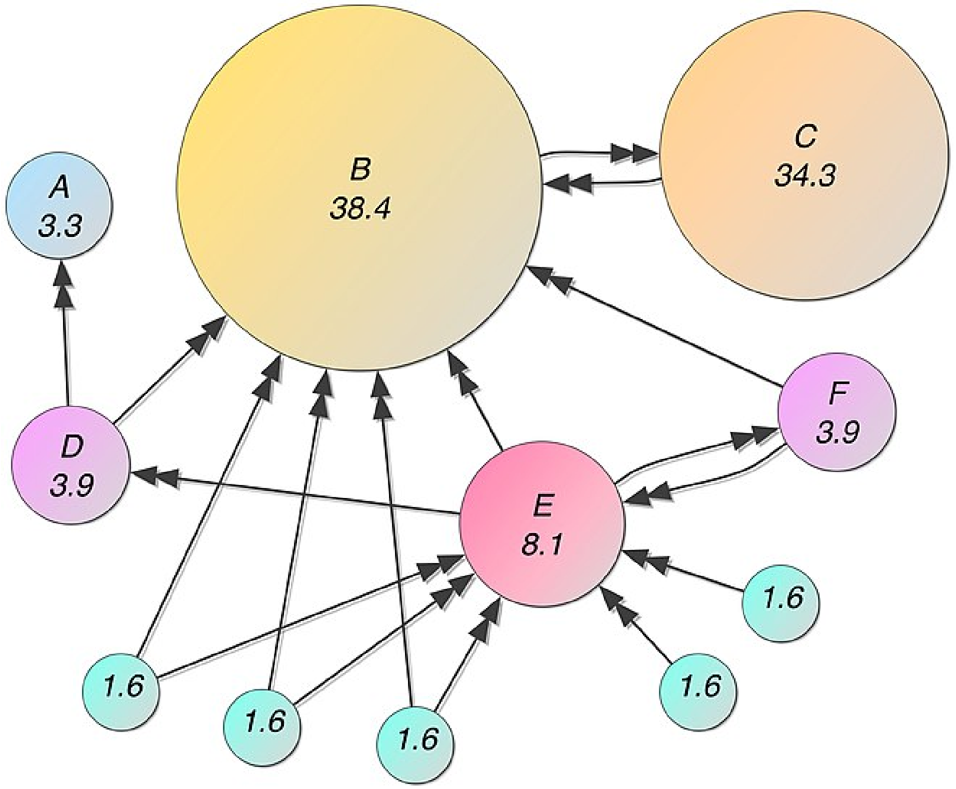
$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

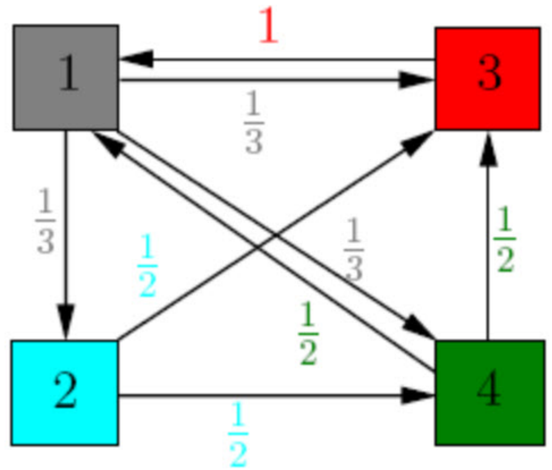
$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Sources:

- https://365datascience.com/tutorials/python-tutorials/principal-components-analysis/

Google PageRank





$$A = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

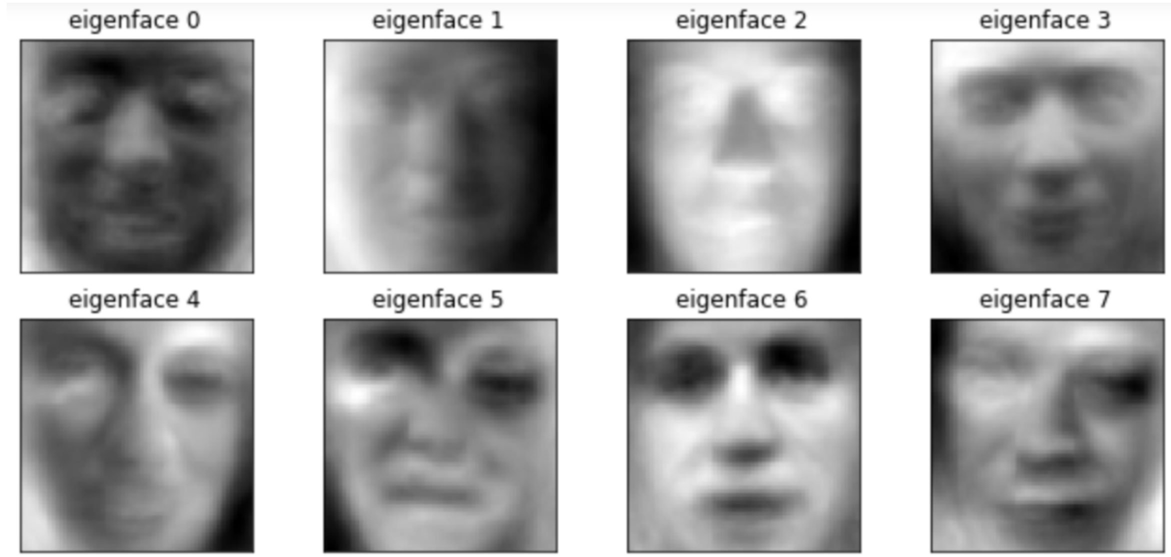
Site importance x_i:

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Sources:

- http://pi.math.cornell.edu/~mec/Winter2009/RalucaRemus/Lecture3/lecture3.html
- https://en.wikipedia.org/wiki/PageRank

Image Processing: Eigenfaces



$$\text{Sheryl Crow} \approx 0.225\ e_0 + 0.092\ e_1 + \ldots - 0.143\ e_{1000}$$

$$\text{Your face's image} \approx 0.418\ e_0 + 0.078\ e_1 + \ldots + 0.972\ e_{1000}$$

Sources:

- https://365datascience.com/tutorials/python-tutorials/principal-components-analysis/

Physics of Waves and Quantum: Eigenfunctions

$$\frac{d^2}{dx^2}X(x) = -\frac{\omega^2}{c^2}X(x)$$
$$\frac{d^2}{dt^2}T(t) = -\omega^2 T(t)$$

Vibrating Strings

$$i\hbar\frac{\partial}{\partial t}\Psi(r,t) = H\Psi(r,t)$$

Schrödinger Equation

Sources:

- https://en.wikipedia.org/wiki/Eigenfunction

And we're done: Good luck on all your final exams!